EXERCISES ON RATIONAL CURVES

The following list of exercises will allow you to work through some examples of rational curves on smooth projective varieties. The exercises are organized according to the most relevant lecture. They are not organized in order of difficulty; feel free to skip around. Unless otherwise noted we are always working over the ground field $\mathbb{C}$.

1. Lecture 1

If you like, you can try to prove more carefully some of the assertions from the lecture:

Exercise 1.1. Let X be a smooth projective variety.

(1) Explain why the image of a non-trivial morphism $f : \mathbb{P}^1 \to X$ is a rational curve on X.
(2) Explain carefully why $\mathrm{Mor}(\mathbb{P}^1, X)$ is an open subscheme of $\mathrm{Hilb}(\mathbb{P}^1 \times X)$.
(3) Explain why $\mathrm{Mor}(\mathbb{P}^1, X)_\alpha$ is a finite type scheme for any numerical class α.
(4) Show that the universal family $\mathcal{U} \to \mathrm{Mor}(\mathbb{P}^1, X)$ is isomorphic to the product $\mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X)$.
(5) Suppose that $Z \subset \mathbb{P}^1 \times X$ is a section corresponding to a morphism $f : \mathbb{P}^1 \to X$. Explain why the normal bundle of Z is isomorphic to f^*T_X.

1.1. **Curves in projective space.** We start with some more calculations concerning curves in projective space. Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^n$ is a degree d morphism with equations $(f_0 : f_1 : \ldots : f_d)$. Using the Euler sequence we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \arrow[r] \arrow[d, "g"] & T_{\mathbb{P}^1} \arrow[r] \arrow[d, "Tf"] & 0 \\
0 \arrow[r] & f^*\mathcal{O}_{\mathbb{P}^n} \arrow[r] \arrow[d, equal] & f^*\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \arrow[r] \arrow[d, equal] & f^*T_{\mathbb{P}^n} \arrow[r] & 0 \\
 & \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus n+1} & &
\end{tikzcd}$$

where the map g is given by the matrix with entries $\partial f_i / \partial x_j$. This is also useful for computing the normal bundle; the cokernel of $T_{\mathbb{P}^1} \to f^*T_X$ is isomorphic to the cokernel of g.

(While this calculation method is very general, you may prefer to find simpler direct arguments for the following examples.)

Exercise 1.2. (1) Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^n$ is a line. Prove that $f^*T_{\mathbb{P}^n} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$.
(2) Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^n$ is a smooth conic. Compute $f^*T_{\mathbb{P}^n}$.
(3) Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^n$ is a twisted cubic (i.e. a rational normal curve of degree 3). Compute $f^*T_{\mathbb{P}^n}$.

Exercise 1.3. Prove that every rational curve on $\mathbb{P}^n$ is a free curve.

Exercise 1.4. Consider the Hilbert scheme of conics in $\mathbb{P}^n$. It turns out that the closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial $2t + 1$ are either:

- smooth conics,
- unions of two lines meeting at a point, or
- planar double lines

(1) Show (or assume) the above assertion.
(2) Deduce that the Hilbert scheme N of conics admits a morphism $\pi : N \to G(3, n+1)$.
(3) Let $\mathcal{F}$ denote the tautological rank 3 sheaf on $G(3, n+1)$. Prove that $\pi : N \to G(3, n+1)$ realizes N as the projective bundle $\mathbb{P}_{G(3,n+1)}(\mathrm{Sym}^2 \mathcal{F}^\vee)$.

1.2. **Hirzebruch surfaces.** Recall that the Hirzebruch surface $\mathbb{F}_e$ is the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-e)$. We will use $\pi : \mathbb{F}_e \to \mathbb{P}^1$ to denote the projective bundle map. The Picard group of $\mathbb{F}_e$ is isomorphic to $\mathbb{Z}^2$; it is generated by the class F of a fiber of π and the "rigid" section C_0 defined by the surjection $\mathcal{E} \to \mathcal{O}(-e)$. A key property of the rigid section is that $C_0^2 = -e$.

Exercise 1.5 (Sections)**.** A section of π is a map $f : \mathbb{P}^1 \to \mathbb{F}_e$ such that $\pi \circ f = id$. Note that C will be a section of π if and only if $C \cdot F = 1$. Thus the numerical class of a section will have the form $C = C_0 + bF$ for some integer b.

Show that $\mathrm{Mor}(\mathbb{P}^1, \mathbb{F}_e)_{C_0+bF}$ is either empty or an open set of a projective space. For which values of b is the space empty? How does the dimension of the projective space depend on b?

Exercise 1.6. More generally, suppose we fix the numerical class $aC_0 + bF$ for integers a, b. Using the universal property of the Proj construction, show that the parameter space of rational curves is either empty or is an open set of a projective bundle over a projective space. How does the dimension of $\mathrm{Mor}(\mathbb{P}^1, \mathbb{F}_e)_{aC_0+bF}$ depend on a and b? Explicitly describe the projective bundle which is the closure of this space.

1.3. **Hypersurfaces.** In this section we will analyze the families of rational curves on hypersurfaces in $\mathbb{P}^n$. In general this is a difficult task – there are still many open questions about the properties of rational curves on hypersurfaces. We will focus on curves of low degree which are more amenable to computations.

We will start by analyzing lines. You may know that a general degree d hypersurface $X \subset \mathbb{P}^n$ will contain a line if and only if $d \leq 2n-3$ (see Vakil, "Rising Sea", Section 11.2.17). We will use a different construction which elucidates the structure of the space of lines on a low degree hypersurface.

Exercise 1.7. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Consider the universal family of lines

$$\begin{tikzcd} \mathcal{U} \arrow[r, "s"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(1,n) & \end{tikzcd}$$

We will use this structure to analyze the lines on X. Note that in this exercise we are *not* working with the space of morphisms, but with the Hilbert scheme of lines in X.

(1) Show that
$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \cong H^0(\mathcal{U}, s^*\mathcal{O}(d)) \cong H^0(\mathbb{G}(1,n), p_*s^*\mathcal{O}(d)).$$
In particular the hypersurface X defines a global section σ_X of $p_*s^*\mathcal{O}(d)$.

(2) Explain why the vanishing locus of σ_X coincides with the subscheme of $\mathbb{G}(1,n)$ that parametrizes the lines contained in X.
(3) Prove that the sheaf $p_* s^* \mathcal{O}(d)$ is a globally generated locally free sheaf on $\mathbb{G}(1,n)$. Compute the rank of this sheaf. Deduce that when $d \leq 2n-3$ the space of lines on a general hypersurface X of degree d is a smooth variety of dimension $2n-d-3$.
(4) Prove that the expected dimension of $\mathrm{Mor}(\mathbb{P}^1, X)_1$ is $2n-d$. Thus verify that the family of lines on a general hypersurface has the expected dimension. (The discrepancy of 3 with regards to the calculation in the previous step is to account for the PGL_2-worth of morphisms associated with a single line.)
(5) (For those who know some intersection theory) Prove that the number of lines on a general cubic surface is 27.

Exercise 1.8. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Try to repeat the argument of Exercise 1.7 for rational curves of higher degree. Which aspects of this argument continue to work well? Which aspects fail?

Exercise 1.9. In this example we will see a dominant family of curves such that no member is a free curve.

Let K be an algebraically closed field of characteristic 2. Let X denote the Fermat fourfold of degree 5 defined by the equation

$$\sum_{i=0}^{5} x_i^5 = 0.$$

Prove that the family of lines on X gives a dominant family of curves with larger than the expected dimension. Conclude that no line on X is a free curve.

Another technique for analyzing rational curves on hypersurfaces is to compute $f^* T_X$.

Exercise 1.10. Let X be a smooth quadric hypersurface in projective space.

(1) Compute the restricted tangent bundle for a line in X.
(2) Prove that every rational curve on X is a free curve.

Exercise 1.11. Let X be a smooth cubic hypersurface in projective space of dimension ≥ 3. Prove that *every* line in X is a smooth point of $\mathrm{Mor}(\mathbb{P}^1, X)$. (Hint: by relating $N_{\ell/X}$ to $N_{\ell/\mathbb{P}^n}$, show that $N_{\ell/X}|_\ell$ injects into $\mathcal{O}(1)^{\oplus n-1}$. On the other hand, we have a lower bound on $h^0(\ell, N_{\ell/X}|_\ell)$. Combine these to show that the lowest summand in the restricted tangent bundle has degree ≥ -1.)

However, not every line in X needs to be a free curve on X. For example, consider the line ℓ defined by the equations $x_2 = x_3 = x_4 = 0$ inside the cubic threefold X defined by $x_0^2 x_2 + x_1^2 x_3 + x_2^3 + x_3^3 + x_4^3 = 0$. Prove that X is smooth and ℓ is not a free curve.

1.4. **Del Pezzo surfaces.** A del Pezzo surface is a smooth projective surface S such that $-K_S$ is ample. Such surfaces are completely classified: aside from $\mathbb{P}^1 \times \mathbb{P}^1$, each del Pezzo surface is a blow-up of $\mathbb{P}^2$ at a set of distinct points $\{p_1, \ldots, p_r\}$ where:

- $0 \leq r \leq 8$
- No three of the points are contained in a line in $\mathbb{P}^2$.
- No six of the points are contained in a conic in $\mathbb{P}^2$.

The degree of a del Pezzo surface is $(-K_S)^2$.

Exercise 1.12. Show that if a del Pezzo surface S is the blow-up of $\mathbb{P}^2$ at r points then the degree of S is $9-r$.

We will now analyze some rational curves on del Pezzo surfaces. Since $-K_S$ is ample, each curve on S will have positive intersection against $-K_S$. It is easiest to analyze the curves with low intersection number.

Exercise 1.13. Let S be a del Pezzo surface of degree $d \geq 2$. Suppose that $C \subset S$ is a rational curve such that $-K_S \cdot C = 1$.

(1) By applying the Hodge Index Theorem for $-K_S$ and C, identify a list of possible values of C^2.
(2) Using adjunction, prove that the arithmetic genus of C is equal to 0. (Hint: the arithmetic genus must be a non-negative integer.)
(3) Conclude that C is a smooth rational curve satisfying $-K_S \cdot C = 1$ and $C^2 = -1$. Deduce that C is a (-1)-curve, and in particular, that C cannot deform.

Altogether, we see that when $d \geq 2$ the set of rational curves of anticanonical degree 1 is parametrized by a 0-dimensional subscheme. (What happens when $d = 1$?)

Exercise 1.14. Let S be a del Pezzo surface of degree $d \geq 3$. Suppose that $C \subset S$ is a rational curve such that $-K_S \cdot C = 3$.

(1) By applying the Hodge Index Theorem for $-K_S$ and C, identify a list of possible values of C^2.
(2) Using adjunction, prove that the arithmetic genus of C is equal to 0. (Hint: the arithmetic genus must be a non-negative integer.)
(3) Conclude that C is a smooth rational curve satisfying $-K_S \cdot C = 1$ and $C^2 = 0$. Deduce that C is a fiber of a morphism $f : S \to \mathbb{P}^1$.

Altogether, we see that when $d \geq 2$ the set of rational curves of anticanonical degree 2 is parametrized by a union of open subsets of rational curves. (What happens when $d \leq 2$?)

1.5. **Uniruled varieties.**

Exercise 1.15. Suppose our ground field has characteristic 0. Decide which of the following types of varieties are uniruled:

(1) Grassmannians
(2) Abelian varieties
(3) Toric varieties
(4) K3 surfaces

Does your answer change if the ground field has characteristic p?

2. Lecture 2

2.1. **Kodaira dimension and rational curves.**

Exercise 2.1. Suppose our ground field has characteristic 0. Show that for any dimension n and Kodaira dimension k satisfying $0 \leq k \leq n$ there is a smooth projective variety of dimension n and Kodaira dimension k which does not admit any rational curves.

For what range of Kodaira dimensions can you find a Zariski dense set of rational curves? (Remember, the definitive answer to this question is still conjectural. Here the goal is to motivate the conjecture by thinking about which examples you *can* construct.)

Exercise 2.2. Let X be a smooth projective variety over an algebraically closed field of characteristic 0. Recall that the pseudo-effective cone of divisors $\overline{\mathrm{Eff}}^1(X)$ is the closure of the cone in $N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$ generated by effective Cartier divisors. Boucksom-Demailly-Paŭn-Peternell proved a fundamental property of the pseudo-effective cone:

Theorem 2.3 (Boucksom-Demailly-Paŭn-Peternell)**.** *A Cartier divisor L on X fails to be pseudo-effective if and only if there is a dominant family of curves C on X satisfying $L \cdot C < 0$.*

This theorem has interesting implications for rational curves.

(1) Using the theorem above, show that a smooth projective variety is uniruled if and only if K_X fails to be pseudo-effective.
(2) Using the Abundance Conjecture for dimension ≤ 3, prove that a smooth projective variety of dimension ≤ 3 is uniruled if and only if it has Kodaira dimension $-\infty$.

Exercise 2.4. The relationship between uniruledness and Kodaira dimension breaks down in characteristic p.

Choose positive integers $p > n$. Fix an algebraically closed field of characteristic p and consider the product $\mathbb{P}^1 \times \mathbb{P}^n$ equipped with the two projection maps π_1, π_2. Let $D = p\pi_1^*H + \pi_2^*H$.

(1) Show that a general element X of the linear series $|D|$ is a smooth projective variety with Kodaira dimension n. (Hint: to show that a general element of $|D|$ is smooth, one option is simply to write down a single example of a smooth hypersurface in $|D|$.)
(2) Show that the map $\pi_2|_X : X \to \mathbb{P}^n$ is purely inseparable.
(3) Let $g : X' \to \mathbb{P}^n$ be the finite part of the Stein factorization of $\pi_2|_X$. Using properties of the Frobenius map, show that we have a purely inseparable morphism $\mathbb{P}^n \to X'$. Deduce that X' is uniruled, and thus that X is also uniruled.

2.2. **Bend-and-Break.**

Exercise 2.5. Let X be a smooth projective variety. Suppose that $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ is an irreducible component of dimension $> 2\dim(X) + 1$. Prove that for every point $x \in X$ the rational curves parametrized by M through x deform to a non-integral curve whose components have smaller degree.

Conclude that if X is uniruled then there is a rational curve of anticanonical degree $\leq \dim(X) + 1$ through every point of X.

Exercise 2.6. For any positive integer r, give an example of a smooth projective variety X, a morphism $f : \mathbb{P}^1 \to X$, and a point $p \in \mathbb{P}^1$ such that $\mathrm{Mor}(\mathbb{P}^1, X; f|_p)$ has dimension r but the cycles underlying the image of f do not deform to a non-integral curve.

(This shows that we *must* distinguish between the rational curve case and the $g \geq 1$ case when applying Bend-and-Break.)

Exercise 2.7. Consider the Hirzebruch surface $\mathbb{F}_e$ for $e \geq 3$. As before let C_0 denote the class of the rigid section and let F denote the class of a fiber of the projective bundle map. Let $\alpha = C_0 + eF$ denote the class of the sections which do not intersect C_0.

(1) Show that there is a one-parameter family of rational curves of class $C_0 + eF$ through any 2 general points of $\mathbb{F}_e$.
(2) Show that the only non-integral curves of class $C_0 + eF$ consist of the union of C_0 with the π-preimage of a degree e divisor on $\mathbb{P}^1$.

This example shows that it is difficult to control the curves resulting from Bend-and-Break. Even in simple examples, the resulting curves can have many components, components with very negative normal bundles, etc.

3. Lecture 3

3.1. **Fujita invariant.**

Exercise 3.1. Let X be a smooth degree d hypersurface in $\mathbb{P}^n$. Compute $a(X, H)$ where H is the hyperplane class.

Exercise 3.2. For each of the following projective varieties, execute the following steps:

- Compute the Picard group and $N^1(X)_{\mathbb{Z}}$.
- Compute the pseudo-effective cone of divisors $\overline{\mathrm{Eff}}^1(X)$.
- Write an expression which computes $a(X, L)$ for every ample divisor L. (Note that $a(X, L)$ will be a piecewise linear function on the nef cone.)

(1) $\mathbb{P}^n \times \mathbb{P}^m$.
(2) A Hirzebruch surface $\mathbb{F}_e$.
(3) The blow-up of $\mathbb{P}^n$ at a point.

Exercise 3.3. Let S be a del Pezzo surface and let C be a curve in S.

(1) Suppose that the degree of S is ≥ 2. Show that $a(C, -K_S) > a(S, -K_S)$ if and only if C is a (-1)-curve.
(2) Suppose that the degree of S is ≥ 3. Show that $a(C, -K_S) = a(S, -K_S)$ if and only if C is a fiber of a morphism $f : S \to \mathbb{P}^1$.

Exercise 3.4. Suppose that X is a smooth Fano variety of dimension n and that L is an ample divisor on X. Prove that $a(X, L) \leq n + 1$. (Hint: the function $\mathbb{Z} \to \mathbb{Z}$ defined by $d \mapsto \chi(\mathcal{O}_X(K_X + dL))$ is a polynomial function. Consider the values of this function for $1 \leq d \leq n + 1$.)

Prove that $a(X, L) = n + 1$ if and only if $X \cong \mathbb{P}^n$ and L is the hyperplane class.

3.2. **Non-dominant families of rational curves.**

Exercise 3.5. Let X be an n-dimensional quadric hypersurface with $n \geq 2$.

(1) Prove that $a(X, H) = n$.
(2) Prove that X does not contain any subvariety $Y \subset X$ such that $a(Y, H) > a(X, H)$.

This implies that every family of rational curves on a quadric hypersurface has the expected dimension.

Exercise 3.6. Let X be an n-dimensional cubic hypersurface with $n \geq 3$.

(1) Prove that $a(X, H) = n - 1$.
(2) Prove that X does not contain any subvariety $Y \subset X$ such that $a(Y, H) > a(X, H)$.

This implies that every family of rational curves on a cubic hypersurface has the expected dimension. (What goes wrong when $n = 2$?)

Exercise 3.7 (Taken from a note by Christian Schnell)**.** Recall from the lecture that when X is a Fano variety there is a proper closed subvariety $V \subset X$ such that if $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ is a component that does not parametrize free curves then the curves parametrized by M

are contained in V. The following example shows that when X is a rational variety which is not Fano the analogous statement need not be true.

Fix a pencil of elliptic curves in $\mathbb{P}^2$ such that every member of the pencil is irreducible. By blowing up the 9 basepoints of the pencil we obtain a birational model $\phi : X \to \mathbb{P}^2$ equipped with a morphism $\pi : X \to \mathbb{P}^1$ whose fibers are the elliptic curves in our pencil.

(1) Show that $N^1(X)_{\mathbb{Z}}$ has rank 10 and is generated by the pullback H of the hyperplane class on $\mathbb{P}^2$ and the 9 exceptional curves $E_1, \ldots, E_9$. Show that the canonical divisor of X is $-3H + \sum_{i=1}^{9} E_i$.

(2) Show that a curve $C \subset X$ is a (-1)-curve if and only if it has class $bH - \sum a_i E_i$ where
$$\sum_{i=1}^{9} a_i = 3b - 1 \qquad\qquad \sum_{i=1}^{9} a_i^2 = b^2 + 1$$
In particular show that every (-1)-curve is a section of π.

(3) We have a group action on the sections of π in the following way. Suppose C_1, C_2 are sections of π. For any smooth fiber F of π, we can apply the group law of F to the points $(C_1 \cap F)$ and $(C_2 \cap F)$ to get a new point of F. We can apply this operation to every smooth fiber simultaneously and take a closure to obtain a section $C_1 * C_2 = C_3$. In this way we can obtain infinitely many (-1)-curves on $\mathbb{P}^2$.

Here is an explicit formula. Suppose that C is a (-1)-curve with class $bH - \sum a_i E_i$. Assume for simplicity that C is not any of the E_j. Prove that $C' = C * E_j$ has class determined by the formulas
$$a_i' = b - a_i - a_j + \delta_{ij} \qquad\qquad b' = 2b - 3a_j + 1$$
(Hint: for $i \neq j$, show that C' and E_i will intersect along a fiber F if and only if $\phi(C' \cap F)$, $\phi(E_i)$ and $\phi(E_j)$ are collinear in $\mathbb{P}^2$.)

3.3. **Free curves.**

Exercise 3.8. Prove the Movable Bend-and-Break conjecture for a del Pezzo surface X. (Hint: suppose that $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ is a component parametrizing free curves and that $\dim(M) = r$. Consider the sublocus in M parametrizing rational curves through $r-1$ general points of X. What are the possible outcomes for Bend-and-Break applied to this family?)

Deduce that the monoid of non-pathological components of $\mathrm{Mor}(\mathbb{P}^1, X)$ is finitely generated.