Entropy theory in the nonamenable setting

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Current Trends in Dynamical Systems and the Mathematical
Legacy of Rufus Bowen,
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Overview

Classical entropy theory is concerned with systems that evolve in time.

Time is usually represented by either $\mathbb{Z}, \mathbb{N}, \mathbb{R}$ or $\mathbb{R}^{>0}$ but more general groups such as $\mathbb{Z}^d, \mathbb{R}^d$ can be and have been considered.

What happens if we replace the acting group with a free group $\mathbb{F}_2 = \langle a, b \rangle$?

The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $\mathbb{F} = \langle a, b \rangle$ is the rank 2 free group then the full 2-shift over $\mathbb{F}$ factors onto the full 4-shift over $\mathbb{F}$.

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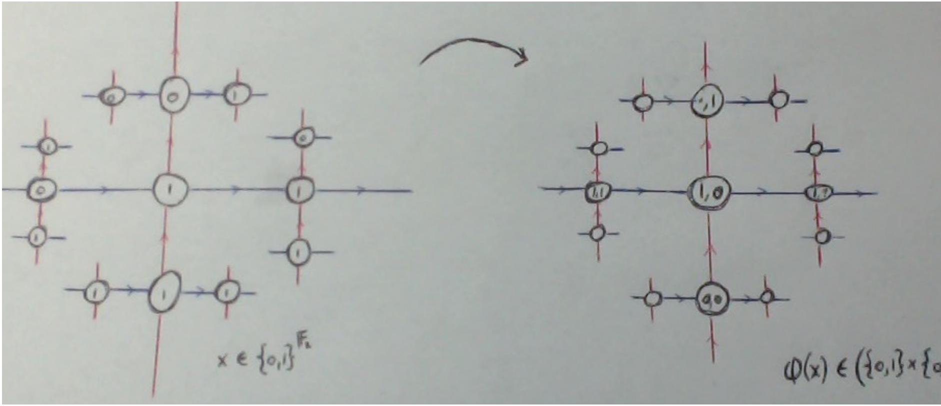
This is surjective, shift-equivariant, 2-1, continuous and a homomorphism of compact abelian groups!

(Ornstein-Weiss, 1987): Is the full 2-shift over $\mathbb{F}$ isomorphic to the full 4-shift?

The Ornstein-Weiss map



Factors between Bernoulli shifts

A **Bernoulli shift over a countable group** Γ is an action of the form $\Gamma \curvearrowright (K^\Gamma, \kappa^\Gamma)$ where K is a Borel space, κ is a probability measure on K and $\Gamma \curvearrowright K^\Gamma$ by

$$(gx)_f = x_{g^{-1}f} \text{ for } g, f \in \Gamma, x \in K^\Gamma.$$

Theorem (B., 2017)

If Γ is any non-amenable group then every Bernoulli shift over G factors onto every Bernoulli shift over G.

Topological entropy ala Rufus Bowen

Let $T : X \to X$ be a homeomorphism of a compact metrizable space X.

The **topological entropy of** (X, T) is the exponential growth rate of the number of length n partial orbits that can be distinguished at scale ϵ (and then send $\epsilon \searrow 0$).

Topological entropy ala Rufus Bowen

Let ρ be a metric on X.

A **length-n partial orbit** is an n-tuple of the form $\underline{x} = (x, Tx, T^2x, \ldots, T^{n-1}x)$.

The **ρ_∞-distance** on length-n partial orbits is

$$\rho_\infty(\underline{x}, \underline{y}) = \max_{0 \le i \le n-1} \rho(T^i x, T^i y).$$

Let $\mathrm{cov}_\epsilon(n, \rho_\infty)$ be the minimum cardinality of a collection $\mathcal{C}$ of length-n partial orbits that is **(ρ_∞, ϵ)-covering** in the sense that every length-n partial orbit is (ρ_∞, ϵ)-close to some partial orbit in $\mathcal{C}$.

$$h(X, T) := \sup_{\epsilon > 0} \limsup_{n \to \infty} n^{-1} \log \mathrm{cov}_\epsilon(n, \rho_\infty)$$

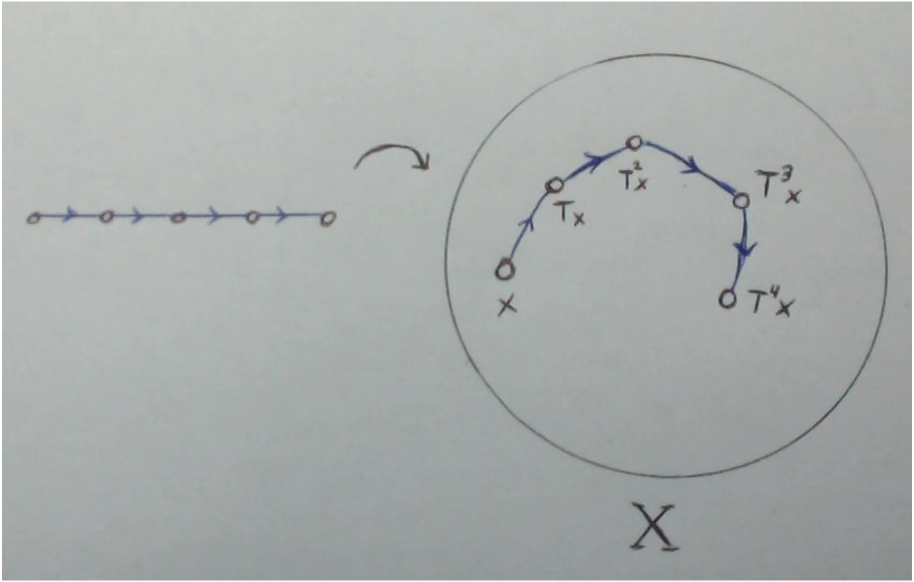
Main Results

1. If (X, T) embeds into (Y, S) then $h(X, T) \leq h(Y, S)$.

2. In particular, entropy is a topological conjugacy invariant.

3. (Topological entropy was defined earlier in a different way by Adler, Konheim and McAndrew in 1965).

A partial orbit



Pseudo-orbits

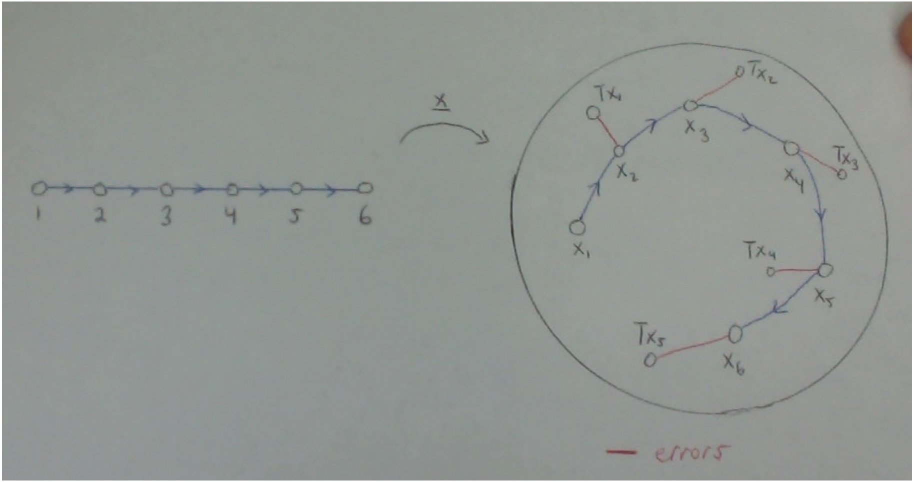
Consider softening the notion of partial orbit.

An (n, δ)**-pseudo orbit** is a tuple $\underline{x} = (x_1, \ldots, x_n) \in X^n$ such that

$$\frac{1}{n} \sum_{i=1}^{n-1} \rho(Tx_i, x_{i+1}) < \delta.$$

Note: we are using an ℓ^1 metric instead of an ℓ^∞ metric.

A pseudo-orbit



Entropy via pseudo-orbits

Let $\mathrm{cov}_\epsilon(n, \delta, \rho_\infty)$ be the minimum cardinality of a collection $\mathcal{C}$ of (n, δ)-pseudo orbits that is (ρ_∞, ϵ)**-covering** in the sense that every (n, δ)-pseudo orbit is (ρ_∞, ϵ)-close to something in $\mathcal{C}$.

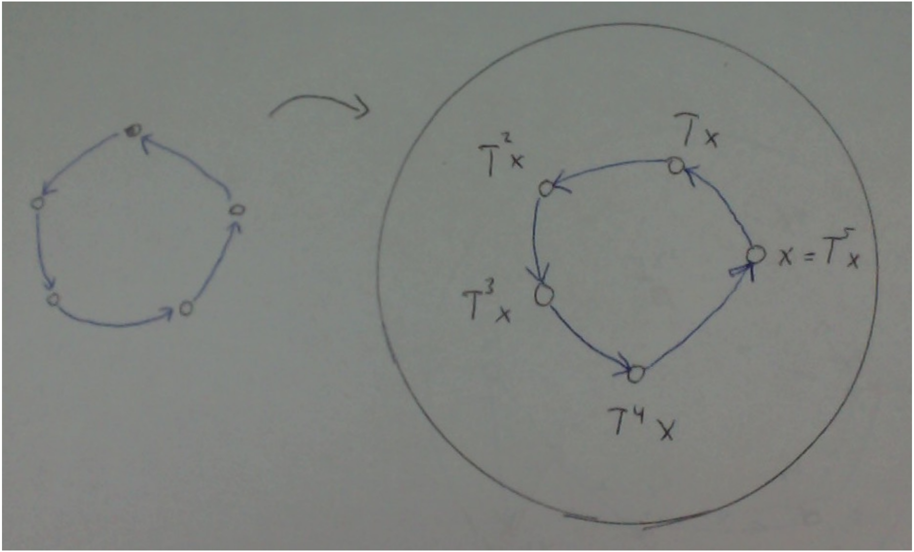
Theorem

$$h(X, T) = \sup_{\epsilon>0} \inf_{\delta>0} \limsup_{n\to\infty} n^{-1} \log \mathrm{cov}_\epsilon(n, \delta, \rho_\infty)$$

Periodic orbits

A **periodic orbit with period** n is a tuple $(x, Tx, \ldots, T^{n-1}x)$ such that $T^n x = x$ (up to cyclic reordering).

A periodic orbit



Entropy via periodic orbits?

The exponential rate of growth of the number of periodic points that can be distinguished at scale ϵ (and then send $\epsilon \searrow 0$) is a lower bound for entropy.

But in general, it is not equal to entropy.

How to compute entropy

partial orbits

pseudo orbits

periodic orbits

pseudo periodic orbits

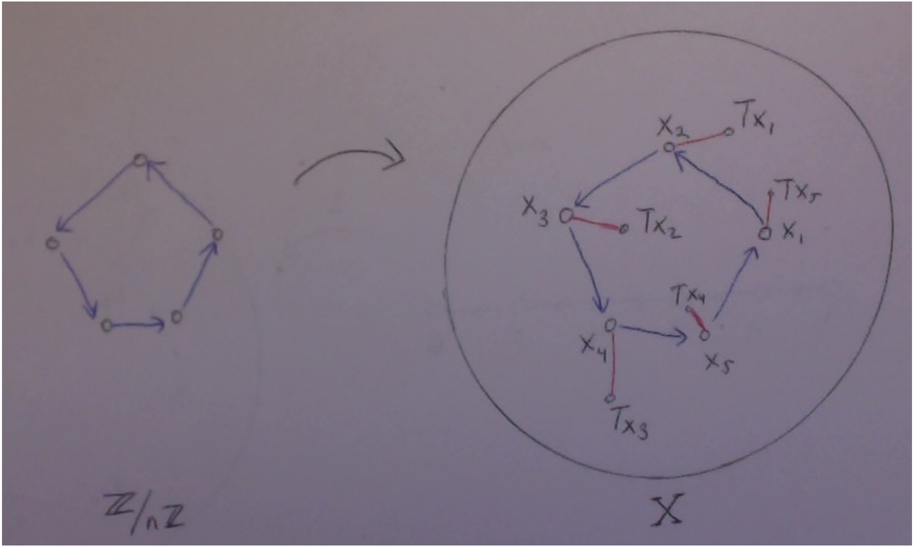
Pseudo periodic orbits

An (n, δ)**-pseudo periodic orbit** is a tuple $\underline{x} = (x_1, \ldots, x_n) \in X^n$ (up to cyclic order) such that

$$\frac{1}{n} \sum_{i=1}^{n} \rho(Tx_i, x_{i+1}) < \delta$$

(indices mod n).

Pseudo periodic orbits



Entropy via pseudo periodic orbits

Let $\text{cov}_\epsilon^{per}(n, \delta, \rho_\infty)$ be the minimum cardinality of a collection $\mathcal{C}$ of (n, δ)-pseudo periodic orbits that is (ρ_∞, ϵ)**-covering** in the sense that every (n, δ)-pseudo periodic orbit is (ρ_∞, ϵ)-close to something in $\mathcal{C}$.

Theorem

$$h(X, T) = \sup_{\epsilon>0} \inf_{\delta>0} \limsup_{n\to\infty} n^{-1} \log \text{cov}_\epsilon^{per}(n, \delta, \rho_\infty)$$

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(pseudo periodic orbits are also called **microstates**)

A first step towards sofic entropy

Let Γ be a countable group, $\Gamma \curvearrowright X$ an action by homeomorphisms.

Preliminary definition : an **pseudo periodic orbit** consists of an action $\Gamma \curvearrowright^{\sigma} V$ on a finite set and a map $\phi : V \to X$ that is approximately equivariant in the following ℓ^1-sense:

$$|V|^{-1} \sum_{v \in V} \rho\Big(\phi(\sigma(g)v), g\phi(v)\Big) < \delta \quad \forall g \in F$$

where $F \subset \Gamma$ is finite.

More precisely, this is a (σ, δ, F)**-pseudo periodic orbit** .

A first step towards sofic entropy

Let $\Sigma = \{\Gamma \curvearrowright^{\sigma_n} V_n\}$ be a sequence of actions on finite sets.

Preliminary definition : the **sofic entropy of $\Gamma \curvearrowright X$ with respect to Σ** is

$$h_\Sigma(\Gamma \curvearrowright X) := \sup_{\epsilon>0} \inf_{\delta>0, F \Subset \Gamma} \limsup_{n\to\infty} |V_n|^{-1} \log \mathrm{cov}_\epsilon^{per}(\sigma_n, \delta, F, \rho_\infty)$$

where $\mathrm{cov}_\epsilon^{per}(\sigma_n, \delta, F, \rho_\infty)$ is the minimum cardinality of a collection $\mathcal{C}$ of (σ_n, δ, F)-pseudo periodic orbits that is **(ρ_∞, ϵ)-covering** in the sense that every (σ_n, δ, F)-pseudo periodic orbit is (ρ_∞, ϵ)-close to something in $\mathcal{C}$.

Main Results (Kerr-Li, 2010)

1. If $\Gamma \curvearrowright X$ embeds into $\Gamma \curvearrowright Y$ then $h_\Sigma(\Gamma \curvearrowright X) \le h_\Sigma(\Gamma \curvearrowright Y)$.
2. In particular, Σ-entropy is a topological conjugacy invariant.
3. $h_\Sigma(\Gamma \curvearrowright K^\Gamma) = \log |K|$.

A boring example

Suppose V_n is a single point for all n.

Then $h_\Sigma(\Gamma \curvearrowright X) = \log \#$ (fixed points).

This isn't what is usually meant by entropy.

To fix this, require that the actions $\Gamma \curvearrowright^{\sigma_n} V_n$ witness Γ in the sense that: for all $g \in \Gamma \setminus \{1_\Gamma\}$,

$$|V_n|^{-1} \# \{v \in V_n : \sigma_n(g)v \neq v\} \to 1 \text{ as } n \to \infty.$$

With this assumption, Σ is said to be a **sofic approximation** to Γ.

A curious example

Let $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright \{0, 1\}$ so that each of a, b act nontrivially.

Any action $\mathbb{F}_2 \curvearrowright^{\sigma} V$ on a finite set V determines a graph $G = (V, E)$ where

$$E = \{(v, \sigma(a)v), (v, \sigma(b)v) : \ v \in V\}.$$

If the graphs corresponding to the actions in $\Sigma = \{\Gamma \curvearrowright V_n\}$ are bi-partite then

$$h_\Sigma(\mathbb{F}_2 \curvearrowright \{0, 1\}) = 0.$$

If they are far from bi-partite (e.g. if $\sigma \in \mathrm{Hom}(\mathbb{F}_2, \mathrm{sym}(V))$ is uniformly random) then there are no pseudo periodic orbits and

$$h_\Sigma(\mathbb{F}_2 \curvearrowright \{0, 1\}) = -\infty.$$

What is this good for?

Gottschalk's Surjunctivity Conjecture (1973): Let k be a finite set and $\Phi : k^\Gamma \to k^\Gamma$ a continuous shift-equivariant injective map. Then Φ is surjective.

Theorem (Gromov, 1999)

If Γ is sofic then the conjecture is true.

Proof by Kerr-Li, 2010.

- $h_\Sigma(\Gamma \curvearrowright k^\Gamma) = \log |k|$.
- $h_\Sigma(\Gamma \curvearrowright \Phi(k^\Gamma)) = \log |k|$.
- The sofic entropy of any proper subshift of k^Γ is $< \log |k|$.

□

Partial actions

We don't actually need $\Gamma \curvearrowright^{\sigma_n} V_n$ to be actions.

Instead we require $\{\sigma_n : \Gamma \to \mathrm{sym}(V_n)\}$ to be a sequence of maps (not necessarily homomorphisms!) such that

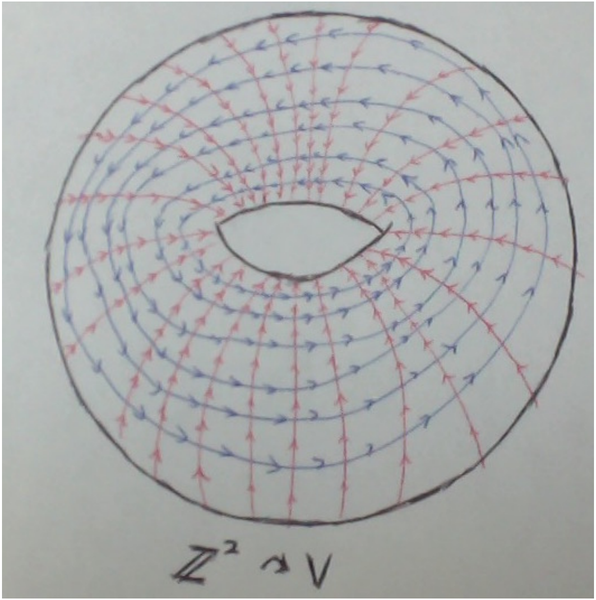
$$\forall g, h \in \Gamma, \quad |V_n|^{-1} \#\{v \in V_n : \ \sigma_n(gh)v = \sigma_n(g)\sigma_n(h)v\} \to 1 \text{ as } n \to \infty$$

$$\forall g \in \Gamma \setminus \{1_\Gamma\}, \quad |V_n|^{-1} \#\{v \in V_n : \ \sigma_n(g)v \neq v\} \to 1 \text{ as } n \to \infty.$$

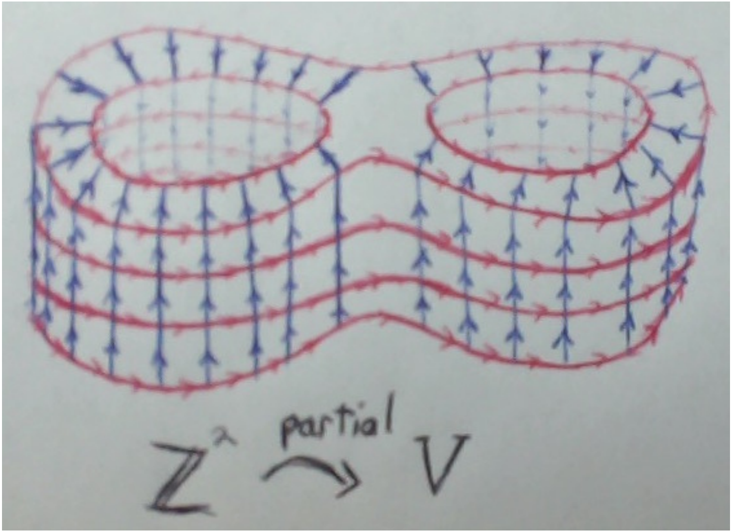
Such a sequence is a **sofic approximation** and Γ is **sofic** it has one.

Definition due to Gromov (1999), named and made accessible by Weiss (2000).

An action of $\mathbb{Z}^2$



A partial action of $\mathbb{Z}^2$



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- The class of sofic groups is closed under: subgroups, direct limits, inverse limits, direct products, extensions by amenable groups, free products with amenable amalgamation, wreathe products. (Elek-Szabo, Dykema-Kerr-Pichot, Paunescu, Hayes-Sale)

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- Open: Is every countable group sofic?

Measure entropy ala Kerr-Li

Let $\Gamma \curvearrowright (X, \mu)$ be an action by homeomorphisms and μ an invariant probability measure.

The **measure sofic entropy of $\Gamma \curvearrowright (X, \mu)$** is the exponential growth rate of the number of approximately equidistributed periodic orbits that can be distinguished at scale ϵ (and then send $\epsilon \searrow 0$).

Measure entropy ala Kerr-Li

The **empirical distribution** of a map $\phi : V \to X$ is the probability measure

$$P_\phi := \frac{1}{|V|} \sum_{v \in V} \delta_{\phi(v)} \in \mathrm{Prob}(X).$$

If $\mathcal{O} \subset \mathrm{Prob}(X)$ is an open neighborhood of μ then a **$(\sigma, \delta, F, \mathcal{O})$-pseudo periodic orbit** is a map $\phi : V \to X$ such that ϕ is a (σ, δ, F)-pseudo periodic orbit and $P_\phi \in \mathcal{O}$.

Measure entropy ala Kerr-Li

Let $\mathrm{cov}_\epsilon(\sigma, \delta, F, \mathcal{O}, \rho_\infty)$ be the minimum cardinality of a collection $\mathcal{C}$ of $(\sigma, \delta, F, \mathcal{O})$-pseudo periodic orbits that (ρ_∞, ϵ)-cover the set of all $(\sigma, \delta, F, \mathcal{O})$-pseudo periodic orbits.

$$h_\Sigma(\Gamma \curvearrowright (X, \mu)) := \sup_{\epsilon>0} \inf_{\delta, F, \mathcal{O}} \limsup_{n\to\infty} |V_n|^{-1} \log \mathrm{cov}_\epsilon(\sigma_n, \delta, F, \mathcal{O}, \rho_\infty).$$

Main Results

1. (Variational Principle, Kerr-Li) $h_\Sigma(\Gamma \curvearrowright X) = \sup_\mu h_\Sigma(\Gamma \curvearrowright (X, \mu))$.

2. (B., Kerr-Li) Measure sofic entropy is a measure conjugacy invariant.

3. If (K, κ) is any probability space and $\Gamma \curvearrowright K^{\mathbb{Z}}$ is the shift action $(gx)_h = x_{g^{-1}h}$ then

$$h_\Sigma(\Gamma \curvearrowright (K, \kappa)^\Gamma) = H(\kappa) := \sum_{k \in K} -\kappa(\{k\}) \log \kappa(\{k\}).$$

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4. This is the **Bernoulli shift over $\mathbb{Z}$ with base (K, κ)** .

5. So the 2-shift over $\mathbb{F}_2$ is not isomorphic to the 4-shift over $\mathbb{F}_2$!

Classification of Bernoulli shifts

Conjecture: Assume $|\Gamma| = \infty$. Then

$$\Gamma \curvearrowright (K, \kappa)^\Gamma \cong \Gamma \curvearrowright (L, \lambda)^\Gamma \Leftrightarrow H(\kappa) = H(\lambda).$$

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- Γ amenable groups (folklore or Kieffer?, 1970s)
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$\Leftarrow$

- $\Gamma = \mathbb{Z}$ (Ornstein, 1970)
- Γ amenable (Ornstein-Weiss, 1980)
- $\mathbb{Z} \le \Gamma$ (Stepin, 1975)
- $\forall \Gamma$, $|K| > 2$ and $|L| > 2$ (B. 2012)
- $\forall \Gamma$ (Seward, 2017)

Further topics

- Markov chains over free groups (B. 2010) : a variant of sofic entropy theory via random permutations yields exact computations and structural results although a classification remains elusive.

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- Rokhlin entropy is an upper bound for sofic entropy. Brandon Seward has used it to prove generalizations of Krieger's generator theorem and Sinai's Factor Theorem for all countable groups.
- Weak Pinsker Conjecture: Tim Austin recently posted a solution for actions of amenable groups. I have a counterexample in the case of free group actions based on sofic entropy theory and probabilistic combinatorics via constraint satisfaction problems.

Algebraic Dynamics

Theorem (Ben Hayes)

Consider an action of a sofic group Γ *on a compact group* X *by automorphisms.*

1. *Under mild hypotheses, the topological sofic entropy and measure sofic entropy agree.*
2. *For any* $f \in \mathbb{Z}\Gamma$ *such that left convolution with* f *is injective as an operator on* $\ell^2(\Gamma)$*, the sofic entropy of* $\Gamma \curvearrowright \widehat{(\mathbb{Z}/f\mathbb{Z})}$ *is* $\log \det^+ |f|$.
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earlier work due to: Rufus Bowen, B., Deninger, Kerr, Li, Lind, Schmidt, Ward, Yuzvinskii, . . .