Bowen's dimension formula and rigorous estimates

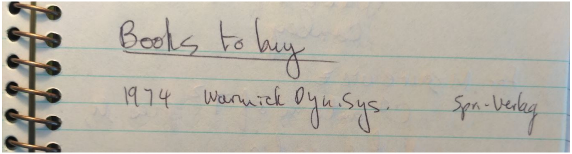
Mark Pollicott, Warwick University

4 August, 2017

Bowen's dimension formula and rigorous estimates

Mark Pollicott, Warwick University

4 August, 2017



Overview

- In one of his final papers Bowen introduced a method for describing the Hausdorff Dimension of certain dynamically defined sets (Quasi-Circles).

Overview

- In one of his final papers Bowen introduced a method for describing the Hausdorff Dimension of certain dynamically defined sets (Quasi-Circles).
- The method was subsequently developed by Ruelle to describe the Hausdorff Dimension of hyperbolic Julia sets and, more generally, conformal repellors. This has now become a standard technique.

Overview

- In one of his final papers Bowen introduced a method for describing the Hausdorff Dimension of certain dynamically defined sets (Quasi-Circles).
- The method was subsequently developed by Ruelle to describe the Hausdorff Dimension of hyperbolic Julia sets and, more generally, conformal repellors. This has now become a standard technique.
- Eventually I want to discuss how this method can be useful in rigorous numerical estimates (joint work with O.Jenkinson).

Overview

- In one of his final papers Bowen introduced a method for describing the Hausdorff Dimension of certain dynamically defined sets (Quasi-Circles).
- The method was subsequently developed by Ruelle to describe the Hausdorff Dimension of hyperbolic Julia sets and, more generally, conformal repellors. This has now become a standard technique.
- Eventually I want to discuss how this method can be useful in rigorous numerical estimates (joint work with O.Jenkinson).

Question

Why should we be interested in numerically computing dimension?

Overview

- In one of his final papers Bowen introduced a method for describing the Hausdorff Dimension of certain dynamically defined sets (Quasi-Circles).
- The method was subsequently developed by Ruelle to describe the Hausdorff Dimension of hyperbolic Julia sets and, more generally, conformal repellors. This has now become a standard technique.
- Eventually I want to discuss how this method can be useful in rigorous numerical estimates (joint work with O.Jenkinson).

Question

Why should we be interested in numerically computing dimension?

This may be a matter of taste. *However, the most interesting aspect is the mathematical method of getting good bounds on the error.*

Overview

- In one of his final papers Bowen introduced a method for describing the Hausdorff Dimension of certain dynamically defined sets (Quasi-Circles).
- The method was subsequently developed by Ruelle to describe the Hausdorff Dimension of hyperbolic Julia sets and, more generally, conformal repellors. This has now become a standard technique.
- Eventually I want to discuss how this method can be useful in rigorous numerical estimates (joint work with O.Jenkinson).

Question

Why should we be interested in numerically computing dimension?

This may be a matter of taste. *However, the most interesting aspect is the mathematical method of getting good bounds on the error.*

However, before all this I will mention two connections Rufus Bowen had with Warwick.

Warwick Connection I

Bowen was at Warwick University, U.K., in 1969-70.

Warwick Connection I

Bowen was at Warwick University, U.K., in 1969-70.



There was a tree planted in his memory nearby.

Warwick Connection I

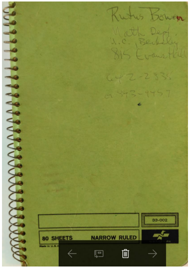
Bowen was at Warwick University, U.K., in 1969-70.

There was a tree planted in his memory nearby.

This tree had to be moved twice, because of building work and ultimately the tree had to be replaced by a newer/healthier one.

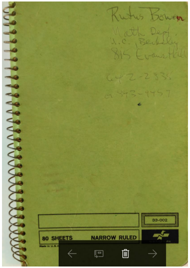
Warwick Connection II

The notebook of Rufus Bowen listing 157 problems has been in Warwick during recent years.



Warwick Connection II

The notebook of Rufus Bowen listing 157 problems has been in Warwick during recent years.



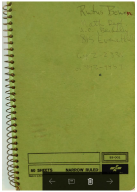
I brought it with me on my flight from the UK last Friday.

Aside: A tale of two notebooks

There is an interesting parallel with another famous notebook.

Aside: A tale of two notebooks

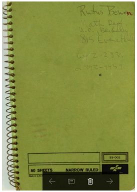
There is an interesting parallel with another famous notebook.



The Bowen notebook was brought
to the UK by Peter Walters in 1978.

Aside: A tale of two notebooks

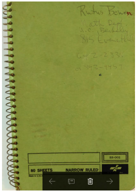
There is an interesting parallel with another famous notebook.

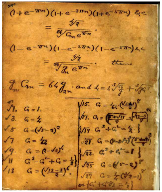


The Bowen notebook was brought to the UK by Peter Walters in 1978. For the next 39 years this notebook was in Peter's house in Kenilworth (5,256 miles from Berkeley and 3 miles from Warwick University)

Aside: A tale of two notebooks

There is an interesting parallel with another famous notebook.



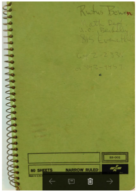


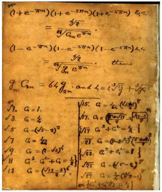
The Bowen notebook was brought to the UK by Peter Walters in 1978. For the next 39 years this notebook was in Peter's house in Kenilworth (5,256 miles from Berkeley and 3 miles from Warwick University)

After Ramanujan's death in 1920, his "lost" notebook was sent from Madras to Hardy, in England, who passed it to Watson.

Aside: A tale of two notebooks

There is an interesting parallel with another famous notebook.





The Bowen notebook was brought to the UK by Peter Walters in 1978. For the next 39 years this notebook was in Peter's house in Kenilworth (5,256 miles from Berkeley and 3 miles from Warwick University)

After Ramanujan's death in 1920, his "lost" notebook was sent from Madras to Hardy, in England, who passed it to Watson. For the next 42 years this notebook stayed in his house in Leamington Spa (8,299 miles from Madras and 7 miles from Warwick University)

Dimension of sets

The *Bowen dimension formula* deals with the dimension of certain sets $X \subset \mathbb{R}^D$.

Dimension of sets

The *Bowen dimension formula* deals with the dimension of certain sets $X \subset \mathbb{R}^D$.

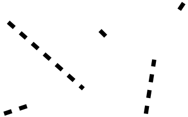
In all our examples the Hausdorff Dimension will be the same as the Box Dimension, so we can cheat and recall its (simpler) definition instead.

Dimension of sets

The *Bowen dimension formula* deals with the dimension of certain sets $X \subset \mathbb{R}^D$.

In all our examples the Hausdorff Dimension will be the same as the Box Dimension, so we can cheat and recall its (simpler) definition instead.

For $\epsilon > 0$ we let $N(\epsilon)$ be the smallest number of D-dimensional ϵ-boxes needed to cover X.

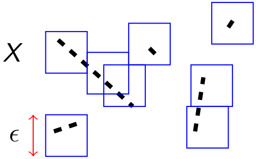
X


Dimension of sets

The *Bowen dimension formula* deals with the dimension of certain sets $X \subset \mathbb{R}^D$.

In all our examples the Hausdorff Dimension will be the same as the Box Dimension, so we can cheat and recall its (simpler) definition instead.

For $\epsilon > 0$ we let $N(\epsilon)$ be the smallest number of D-dimensional ϵ-boxes needed to cover X.

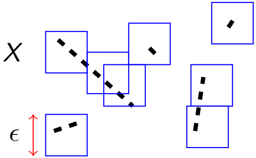


Dimension of sets

The *Bowen dimension formula* deals with the dimension of certain sets $X \subset \mathbb{R}^D$.

In all our examples the Hausdorff Dimension will be the same as the Box Dimension, so we can cheat and recall its (simpler) definition instead.

For $\epsilon > 0$ we let $N(\epsilon)$ be the smallest number of D-dimensional ϵ-boxes needed to cover X.



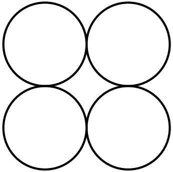
Definition

We define the dimension by: $\dim(X) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?

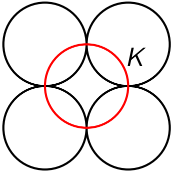


Consider four touching circles in the plane

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?

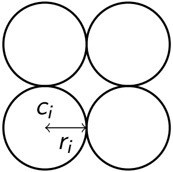


Consider four touching circles in the plane chosen such that there is a circle K passing through the 4 points of contact.

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?



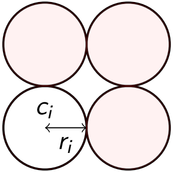
Consider four touching circles in the plane chosen such that there is a circle K passing through the 4 points of contact. The inversions $\gamma_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$\gamma_i(z) = \frac{r_i^2(z - c_i)}{|z - c_i|^2} + c_i, \qquad \text{for } i = 1, 2, 3, 4,$$

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?



Consider four touching circles in the plane chosen such that there is a circle K passing through the 4 points of contact. The inversions $\gamma_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

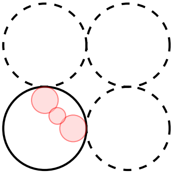
$$\gamma_i(z) = \frac{r_i^2(z - c_i)}{|z - c_i|^2} + c_i, \quad \text{for } i = 1, 2, 3, 4,$$

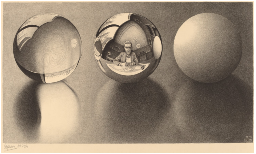
map the inside of C_i to the outside,

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?





Consider four touching circles in the plane chosen such that there is a circle K passing through the 4 points of contact. The inversions $\gamma_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

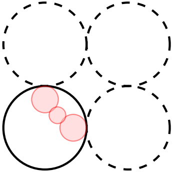
$$\gamma_i(z) = \frac{r_i^2(z - c_i)}{|z - c_i|^2} + c_i, \quad \text{for } i = 1, 2, 3, 4,$$

map the inside of C_i to the outside, and vica versa.

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?



Consider four touching circles in the plane chosen such that there is a circle K passing through the 4 points of contact. The inversions $\gamma_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

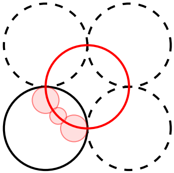
$$\gamma_i(z) = \frac{r_i^2(z - c_i)}{|z - c_i|^2} + c_i, \quad \text{for } i = 1, 2, 3, 4,$$

map the inside of C_i to the outside, and vica versa.
Trivial Observation: K is fixed by the transformations, i.e., $\gamma_i K = K$.

Quasi-Circles: A simple example

Question

What was Bowen's quasi-circle result about?



Consider four touching circles in the plane chosen such that there is a circle K passing through the 4 points of contact. The inversions $\gamma_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$\gamma_i(z) = \frac{r_i^2(z - c_i)}{|z - c_i|^2} + c_i, \quad \text{for } i = 1, 2, 3, 4,$$

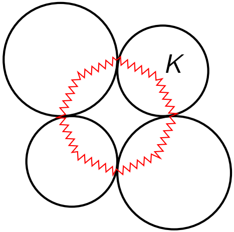
map the inside of C_i to the outside, and vica versa.
Trivial Observation: K is fixed by the transformations, i.e., $\gamma_i K = K$.

Quasi-Circles

If the four contact points don't lie on a circle then there is no longer a circle fixed under the corresponding 4 transformations $R_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ $(i = 1, 2, 3, 4)$

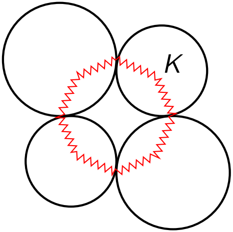
Quasi-Circles

If the four contact points don't lie on a circle then there is no longer a circle fixed under the corresponding 4 transformations $R_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ $(i = 1, 2, 3, 4)$ however there is still a quasi-circle K (topological circle)



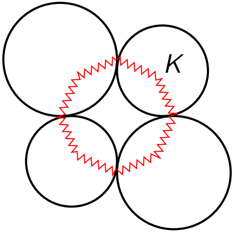
Quasi-Circles

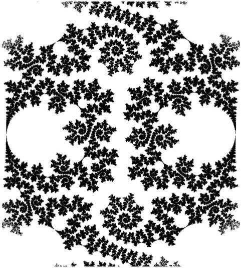
If the four contact points don't lie on a circle then there is no longer a circle fixed under the corresponding 4 transformations $R_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ $(i = 1, 2, 3, 4)$ however there is still a quasi-circle K (topological circle)



Quasi-Circles

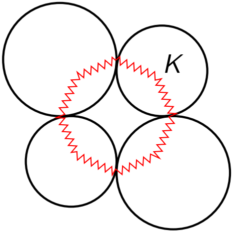
If the four contact points don't lie on a circle then there is no longer a circle fixed under the corresponding 4 transformations $R_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ $(i = 1, 2, 3, 4)$ however there is still a quasi-circle K (topological circle)





Quasi-Circles

If the four contact points don't lie on a circle then there is no longer a circle fixed under the corresponding 4 transformations $R_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ $(i = 1, 2, 3, 4)$ however there is still a quasi-circle K (topological circle)



Claim

When K isn't a circle, then it has Hausdorff Dimension $\dim_H(K) > 1$.

The Bowen paper on Quasi-Circles

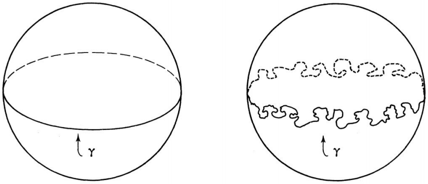
The Bowen paper dealt with a similar problem for Quasi-Fuchsian groups.
Let $\Gamma_0 < PSL(2, \mathbb{C})$ be a discrete group of Möbius transformations of $\widehat{\mathbb{C}}$

The Bowen paper on Quasi-Circles

The Bowen paper dealt with a similar problem for Quasi-Fuchsian groups.
Let $\Gamma_0 < PSL(2, \mathbb{C})$ be a discrete group of Möbius transformations of $\widehat{\mathbb{C}}$ which preserve a circle K_0 (i.e., a Fuchsian group).

The Bowen paper on Quasi-Circles

The Bowen paper dealt with a similar problem for Quasi-Fuchsian groups.
Let $\Gamma_0 < PSL(2, \mathbb{C})$ be a discrete group of Möbius transformations of $\widehat{\mathbb{C}}$ which preserve a circle K_0 (i.e., a Fuchsian group).

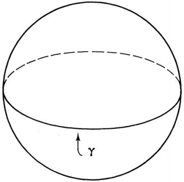


γ is a round circle (Theorem 1) γ is a genuine quasi-circle (Theorem 2)

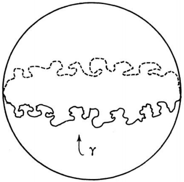
For a nearby discrete group Γ there is still a quasi-circle K fixed by each $\gamma \in \Gamma$.

The Bowen paper on Quasi-Circles

The Bowen paper dealt with a similar problem for Quasi-Fuchsian groups.
Let $\Gamma_0 < PSL(2, \mathbb{C})$ be a discrete group of Möbius transformations of $\widehat{\mathbb{C}}$ which preserve a circle K_0 (i.e., a Fuchsian group).



γ is a round circle (Theorem 1)



γ is a genuine quasi-circle (Theorem 2)

For a nearby discrete group Γ there is still a quasi-circle K fixed by each $\gamma \in \Gamma$.

Theorem (Bowen, 1979)

If Γ_0 is cocompact then either

1. *K is still a genuine circle, or*
2. *K has Hausdorff Dimension > 1.*

Bowen Paper

This paper was published posthumously in 1979 and is his 4th most cited publication.

Most Cited Publications	
Citations	**Publication**
821	**MR0442989 (56 #1364)** Bowen, Rufus Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. *Springer-Verlag, Berlin-New York,* 1975. i+108 pp. (Reviewer: L. A. Bunimovic) 58F10 (28A65) **Book**
223	**MR0274707 (43 #469)** Bowen, Rufus Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** 1971 401–414. (Reviewer: G. Della Riccia) 28.70 (22.00)
192	**MR0380889 (52 #1786)** Bowen, Rufus; Ruelle, David The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), no. 3, 181–202. (Reviewer: L. A. Bunimovic) 58F15 (28A65)
165	**MR0556580 (81g:57023)** Bowen, Rufus Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* No. 50 (1979), 11–25. (Reviewer: L. A. Bunimovich) 57S05 (30C20 32G15 58F11)

Bowen Paper

This paper was published posthumously in 1979 and is his 4th most cited publication.

Most Cited Publications		
Citations	**Publication**	
821	**MR0442989 (56 #1364)** Bowen, Rufus Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. *Springer-Verlag, Berlin-New York,* 1975. i+108 pp. (Reviewer: L. A. Bunimovic) 58F10 (28A65)	**Book**
223	**MR0274707 (43 #469)** Bowen, Rufus Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** 1971 401–414. (Reviewer: G. Della Riccia) 28.70 (22.00)	
192	**MR0380889 (52 #1786)** Bowen, Rufus; Ruelle, David The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), no. 3, 181–202. (Reviewer: L. A. Bunimovic) 58F15 (28A65)	
165	**MR0556580 (81g:57023)** Bowen, Rufus Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* No. 50 (1979), 11–25. (Reviewer: L. A. Bunimovich) 57S05 (30C20 32G15 58F11)	

These include generalizations to

- more general groups (Bishop and Jones) and
- higher dimensions (C.-B. Yue).

Bowen Paper

This paper was published posthumously in 1979 and is his 4th most cited publication.

Most Cited Publications		
Citations	**Publication**	
821	**MR0442989 (56 #1364)** Bowen, Rufus Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. *Springer-Verlag, Berlin-New York,* 1975. i+108 pp. (Reviewer: L. A. Bunimovic) 58F10 (28A65)	**Book**
223	**MR0274707 (43 #469)** Bowen, Rufus Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** 1971 401–414. (Reviewer: G. Della Riccia) 28.70 (22.00)	
192	**MR0380889 (52 #1786)** Bowen, Rufus; Ruelle, David The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), no. 3, 181–202. (Reviewer: L. A. Bunimovic) 58F15 (28A65)	
165	**MR0556580 (81g:57023)** Bowen, Rufus Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* No. 50 (1979), 11–25. (Reviewer: L. A. Bunimovich) 57S05 (30C20 32G15 58F11)	

These include generalizations to

- more general groups (Bishop and Jones) and
- higher dimensions (C.-B. Yue).

But perhaps the reason for its influence is that Bowen's original idea has proved useful in a multitude of similar settings.

Bowen Paper

This paper was published posthumously in 1979 and is his 4th most cited publication.

Most Cited Publications	
Citations	**Publication**
821	**MR0442989 (56 #1364)** Bowen, Rufus Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. *Springer-Verlag, Berlin-New York,* 1975. i+108 pp. (Reviewer: L. A. Bunimovic) 58F10 (28A65) **Book**
223	**MR0274707 (43 #469)** Bowen, Rufus Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** 1971 401–414. (Reviewer: G. Della Riccia) 28.70 (22.00)
192	**MR0380889 (52 #1786)** Bowen, Rufus; Ruelle, David The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), no. 3, 181–202. (Reviewer: L. A. Bunimovic) 58F15 (28A65)
165	**MR0556580 (81g:57023)** Bowen, Rufus Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* No. 50 (1979), 11–25. (Reviewer: L. A. Bunimovich) 57S05 (30C20 32G15 58F11)

These include generalizations to

- more general groups (Bishop and Jones) and
- higher dimensions (C.-B. Yue).

But perhaps the reason for its influence is that Bowen's original idea has proved useful in a multitude of similar settings. Let us consider a particularly simple one.

A simple application: Iterated Function Schemes

Consider an iterated function scheme given by $T_1, T_2 : [0,1] \to [0,1]$

A simple application: Iterated Function Schemes

Consider an iterated function scheme given by $T_1, T_2 : [0,1] \to [0,1]$ where

1. Each T_i is a C^ω contraction.

A simple application: Iterated Function Schemes

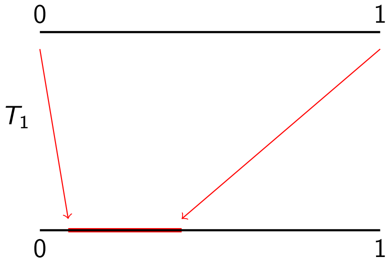
Consider an iterated function scheme given by $T_1, T_2 : [0,1] \to [0,1]$ where

1. Each T_i is a C^ω contraction.
2. The images are disjoint (i.e., $T_1[0,1] \cap T_2[0,1] = \emptyset$).

A simple application: Iterated Function Schemes

Consider an iterated function scheme given by $T_1, T_2 : [0,1] \to [0,1]$ where

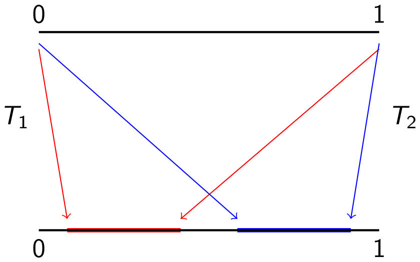
1. Each T_i is a C^ω contraction.
2. The images are disjoint (i.e., $T_1[0,1] \cap T_2[0,1] = \emptyset$).



A simple application: Iterated Function Schemes

Consider an iterated function scheme given by $T_1, T_2 : [0,1] \to [0,1]$ where

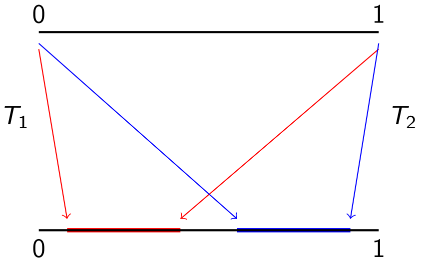
1. Each T_i is a C^ω contraction.
2. The images are disjoint (i.e., $T_1[0,1] \cap T_2[0,1] = \emptyset$).



A simple application: Iterated Function Schemes

Consider an iterated function scheme given by $T_1, T_2 : [0,1] \to [0,1]$ where

1. Each T_i is a C^ω contraction.
2. The images are disjoint (i.e., $T_1[0,1] \cap T_2[0,1] = \emptyset$).



The limit set Λ is the Cantor set of limit points

$$\Lambda = \left\{ \lim_{n \to +\infty} T_{i_1} T_{i_2} \cdots T_{i_n}(x_0) \; : \; i_1, i_2, \cdots \{1,2\} \right\} \text{ for any } x_0 \in [0,1].$$

Example 1: Middle third Cantor set

Let us begin with a trivial example.
Consider the contractions $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{1}{3}$$

Example 1: Middle third Cantor set

Let us begin with a trivial example.
Consider the contractions $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{1}{3}$$

The limit set Λ is the usual middle third Cantor set, i.e.,

$$\Lambda = \left\{ \sum_{n=1}^{\infty} \frac{j_n}{3^n} : j_1, j_2, j_3, \cdots \in \{0,2\} \right\}.$$

Example 1: Middle third Cantor set

Let us begin with a trivial example.
Consider the contractions $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{1}{3}$$

The limit set Λ is the usual middle third Cantor set, i.e.,

$$\Lambda = \left\{ \sum_{n=1}^{\infty} \frac{j_n}{3^n} : j_1, j_2, j_3, \cdots \in \{0, 2\} \right\}.$$

We might describe Λ *to be "linear Cantor set"* since T_1, T_2 are affine maps.

Example 1: Middle third Cantor set

Let us begin with a trivial example.
Consider the contractions $T_1, T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{1}{3}$$

The limit set Λ is the usual middle third Cantor set, i.e.,

$$\Lambda = \left\{ \sum_{n=1}^{\infty} \frac{j_n}{3^n} : j_1, j_2, j_3, \cdots \in \{0,2\} \right\}.$$

We might describe Λ to be "linear Cantor set" since T_1, T_2 are affine maps.

It is easy to see from the definitions that $\dim(\Lambda) = \frac{\log 2}{\log 3}$.

Example 2 (after I.J.Good): E_2

Consider the Cantor set associated to those points $0 < x < 1$ whose continued fraction expansions contain only the digits 1 and 2,

Example 2 (after I.J.Good): E_2

Consider the Cantor set associated to those points $0 < x < 1$ whose continued fraction expansions contain only the digits 1 and 2, i.e.,

$$E_2 := \Lambda = \{[a_1, a_2, a_3, \cdots] : a_1, a_2, a_3, \cdots \in \{1, 2\}\}.$$

Example 2 (after I.J.Good): E_2

Consider the Cantor set associated to those points $0 < x < 1$ whose continued fraction expansions contain only the digits 1 and 2, i.e.,

$$E_2 := \Lambda = \{[a_1, a_2, a_3, \cdots] : a_1, a_2, a_3, \cdots \in \{1, 2\}\}.$$

This corresponds to the limit set for $T_1, T_2 : [0,1] \to [0,1]$ where

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2+x}.$$

Example 2 (after I.J.Good): E_2

Consider the Cantor set associated to those points $0 < x < 1$ whose continued fraction expansions contain only the digits 1 and 2, i.e.,

$$E_2 := \Lambda = \{[a_1, a_2, a_3, \cdots] : a_1, a_2, a_3, \cdots \in \{1, 2\}\}.$$

This corresponds to the limit set for $T_1, T_2 : [0,1] \to [0,1]$ where

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2+x}.$$

We might describe Λ to be "nonlinear Cantor set".

Example 2 (after I.J.Good): E_2

Consider the Cantor set associated to those points $0 < x < 1$ whose continued fraction expansions contain only the digits 1 and 2, i.e.,

$$E_2 := \Lambda = \{[a_1, a_2, a_3, \cdots] : a_1, a_2, a_3, \cdots \in \{1, 2\}\}.$$

This corresponds to the limit set for $T_1, T_2 : [0,1] \to [0,1]$ where

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2+x}.$$

We might describe Λ to be "nonlinear Cantor set".

Unfortunately there is no explicit closed form expression for $\dim(E_2)$, and so we have to resort to calculating its value numerically,

Example 2 (after I.J.Good): E_2

Consider the Cantor set associated to those points $0 < x < 1$ whose continued fraction expansions contain only the digits 1 and 2, i.e.,

$$E_2 := \Lambda = \{[a_1, a_2, a_3, \cdots] : a_1, a_2, a_3, \cdots \in \{1, 2\}\}.$$

This corresponds to the limit set for $T_1, T_2 : [0,1] \to [0,1]$ where

$$T_1(x) = \frac{1}{1+x} \text{ and } T_2(x) = \frac{1}{2+x}.$$

We might describe Λ *to be "nonlinear Cantor set".*

Unfortunately there is no explicit closed form expression for $\dim(E_2)$, and so we have to resort to calculating its value numerically,

Question

How accurately can one estimate $\dim(E_2)$*?*

A Good estimate

The first estimate on this value was in an article by Jack Good published in the Proceedings of the Cambridge Philosophical Society in 1941:

$$0.5306 < \dim(E_2) < 0.5320$$

A Good estimate

The first estimate on this value was in an article by Jack Good published in the Proceedings of the Cambridge Philosophical Society in 1941:

$$0.5306 < \dim(E_2) < 0.5320$$

This was in work from Good's thesis, under the supervision of Hardy and Besicovitch.

A Good estimate

The first estimate on this value was in an article by Jack Good published in the Proceedings of the Cambridge Philosophical Society in 1941:

$$0.5306 < \dim(E_2) < 0.5320$$

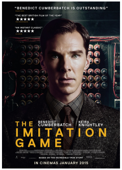
This was in work from Good's thesis, under the supervision of Hardy and Besicovitch. It was awarded the annual "Smith's prize" for research students at Cambridge (established in 1769).

Aside: Good's war

During the Second World War Jack Good worked at Bletchley Park, breaking the german enigma codes.

Aside: Good's war

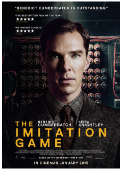
During the Second World War Jack Good worked at Bletchley Park, breaking the german enigma codes.



Good featured as a character in the 2014 movie about the life of Alan Turing

Aside: Good's war

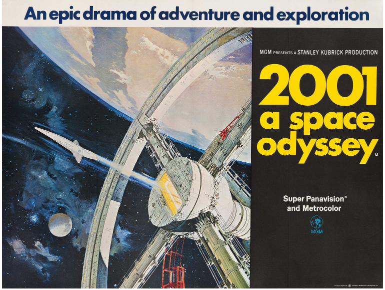
During the Second World War Jack Good worked at Bletchley Park, breaking the german enigma codes.



Good featured as a character in the 2014 movie about the life of Alan Turing, as the guy in glasses who solves the recruitment puzzle at the same time as Kiera Knightley.

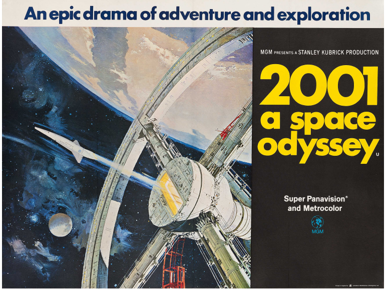
Aside: Good's film career

Moreover, Jack Good had a more direct connection with the film industry. He worked with Stanley Kubrick as an advisor on the movie *2001: A space odyssey*



Aside: Good's film career

Moreover, Jack Good had a more direct connection with the film industry. He worked with Stanley Kubrick as an advisor on the movie *2001: A space odyssey*



A photograph of Jack Good on the set of the movie.

A pressure function

We can try to get better estimates on $\dim(\Lambda)$ *using the Bowen approach.*

A pressure function

We can try to get better estimates on $\dim(\Lambda)$ *using the Bowen approach.*
To define the pressure function we denote:

- For $n \geq 1$, let $\underline{i} = (i_1, \cdots, i_n) \in \{1,2\}^n$ and $|\underline{i}| = n$;

A pressure function

We can try to get better estimates on $\dim(\Lambda)$ *using the Bowen approach.*
To define the pressure function we denote:

- For $n \geq 1$, let $\underline{i} = (i_1, \cdots, i_n) \in \{1, 2\}^n$ and $|\underline{i}| = n$; and
- Let $x_{\underline{i}} = T_{\underline{i}}(x_{\underline{i}})$ be the fixed point for

$$T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n} : [0, 1] \to [0, 1].$$

A pressure function

We can try to get better estimates on $\dim(\Lambda)$ *using the Bowen approach.*
To define the pressure function we denote:

- For $n \geq 1$, let $\underline{i} = (i_1, \cdots, i_n) \in \{1,2\}^n$ and $|\underline{i}| = n$; and
- Let $x_{\underline{i}} = T_{\underline{i}}(x_{\underline{i}})$ be the fixed point for

$$T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n} : [0,1] \to [0,1].$$

Definition

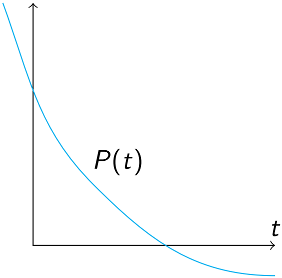
We can define a pressure function $P : \mathbb{R} \to \mathbb{R}$ by

$$P(t) = \lim_{n \to +\infty} \frac{1}{n} \sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x_{\underline{i}})|^t$$

where $t \in \mathbb{R}$.

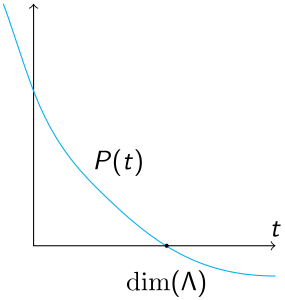
Pressure and dimension

This pressure function $P : \mathbb{R} \to \mathbb{R}$ is analytic and convex.



Pressure and dimension

This pressure function $P : \mathbb{R} \to \mathbb{R}$ is analytic and convex.



The connection with the dimension is given by:

Theorem (Bowen, Ruelle)

The dimension of the limit set is the zero $t = \dim(\Lambda)$: $P(t) = 0$.

Bowen's original formulation

The original statement in Bowen's paper is rather modestly presented as "Lemma 10":

and when a is sufficiently large $P(a\varphi)<0$ (since $S_N \leqslant -\varepsilon$). The formula shows that $P(a\varphi)$ strictly decreases as a increases; since $P(a\varphi)$ is continuous in a, there is a unique a with $P(a\varphi)=0$.

Lemma 10. — The Hausdorff dimension of γ is a. The a-dimensional Hausdorff measure ν_a on γ is finite and equivalent to $\pi_\Lambda^* \mu_{a\varphi}$.

Bowen's original formulation

The original statement in Bowen's paper is rather modestly presented as "Lemma 10":

> and when a is sufficiently large $P(a\varphi) < 0$ (since $S_N \leqslant -\varepsilon$). The formula shows that $P(a\varphi)$ strictly decreases as a increases; since $P(a\varphi)$ is continuous in a, there is a unique a with $P(a\varphi) = 0$.
>
> *Lemma 10. — The Hausdorff dimension of* γ *is* a. *The* a*-dimensional Hausdorff measure* ν_a *on* γ *is finite and equivalent to* $\pi_\Lambda^* \mu_{a\varphi}$.

Returning to the main theme of this lecture:

Question

How can we use the Bowen dimension formula as a computational tool?

Bowen's original formulation

The original statement in Bowen's paper is rather modestly presented as "Lemma 10":

> and when a is sufficiently large $P(a\varphi)<0$ (since $S_N \leqslant -\varepsilon$). The formula shows that $P(a\varphi)$ strictly decreases as a increases; since $P(a\varphi)$ is continuous in a, there is a unique a with $P(a\varphi)=0$.
>
> *Lemma 10. — The Hausdorff dimension of γ is a. The a-dimensional Hausdorff measure ν_a on γ is finite and equivalent to $\pi_\Lambda^* \mu_{a\varphi}$.*

Returning to the main theme of this lecture:

Question

How can we use the Bowen dimension formula as a computational tool?

The first point is that we don't want to use the definition of the pressure given before, but an alternative formulation ...

Bowen's original formulation

The original statement in Bowen's paper is rather modestly presented as "Lemma 10":

and when a is sufficiently large $P(a\varphi)<0$ (since $S_N \leqslant -\varepsilon$). The formula shows that $P(a\varphi)$ strictly decreases as a increases; since $P(a\varphi)$ is continuous in a, there is a unique a with $P(a\varphi)=0$.

Lemma 10. — The Hausdorff dimension of γ is a. The a-dimensional Hausdorff measure ν_a on γ is finite and equivalent to $\pi_\Lambda^* \mu_{a\varphi}$.

Returning to the main theme of this lecture:

Question

How can we use the Bowen dimension formula as a computational tool?

The first point is that we don't want to use the definition of the pressure given before, but an alternative formulation ... in terms of transfer operators.

The transfer operator feels the pressure

For simplicity, we again restrict to iterated function schemes.

The transfer operator feels the pressure

For simplicity, we again restrict to iterated function schemes.

Let $\mathcal{B} = C^0([0,1])$ be the Banach space of continuous functions (with the usual supremum norm).

The transfer operator feels the pressure

For simplicity, we again restrict to iterated function schemes.

Let $\mathcal{B} = C^0([0,1])$ be the Banach space of continuous functions (with the usual supremum norm).

Let $\mathcal{L}_t : \mathcal{B} \to \mathcal{B}$ be the *transfer operator(s)* defined by

$$\mathcal{L}_t f(x) = |T_1'(x)|^t f(T_1 x) + |T_2'(x)|^t f(T_2 x), \quad \text{where } f \in \mathcal{B},$$

for $t \geq 0$.

The transfer operator feels the pressure

For simplicity, we again restrict to iterated function schemes.

Let $\mathcal{B} = C^0([0,1])$ be the Banach space of continuous functions (with the usual supremum norm).

Let $\mathcal{L}_t : \mathcal{B} \to \mathcal{B}$ be the *transfer operator(s)* defined by

$$\mathcal{L}_t f(x) = |T_1'(x)|^t f(T_1 x) + |T_2'(x)|^t f(T_2 x), \quad \text{where } f \in \mathcal{B},$$

for $t \geq 0$.

Lemma (Ruelle Operator Theorem)

$\mathcal{L}_t$ has largest eigenvalue $e^{P(t)}$.

The transfer operator feels the pressure

For simplicity, we again restrict to iterated function schemes.

Let $\mathcal{B} = C^0([0,1])$ be the Banach space of continuous functions (with the usual supremum norm).

Let $\mathcal{L}_t : \mathcal{B} \to \mathcal{B}$ be the *transfer operator(s)* defined by

$$\mathcal{L}_t f(x) = |T_1'(x)|^t f(T_1 x) + |T_2'(x)|^t f(T_2 x), \quad \text{where } f \in \mathcal{B},$$

for $t \geq 0$.

Lemma (Ruelle Operator Theorem)

$\mathcal{L}_t$ has largest eigenvalue $e^{P(t)}$.

Thus the Bowen dimension formula can be reinterpreted as:

Corollary

$t = \dim(\Lambda)$ corresponds to 1 being the largest eigenvalue for $\mathcal{L}_t$

Transfer operator approach to calculating dimension

The standard application of the Bowen dimension formula for computing the dimension $\dim(\Lambda)$ of the limit set has four steps led to better estimates:

Transfer operator approach to calculating dimension

The standard application of the Bowen dimension formula for computing the dimension $\dim(\Lambda)$ of the limit set has four steps led to better estimates:

1. Approximate each operator $\mathcal{L}_t$ by a (large) $N \times N$ matrix $\mathcal{L}_t^{(N)}$;

Transfer operator approach to calculating dimension

The standard application of the Bowen dimension formula for computing the dimension $\dim(\Lambda)$ of the limit set has four steps led to better estimates:

1. Approximate each operator $\mathcal{L}_t$ by a (large) $N \times N$ matrix $\mathcal{L}_t^{(N)}$;
2. Find the maximal eigenvalue $\lambda_t^{(N)}$ for $\mathcal{L}_t^{(N)}$;

Transfer operator approach to calculating dimension

The standard application of the Bowen dimension formula for computing the dimension $\dim(\Lambda)$ of the limit set has four steps led to better estimates:

1. Approximate each operator $\mathcal{L}_t$ by a (large) $N \times N$ matrix $\mathcal{L}_t^{(N)}$;
2. Find the maximal eigenvalue $\lambda_t^{(N)}$ for $\mathcal{L}_t^{(N)}$;
3. Solve for $t = t_N : \lambda_t^{(N)} = 1$;

Transfer operator approach to calculating dimension

The standard application of the Bowen dimension formula for computing the dimension dim(Λ) of the limit set has four steps led to better estimates:

1. Approximate each operator $\mathcal{L}_t$ by a (large) $N \times N$ matrix $\mathcal{L}_t^{(N)}$;
2. Find the maximal eigenvalue $\lambda_t^{(N)}$ for $\mathcal{L}_t^{(N)}$;
3. Solve for $t = t_N : \lambda_t^{(N)} = 1$;
4. Then $t_N \to \mathrm{dim}(X)$ as $N \to +\infty$.

Transfer operator approach to calculating dimension

The standard application of the Bowen dimension formula for computing the dimension dim(Λ) of the limit set has four steps led to better estimates:

1. Approximate each operator $\mathcal{L}_t$ by a (large) $N \times N$ matrix $\mathcal{L}_t^{(N)}$;
2. Find the maximal eigenvalue $\lambda_t^{(N)}$ for $\mathcal{L}_t^{(N)}$;
3. Solve for $t = t_N : \lambda_t^{(N)} = 1$;
4. Then $t_N \to \dim(X)$ as $N \to +\infty$.

Example 2 revisited: This method (essentially) has been used by several authors to estimate dim(E_2), the non-linear Cantor set of numbers whose continued fraction expansion only used the digits 1 and 2...

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Falk and Nussbaum (2016) showed that

$$\dim_H(E_2) = 0.53128050\ldots$$

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Falk and Nussbaum (2016) showed that

$$\dim_H(E_2) = 0.53128050\ldots$$

Where the estimates are presented to the number of places they are known to be accurate.

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Falk and Nussbaum (2016) showed that

$$\dim_H(E_2) = 0.53128050\ldots$$

Where the estimates are presented to the number of places they are known to be accurate.

Question

How can we further improve on these estimates?

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Falk and Nussbaum (2016) showed that

$$\dim_H(E_2) = 0.53128050\ldots$$

Where the estimates are presented to the number of places they are known to be accurate.

Question

How can we further improve on these estimates?

We would like employ the basic Bowen dimension formula using an extra ingredient ...

Estimates $\dim(E_2)$

Using the dimension formula and faster computers/more memory (to choose n larger in the approximation of $\mathcal{L}_t$ by $\mathcal{L}_t^{(N)}$):

Bumby (1985) showed that

$$\dim_H(E_2) = 0.531\ldots$$

Hensley (1989) showed that

$$\dim_H(E_2) = 0.531280\ldots$$

Falk and Nussbaum (2016) showed that

$$\dim_H(E_2) = 0.53128050\ldots$$

Where the estimates are presented to the number of places they are known to be accurate.

Question

How can we further improve on these estimates?

We would like employ the basic Bowen dimension formula using an extra ingredient ... zeta functions.

Zeta functions

We can define a *zeta function* of two variables ($z \in \mathbb{C}$ and $t \in \mathbb{R}$) formally defined by

$$\zeta(z,t) := \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1 - (T_{\underline{i}})'(x_{\underline{i}})} \right).$$

Zeta functions

We can define a *zeta function* of two variables ($z \in \mathbb{C}$ and $t \in \mathbb{R}$) formally defined by

$$\zeta(z,t) := \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1-(T_{\underline{i}})'(x_{\underline{i}})} \right).$$

Given $t > 0$ this converges for $|z|$ sufficiently small,

Zeta functions

We can define a *zeta function* of two variables ($z \in \mathbb{C}$ and $t \in \mathbb{R}$) formally defined by

$$\zeta(z,t) := \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1-(T_{\underline{i}})'(x_{\underline{i}})} \right).$$

Given $t > 0$ this converges for $|z|$ sufficiently small, but, in fact, has an analytic extension, and thus makes sense, for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$ *(as we will see later ...)*.

Zeta functions

We can define a *zeta function* of two variables ($z \in \mathbb{C}$ and $t \in \mathbb{R}$) formally defined by

$$\zeta(z, t) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1 - (T_{\underline{i}})'(x_{\underline{i}})}\right).$$

Given $t > 0$ this converges for $|z|$ sufficiently small, but, in fact, has an analytic extension, and thus makes sense, for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$ *(as we will see later ...)*.

Moreover, setting $z = 1$ the Bowen dimension formula can be reinterpreted in terms of the function $t \mapsto \zeta(1, t)$ (where we set $z = 1$).

Zeta functions

We can define a *zeta function* of two variables ($z \in \mathbb{C}$ and $t \in \mathbb{R}$) formally defined by

$$\zeta(z,t) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|(T_{\underline{i}})'(x_{\underline{i}})|^t}{1-(T_{\underline{i}})'(x_{\underline{i}})} \right).$$

Given $t > 0$ this converges for $|z|$ sufficiently small, but, in fact, has an analytic extension, and thus makes sense, for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$ *(as we will see later ...)*.

Moreover, setting $z = 1$ the Bowen dimension formula can be reinterpreted in terms of the function $t \mapsto \zeta(1,t)$ (where we set $z = 1$).

Lemma (Bowen Formula, version II)

$t = \dim_H(\Lambda)$ *satisfies* $\zeta(1,t) = 0$.

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

1. For each t approximate $z \mapsto \zeta(z, t)$ by a polynomial $z \mapsto \zeta_N(z, t)$;

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

1. For each t approximate $z \mapsto \zeta(z, t)$ by a polynomial $z \mapsto \zeta_N(z, t)$;
2. Set $z = 1$ and consider $t \mapsto \zeta_N(z, 1)$;

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

1. For each t approximate $z \mapsto \zeta(z, t)$ by a polynomial $z \mapsto \zeta_N(z, t)$;
2. Set $z = 1$ and consider $t \mapsto \zeta_N(z, 1)$;
3. Solve for $t_N = t$: $\zeta_N(1, t) = 0$;

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

1. For each t approximate $z \mapsto \zeta(z, t)$ by a polynomial $z \mapsto \zeta_N(z, t)$;
2. Set $z = 1$ and consider $t \mapsto \zeta_N(z, 1)$;
3. Solve for $t_N = t$: $\zeta_N(1, t) = 0$;
4. Then $t_N \to \dim(\Lambda)$ as $N \to +\infty$.

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

1. For each t approximate $z \mapsto \zeta(z, t)$ by a polynomial $z \mapsto \zeta_N(z, t)$;
2. Set $z = 1$ and consider $t \mapsto \zeta_N(z, 1)$;
3. Solve for $t_N = t$: $\zeta_N(1, t) = 0$;
4. Then $t_N \to \dim(\Lambda)$ as $N \to +\infty$.

Question

Is this any better than the previous approach using transfer operators?

Zeta function approach to calculating dimension

Recall that $\zeta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ and the dimension of Λ is given by the solution

$$t = \dim(\Lambda) : \zeta(1, t) = 0.$$

We use the zeta function to calculate the dimension $\dim(\Lambda)$ as follows:

1. For each t approximate $z \mapsto \zeta(z, t)$ by a polynomial $z \mapsto \zeta_N(z, t)$;
2. Set $z = 1$ and consider $t \mapsto \zeta_N(z, 1)$;
3. Solve for $t_N = t$: $\zeta_N(1, t) = 0$;
4. Then $t_N \to \dim(\Lambda)$ as $N \to +\infty$.

Question

Is this any better than the previous approach using transfer operators?

Let us illustrate this (again) with $\dim(E_2)$, the Cantor set of numbers whose continued fraction expansion only used the digits 1 and 2.

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.53128050627720514$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368\\471785493059$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368$$
$$471785493059109018398779888397$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368\\ 471785493059109018398779888397\\ 8039275295356$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368\\ 471785493059109018398779888397\\ 803927529535643831345918109570$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368 \\ 471785493059109018398779888397 \\ 803927529535643831345918109570 \\ 1811852398\cdots$$

A zeta function estimate on $\dim(E_2)$

Recall that of the best estimate for $\dim_H(E_2)$ was by Falk and Nussbaum (2016) who showed that

$$\dim_H(\Lambda) = 0.53128050\ldots$$

Question

What is the corresponding estimate using zeta functions?

Theorem (Jenkinson+P. (2016))

We can estimate

$$\dim_H(E_2) = 0.531280506277205141624468647368\\ 471785493059109018398779888397\\ 803927529535643831345918109570\\ 1811852398\cdots$$

Where the estimate in the theorem is presented to the number of places they are known to be accurate.

My co-author

Oliver Jenkinson, Queen Mary - University of London.

(The photograph was taken in Italy, rather than the East End of London.)

Estimates using zeta functions

Let us write the series expansion

$$\zeta(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n = \underbrace{1 + \sum_{n=1}^{N} a_n(t) z^n}_{=:\zeta_N(z,t)} + \underbrace{\sum_{n=N+1}^{\infty} a_n(t) z^n}_{=:\epsilon_N(z,t)}$$

for some $N \geq 1$.

Estimates using zeta functions

Let us write the series expansion

$$\zeta(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n = 1 + \underbrace{\sum_{n=1}^{N} a_n(t) z^n}_{=:\zeta_N(z,t)} + \underbrace{\sum_{n=N+1}^{\infty} a_n(t) z^n}_{=:\epsilon_N(z,t)}$$

for some $N \geq 1$. In particular, we take for the approximating polynomial

$$\zeta_N(z,t) = 1 + \sum_{n=1}^{N} a_n(t) z^n$$

and choose N:

1. sufficiently large that (with $z = 1$, $0 \leq t \leq 1$) the error ϵ_N is small;

Estimates using zeta functions

Let us write the series expansion

$$\zeta(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n = 1 + \underbrace{\sum_{n=1}^{N} a_n(t) z^n}_{=:\zeta_N(z,t)} + \underbrace{\sum_{n=N+1}^{\infty} a_n(t) z^n}_{=:\epsilon_N(z,t)}$$

for some $N \geq 1$. In particular, we take for the approximating polynomial

$$\zeta_N(z,t) = 1 + \sum_{n=1}^{N} a_n(t) z^n$$

and choose N:

1. sufficiently large that (with $z = 1$, $0 \leq t \leq 1$) the error ϵ_N is small; but
2. sufficiently small that the terms $a_n(t)$, $n = 1, 2, \cdots, N$ can be calculated in a reasonable time.

Choosing N

We can choose N as large as our computer (and our own patience) allows.

- $N = 25$: Takes a week to compute ζ_{25};

Choosing N

We can choose N as large as our computer (and our own patience) allows.

- $N = 25$: Takes a week to compute ζ_{25};
- $N = 33$: Takes a year to compute ζ_{33};

Choosing N

We can choose N as large as our computer (and our own patience) allows.

- $N = 25$: Takes a week to compute ζ_{25};
- $N = 33$: Takes a year to compute ζ_{33};
- $N = 69$: Takes at least "age of the universe" to compute ζ_{69}.

Choosing N

We can choose N as large as our computer (and our own patience) allows.

- $N = 25$: Takes a week to compute ζ_{25};
- $N = 33$: Takes a year to compute ζ_{33};
- $N = 69$: Takes at least "age of the universe" to compute ζ_{69}.

Choosing N

We can choose N as large as our computer (and our own patience) allows.

- $N = 25$: Takes a week to compute ζ_{25};
- $N = 33$: Takes a year to compute ζ_{33};
- $N = 69$: Takes at least "age of the universe" to compute ζ_{69}.

We choose $N = 25$ (one week being the limit of my patience)

Choosing N

We can choose N as large as our computer (and our own patience) allows.

- $N = 25$: Takes a week to compute ζ_{25};
- $N = 33$: Takes a year to compute ζ_{33};
- $N = 69$: Takes at least "age of the universe" to compute ζ_{69}.

We choose $N = 25$ (one week being the limit of my patience) then we need accurate (and small) bounds on ϵ_{25} .

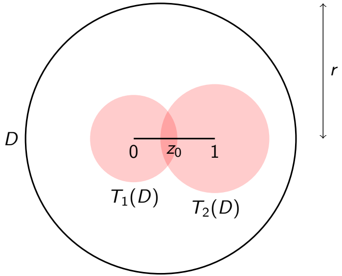
Bounds on the error ϵ_N: Pure Mathematics

Step 1: We need to let the operator $\mathcal{L}_t$ act on a smaller space $\mathcal{H} \subset C([0,1])$ of functions.

Bounds on the error ϵ_N: Pure Mathematics

Step 1: We need to let the operator $\mathcal{L}_t$ act on a smaller space $\mathcal{H} \subset C([0, 1])$ of functions. Choose $z_0 \in \mathbb{R}$ and $r > 0$ such that

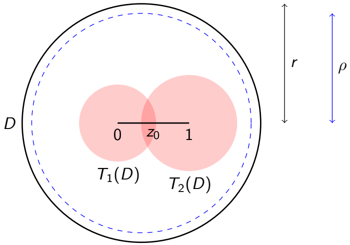
$$D = \{z \in \mathbb{C} : |z - z_0| < r\} \supset [0, 1] \text{ and } T_1 D, T_2 D \subset D.$$



Bounds on the error ϵ_N: Pure Mathematics

Step 1: We need to let the operator $\mathcal{L}_t$ act on a smaller space $\mathcal{H} \subset C([0,1])$ of functions. Choose $z_0 \in \mathbb{R}$ and $r > 0$ such that

$$D = \{z \in \mathbb{C} : |z - z_0| < r\} \supset [0,1] \text{ and } T_1 D, T_2 D \subset D.$$

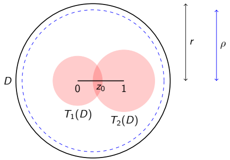


Let $f : D \to \mathbb{C}$ be holomorphic and $||f||^2 = \sup_{\rho<r} \int_0^1 |f(z_0 + \rho e^{2\pi i t})|^2 dt$.

Bounds on the error ϵ_N: Pure Mathematics

Step 1: We need to let the operator $\mathcal{L}_t$ act on a smaller space $\mathcal{H} \subset C([0,1])$ of functions. Choose $z_0 \in \mathbb{R}$ and $r > 0$ such that

$$D = \{z \in \mathbb{C} : |z - z_0| < r\} \supset [0,1] \text{ and } T_1 D, T_2 D \subset D.$$



Let $f : D \to \mathbb{C}$ be holomorphic and $\|f\|^2 = \sup_{\rho<r} \int_0^1 |f(z_0 + \rho e^{2\pi it})|^2 dt$.
Then $\mathcal{H} = \{f : \|f\| < +\infty\}$ is a *Hardy Hilbert space*.

Bounds on ϵ_N

Step 2. We define *approximation numbers* for $\mathcal{L}_t$:

$$s_m = s_m(\mathcal{L}_t) := \sup\{\|\mathcal{L}_t - K\| \ : \ K : \mathcal{H} \to \mathcal{H} \text{ has rank } \leq m-1\} \quad (m \geq 1)$$

Bounds on ϵ_N

Step 2. We define *approximation numbers* for $\mathcal{L}_t$:

$$s_m = s_m(\mathcal{L}_t) := \sup\{\|\mathcal{L}_t - K\| \,:\, K : \mathcal{H} \to \mathcal{H} \text{ has rank } \leq m-1\} \quad (m \geq 1)$$

We can then bound the coefficients a_n $(n > N = 25)$ of $\mapsto \zeta(z,t)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} s_{m_1} s_{m_2} \cdots s_{m_n}$$

Bounds on ϵ_N

Step 2. We define *approximation numbers* for $\mathcal{L}_t$:

$$s_m = s_m(\mathcal{L}_t) := \sup\{\|\mathcal{L}_t - K\| \, : \, K : \mathcal{H} \to \mathcal{H} \text{ has rank } \leq m-1\} \quad (m \geq 1)$$

We can then bound the coefficients a_n $(n > N = 25)$ of $\mapsto \zeta(z, t)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} s_{m_1} s_{m_2} \cdots s_{m_n}$$

Step 3. Finally, using Cauchy-Schwarz inequality we can bound

$$s_m \leq \left(\sum_{k=m-1}^{\infty} \|(q_k)\|^2 \right)^{\frac{1}{2}} \text{ where } q_k = \frac{(z - z_0)^k}{r^k} \in \mathcal{H}.$$

Bounds on ϵ_N

Step 2. We define *approximation numbers* for $\mathcal{L}_t$:

$$s_m = s_m(\mathcal{L}_t) := \sup\{\|\mathcal{L}_t - K\| : K : \mathcal{H} \to \mathcal{H} \text{ has rank } \leq m-1\} \quad (m \geq 1)$$

We can then bound the coefficients a_n $(n > N = 25)$ of $\mapsto \zeta(z, t)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} s_{m_1} s_{m_2} \cdots s_{m_n}$$

Step 3. Finally, using Cauchy-Schwarz inequality we can bound

$$s_m \leq \left(\sum_{k=m-1}^{\infty} \|(q_k)\|^2 \right)^{\frac{1}{2}} \text{ where } q_k = \frac{(z - z_0)^k}{r^k} \in \mathcal{H}.$$

- We can numerically estimate $\|\mathcal{L}(q_k)\|$ for $k \leq 600$, say.

Bounds on ϵ_N

Step 2. We define *approximation numbers* for $\mathcal{L}_t$:

$$s_m = s_m(\mathcal{L}_t) := \sup\{\|\mathcal{L}_t - K\| : K : \mathcal{H} \to \mathcal{H} \text{ has rank } \leq m-1\} \quad (m \geq 1)$$

We can then bound the coefficients a_n $(n > N = 25)$ of $\mapsto \zeta(z,t)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} s_{m_1} s_{m_2} \cdots s_{m_n}$$

Step 3. Finally, using Cauchy-Schwarz inequality we can bound

$$s_m \leq \left(\sum_{k=m-1}^{\infty} \|(q_k)\|^2 \right)^{\frac{1}{2}} \text{ where } q_k = \frac{(z-z_0)^k}{r^k} \in \mathcal{H}.$$

- We can numerically estimate $\|\mathcal{L}(q_k)\|$ for $k \leq 600$, say.
- We can trivially bound $\|\mathcal{L}(q_k)\|$ for $k > 600$, say.

Bounds on ϵ_N

Step 2. We define *approximation numbers* for $\mathcal{L}_t$:

$$s_m = s_m(\mathcal{L}_t) := \sup\{\|\mathcal{L}_t - K\| : K : \mathcal{H} \to \mathcal{H} \text{ has rank } \leq m-1\} \quad (m \geq 1)$$

We can then bound the coefficients a_n $(n > N = 25)$ of $\mapsto \zeta(z,t)$ by

$$|a_n| \leq \sum_{m_1 < \cdots < m_n} s_{m_1} s_{m_2} \cdots s_{m_n}$$

Step 3. Finally, using Cauchy-Schwarz inequality we can bound

$$s_m \leq \left(\sum_{k=m-1}^{\infty} \|(q_k)\|^2 \right)^{\frac{1}{2}} \text{ where } q_k = \frac{(z-z_0)^k}{r^k} \in \mathcal{H}.$$

- We can numerically estimate $\|\mathcal{L}(q_k)\|$ for $k \leq 600$, say.
- We can trivially bound $\|\mathcal{L}(q_k)\|$ for $k > 600$, say.

Combining these bounds (creatively) gives the results.

Aside: Good's formula

We recall another mathematical contribution of I. J. Good (published in 1990) which applies more widely.

Aside: Good's formula

We recall another mathematical contribution of I. J. Good (published in 1990) which applies more widely.

"A very rough guide to the maximum length that a paper should have is given by the formula $10^{9px/2}$ *words where*

- $0 \leq x \leq 1$ *is the importance of the topic,*

Aside: Good's formula

We recall another mathematical contribution of I. J. Good (published in 1990) which applies more widely.

"A very rough guide to the maximum length that a paper should have is given by the formula $10^{9px/2}$ *words where*

- $0 \leq x \leq 1$ *is the importance of the topic, and*
- *a partly-baked idea has a "bakedness" of* $0 \leq p \leq 1$*."*

Aside: Good's formula

We recall another mathematical contribution of I. J. Good (published in 1990) which applies more widely.

"A very rough guide to the maximum length that a paper should have is given by the formula $10^{9px/2}$ *words where*

- $0 \leq x \leq 1$ *is the importance of the topic, and*
- *a partly-baked idea has a "bakedness" of* $0 \leq p \leq 1$*."*

(For calibration we recall that "half-baked idea" ($p = \frac{1}{2}$) means *poorly developed; foolish; unlikely to work*).

Aside: Good's formula

We recall another mathematical contribution of I. J. Good (published in 1990) which applies more widely.

"A very rough guide to the maximum length that a paper should have is given by the formula $10^{9px/2}$ *words where*

- $0 \leq x \leq 1$ *is the importance of the topic, and*
- *a partly-baked idea has a "bakedness" of* $0 \leq p \leq 1$*."*

(For calibration we recall that "half-baked idea" ($p = \frac{1}{2}$) means *poorly developed; foolish; unlikely to work*).

Our article is about 20 pages (or perhaps 6,000 words). Thus even if the idea was fully developed (bakedness $p = 1$) it would need to have an importance factor of 0.83 baked to satisfy this formula!

A final comment on Rufus Bowen

I never had the good fortune to meet Bowen, but like so many people my work was greatly influenced by his.

A final comment on Rufus Bowen

I never had the good fortune to meet Bowen, but like so many people my work was greatly influenced by his.
I will finish with an eloquent quote from a more senior participant than myself who collaborated with Rufus Bowen:

A final comment on Rufus Bowen

I never had the good fortune to meet Bowen, but like so many people my work was greatly influenced by his.
I will finish with an eloquent quote from a more senior participant than myself who collaborated with Rufus Bowen:

"The Greek and Roman gods, supposedly, resented those mortals endowed with superlative gifts and happiness, and punished them. The life and achievements of Rufus Bowen (1947-1978) remind us of this belief of the ancients. When Rufus died unexpectedly, at age thirty-one, from a brain hemorrhage, he was a very happy and successful man. He had great charm, that he did not misuse, and superlative mathematical talent. His mathematical legacy is important, and will not be forgotten, but one wonders what he would have achieved if he had lived longer."

- David Ruelle, Preface to the re-edition of "Equilibrium states and the ergodic theory of Anosov diffeormorphisms"

Finally

Thank you for your attention.