Lecture 2: Rational curves and the canonical divisor

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Introduction

Guiding principle

Recall from last time that the canonical line bundle of a smooth projective variety X is

$$\omega_X = \bigwedge^{\dim X} \Omega_X$$

and the canonical divisor K_X is any divisor representing ω_X.

Principle

The geometry/arithmetic of a smooth projective variety X over a field is controlled by the positivity of K_X.

We will discuss this principle in the context of rational curves. We work over the ground field $\mathbb{C}$ unless otherwise specified.

There are different ways of interpreting the “positivity” of a divisor. To start with we will focus on the three types of “pure” positivity:

negative	torsion	positive
$-K_X$ ample	a multiple of K_X is 0	K_X ample

Of course, most projective varieties will not have one of these three “pure” curvature types. However, the Minimal Model Program predicts that any smooth projective variety can be decomposed into a sequence of fibrations whose fibers have “pure” type.

Low dimensions

Curves

Let's start by analyzing our guiding principle when X is a curve. The basic invariant for classifying curves is the genus, but for our purposes it is better to use (the negative of) the Euler characteristic

$$\deg(K_X) = 2g(C) - 2.$$

With this definition it becomes clear that there is a trichotomy of curves:

$\deg(K_X)$	< 0	$= 0$	> 0
genus	0	1	≥ 2
$\mathrm{Mor}(\mathbb{P}^1, X)_d$	open subset of $\mathbb{P}^{2d+1}$	empty	empty

Note that this same trichotomy occurs in other areas of mathematics as well (Riemann Uniformization Theorem, behavior of rational points, etc.).

Surfaces

We next consider the case when X is a surface. We analyze the behavior of rational curves separately for surfaces with the three types of positivity for the canonical divisor.

A surface X with $-K_X$ ample is known as a del Pezzo surface. These surfaces have been completely classified: with the exception of $\mathbb{P}^1 \times \mathbb{P}^1$, a del Pezzo surface is the blow-up of $\mathbb{P}^2$ along at most 8 points in general position. In particular, each del Pezzo surface is birationally equivalent to $\mathbb{P}^2$.

We can find rational curves through any general point of X by taking the strict transforms of rational curves on $\mathbb{P}^2$. We conclude that a del Pezzo surface X is uniruled.

In the Kodaira-Enriques classification there are four types of surface with K_X torsion.

1) Abelian surfaces.
An abelian surface cannot contain any rational curves. Consider any morphism $f : \mathbb{P}^1 \to X$ and its differential $T_{\mathbb{P}^1} \to f^*T_X$. We have $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and (since an abelian surface has trivial tangent bundle) $f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Thus the map on tangent bundles is the zero map and f contracts $\mathbb{P}^1$ to a point.

2) Hyperelliptic surfaces.
A hyperelliptic surface cannot contain any rational curves. The Albanese map $alb : X \to B$ maps X to an elliptic curve and the fibers of alb are irreducible curves of genus ≥ 1.

3) K3 surfaces.
A K3 surface X can contain a rational curve. (For example, a quartic surface in $\mathbb{P}^3$ can contain a line.) But X is not uniruled: for any non-trivial $f : \mathbb{P}^1 \to X$ the pullback f^*T_X is a rank 2 bundle of degree 0. Since f^*T_X must admit a non-zero map from $\mathcal{O}(2)$, it also must have a negative summand.

In fact much more is true:

Theorem (Chen-Gounelas-Liedtke)

Every complex K3 surface contains infinitely many non-free rational curves.

4) Enriques surfaces.
Every Enriques surface is a quotient of a K3 surface and so has similar behavior.

Surfaces

Finally, we consider the case when K_X is ample.

Conjecture (Algebraic hyperbolicity)

A smooth projective surface with K_X ample will have only finitely many rational curves.

This conjecture has been verified in some cases. For example, one of the early results is:

Theorem (Clemens)

A very general surface of degree ≥ 5 in $\mathbb{P}^3$ contains no rational curves.

Despite some fantastic partial progress, the conjecture remains open in general.

High dimensions

Higher dimensions

When X is a smooth projective variety of dimension ≥ 3 the picture is similar:

$-K_X$ ample	K_X torsion	K_X ample
Thm: (Mori) X is uniruled.	"inbetween"	Conj: The rational curves are contained in a proper Zariski closed subset of X.

Here "inbetween" covers a range of possibilities: X might admit no rational curves at all (abelian variety) or could admit infinitely many rational curves (K3 surface). However we will soon show that if K_X is torsion then X cannot be uniruled. Thus the rational curves on X sweep out at most a countable union of proper closed subvarieties of X.

Before moving on, we discuss one more notion of "positivity" for the canonical divisor. This notion is based around the behavior of sections of the canonical divisor.

Definition

Let X be a smooth projective variety. If $H^0(X, mK_X) = 0$ for every $m > 0$, we say that X has Kodaira dimension $-\infty$. Otherwise, we define the Kodaira dimension to be the smallest non-negative integer r such that

$$\limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^r} < \infty.$$

One can show that the Kodaira dimension of X takes values in the set $\{-\infty, 0, 1, 2, \dots, \dim(X)\}$. If K_X is ample, torsion, or antiample then $\kappa(X) = \dim(X), 0, -\infty$ respectively.

Proposition

If X is uniruled then $\kappa(X) = -\infty$.

Proof: Suppose for a contradiction that $H^0(X, mK_X) \neq 0$ for some $m > 0$. Since X is uniruled, we can find a free rational curve $f : \mathbb{P}^1 \to C \subset X$ and a section $D \in |mK_X|$ such that $D|_C$ does not vanish. In particular $\deg(f^* K_X) \geq 0$.

Consider now the vector bundle $f^* T_X$ of rank $\dim(X)$ on $\mathbb{P}^1$. The calculation above shows that $\deg(f^* T_X) \leq 0$. Since f is free, this must imply that $f^* T_X = \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim(X)}$. However, since f does not contract $\mathbb{P}^1$ to a point there should also be a non-zero map $\mathcal{O}_{\mathbb{P}^1}(2) = T_{\mathbb{P}^1} \to f^* T_X$, yielding a contradiction. □

Higher dimensions

Conversely, the Kodaira dimension should predict the behavior of rational curves. On one extreme, we have:

Conjecture

If $\kappa(X) = -\infty$ then X is uniruled.

On the other extreme, we have:

Conjecture

If $\kappa(X) = \dim(X)$ then there is a proper closed subset of X which contains all the rational curves on X.

This is a birational version of the algebraic hyperbolicity conjecture for rational curves.

Bend-and-Break

For the rest of the lecture, we will focus on Mori's result: a smooth complex variety with $-K_X$ ample is uniruled. In fact, we will sketch the proof of a stronger theorem:

Theorem (Mori)

Let X be a smooth projective variety. Suppose that C is a curve in X satisfying $K_X \cdot C < 0$. Then there is a rational curve in X through every point of C.

This immediately implies the desired result for varieties with $-K_X$ ample.

In order to prove this theorem, we will need to understand the space of morphisms $\operatorname{Mor}(B, X)$ where B is a smooth projective curve of arbitrary genus. Fortunately, the situation is exactly the same:

- $\operatorname{Mor}(B, X)$ can be constructed as a subscheme of $\operatorname{Hilb}(B \times X)$ and thus admits a universal family.
- Given a morphism $f : B \to X$, the tangent space to the morphism scheme at f is $H^0(B, f^* T_X)$.
- The expected dimension

$$\chi(f^* T_X) = -K_X \cdot f_* B + (1 - g(B)) \dim(X)$$

gives a lower bound for the dimension of $\operatorname{Mor}(B, X)$ near f.

We will also need a slight modification: we will consider morphisms $f : B \to X$ which send a fixed point in B to a fixed point in X.

Suppose we fix a map $f : B \to X$ and a point $p \in B$. We denote by $\mathrm{Mor}(B, X; f|_p)$ the sublocus of maps $g \in \mathrm{Mor}(B, X)$ such that $g(p) = f(p)$. We will also need to analyze the tangent space of this subscheme:

- Given a morphism $f : B \to X$ and a point $p \in B$, the tangent space to $\mathrm{Mor}(B, X; f|_p)$ at f is $H^0(B, f^* T_X \otimes \mathcal{O}_B(-p))$.
- The expected dimension

$$\chi(f^* T_X \otimes \mathcal{O}_B(-p)) = -K_X \cdot f_* B - g(B) \cdot \dim(X)$$

gives a lower bound for the dimension of $\mathrm{Mor}(B, X; f|_p)$ near f.

Theorem (Mori's Bend-and-Break)

Let X be a smooth projective variety and let B be a smooth projective curve of genus ≥ 1. Fix a non-trivial map $f : B \to X$ and a point $p \in B$ and suppose we have a curve $T \subset \mathrm{Mor}(B, X; f|_p)$ containing f. Then there is a rational curve through $f(p)$ in X.

Proof: Let $T' \to T$ denote the normalization and let U' denote the base-change of the universal family to T'. Thus $U' \cong B \times T'$ and we have a map $ev : U' \to X$ that contracts the section $\{p\} \times T'$.

We next compactify: we let $\overline{T}$ denote a smooth projective curve containing T' and let $\overline{U}$ denote $B \times \overline{T}$. We now have a rational map $ev : \overline{U} \dashrightarrow X$.

The next step is to appeal to:

Rigidity Lemma: Suppose that $ev : B \times \overline{T} \dashrightarrow X$ is well-defined at every point of the section $\{p\} \times \overline{T}$ and contracts this section to a point. Then ev factors through the projection map to B.

Proof of lemma: A projective curve is contracted by ev if and only if it has vanishing intersection against the pullback of an ample divisor on X. But this is a numerical property; if it is true for one section, it will be true for all of them. □

Since by assumption T parametrizes a family of morphisms which vary in moduli, we see that $ev : B \times \overline{T} \dashrightarrow X$ must fail to be defined at some point (p, t).

The last step is to appeal to the birational geometry of surfaces.

We know that the rational map *ev* can be resolved. That is, there is a birational map $\phi : S \to B \times \overline{T}$ obtained by a sequence of point blow-ups and a morphism $ev_S : S \to X$ which agrees with *ev* on the common locus of definition.

Consider the fiber of ϕ over (p, t); this is a union of rational curves on S. Not all of these curves can be contracted by ev_S; if they were, then our original map *ev* would have been defined at (p, t). Furthermore, the image of this fiber must intersect the image of the strict transform in S of $\{p\} \times \overline{T}$. Altogether, we see that at least one of the rational curves in the fiber of ϕ over (p, t) must survive on X and go through $f(p)$. □

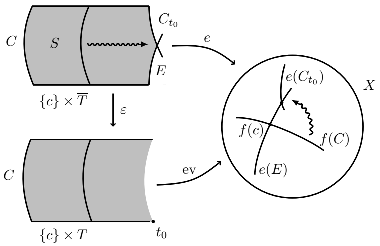

Figure 4: The 1-cycle f_*C degenerates to a 1-cycle with a rational component $e(E)$.

Picture from Debarre, "Bend and Break"

There is also a Bend-and-Break theorem for rational curves.

Given a morphism $f : \mathbb{P}^1 \to X$ and two different points $p, q \in \mathbb{P}^1$, we denote by $\mathrm{Mor}(\mathbb{P}^1, X; f|_{p,q})$ the set of morphisms $g : \mathbb{P}^1 \to X$ such that $g(p) = f(p)$ and $g(q) = f(q)$.

Theorem (Mori's Bend-and-Break)

Let X be a smooth projective variety. Fix a non-trivial map $f : \mathbb{P}^1 \to X$ and points $p, q \in \mathbb{P}^1$. Suppose we have a curve $T \subset \mathrm{Mor}(B, X; f|_{p,q})$ containing f such that the maps parametrized by T sweep out a surface in X. Then the image cycle $f_*(\mathbb{P}^1)$ deforms to a non-integral curve with rational components which contains $f(p)$ and $f(q)$.

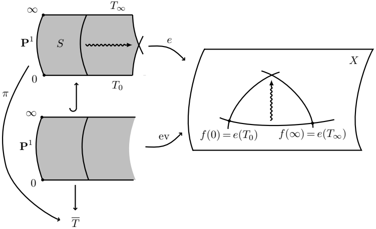

Figure 5: The rational 1-cycle f_*C bends and breaks

Picture from Debarre, "Bend and Break"

We now return to our original goal:

Theorem (Mori)

Let X be a smooth projective variety. Suppose that C is a curve in X satisfying $K_X \cdot C < 0$. Then there is a rational curve in X through every point of C.

Let $f : B \to C \subset X$ denote the normalization map. It suffices to consider the case when $g(B) \geq 1$. Fix any point $p \in B$. If we knew that $\mathrm{Mor}(B, X; f|_p)$ had dimension ≥ 1, then Bend-and-Break would imply the existence of the desired rational curve through p.

Of course, there is no reason to assume that $\dim(\mathrm{Mor}(B, X; f|_p)) \geq 1$. In fact the expected dimension

$$-K_X \cdot f_*(B) - g(B) \cdot \dim(X)$$

might be very negative. Mori found an ingenious way around this obstacle by passing to characteristic p.

Sketch of proof:

Step 1: spreading out
We can choose an algebra Z that is finitely generated over $\mathbb{Z}$ such that every relevant object in our situation is defined over Z. After possibly shrinking $\mathrm{Spec}(Z)$, we can find a smooth map $\mathcal{X} \to \mathrm{Spec}(Z)$ whose fiber over the generic point is isomorphic to X (after extending the base field). We may also ensure that all our constructions extend over all of $\mathcal{X}$.

Thus for every closed point $z \in \mathrm{Spec}(Z)$ we obtain a fiber X_z and a curve C_z satisfying $K_{X_z} \cdot C_z < 0$. Note that each such X_z is defined over a finite field of characteristic $p > 0$.

Sketch of proof:

Step 2: twisting up

Let $f_z : B_z \to C_z \subset X_z$ denote the normalization map. Fix a point $p_z \in B_z$ and consider the scheme $\mathrm{Mor}(B_z, X_z; f_z|_{p_z})$. As remarked earlier, there is no reason to assume that the expected dimension

$$-K_{X_z} \cdot f_{z*}(B_z) - g(B_z) \cdot \dim(X_z)$$

is positive. However, suppose that we now precompose f_z by r iterates of the Frobenius map for B_z. If we let h_z denote the composed map and let p denote the characteristic of the residue field of p_z, the expected dimension is now

$$p^r(-K_{X_z} \cdot f_{z*}(B_z)) - g(B_z) \cdot \dim(X_z)$$

By assumption this will be positive when r is large enough. Applying Bend-and-Break we obtain a rational curve Y_z through every point of C_z.

Sketch of proof:

Step 3: deforming back

For every closed point $z \in \mathrm{Spec}(Z)$ and every point $p_z \in C_z$ we have found a rational curve Y_z through p_z. We would now like to “deform” these rational curves back to the generic fiber to find a rational curve on our original variety X.

If we knew that the rational curves Y_z were bounded – that is, if they were contained in a finite-type subscheme of the relative Hilbert scheme – then there would have to be a single component of the Hilbert scheme that parametrized the curves for a dense open subset of $\mathrm{Spec}(Z)$. By Chevalley’s Theorem, the image of this component in $\mathrm{Spec}(Z)$ would also contain the generic point. Since the geometric genus is constant in the family, we would obtain the desired rational curve through the point $p \in C$.

Introduction
Low dimensions
High dimensions
Bend-and-Break

Sketch of proof:

Step 4: breaking down

Unfortunately Bend-and-Break gives us essentially no control over the rational curves Y_z we constructed in Step 2. In particular, there is no reason to expect that as we vary the closed point $z \in \mathrm{Spec}(Z)$ the rational curves Y_z form a bounded family. In other words, if we fix an ample divisor A on $\mathcal{X}$ then the degrees of the Y_z against A could be unbounded.

Fortunately, Bend-and-Break comes to our rescue again. If the A-degree of Y_z is large enough then the deformation space of Y_z through the point p_z has dimension $> \dim(X) + 1$. In particular, we can find a curve in the moduli space parametrizing deformations of Y_z through p_z and through some other fixed point. Applying the rational curve version of Bend-and-Break, we find a different rational curve Y_z' through p_z of smaller A-degree. Arguing inductively, we eventually find a bounded family of rational curves Y_z which allow us to conclude by the previous step.