The Mathematics of Andrei Suslin

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Andrei Suslin

Andrei Suslin 1950 - 2018.

FRIEND, COLLEAGUE, COAUTHOR.

Introduction

This will be a tour of some of Andrei Suslin's important results.

My goal is to give you some impression of the breadth and importance of his work.

Perhaps you will see an evolution from a brilliant algebraist using ingenious, direct methods to the master/arbiter of the world of algebraic K-theory

Serre's Conjecture

Theorem

(Quillen, Suslin; 1976)
Let k be a field, $n, N > 0$, $R = k[x_1, \ldots, x_n]$. Then any R-module direct summand

$$P \subset R^{\oplus N}$$

is a free R-module.

Reduce to special case: Let P be a finitely generated projective $A[x]$-module, for A "sufficiently general." Assume that P becomes extended from A after we invert all monic polynomials. Then P is extended from A.

Reduction argument

Suslin's explanation:

We proceed by induction on n. We first note that P obviously becomes free after we invert all non-zero polynomials. This implies that there exists a non-zero polynomial f such that P becomes free after we invert f . A classical lemma of Noether shows that making an appropriate change of variables we may assume that f is monic in x_n. Thus setting A equal to $k[x_1, \ldots, x_n]$ and applying the above theorem we conclude that P is extended from A.

Merkurjev-Suslin Theorem

Theorem

(Merkurjev, Suslin1982)
Let F be a field and $n > 0$ invertible in F. Then the norm residue homomorphism

$$K_2(F)/n \overset{\sim}{\to} Br(F)/n$$

(sending a symbol $\{a, b\}$ to the cyclic algebra $A_\zeta(a, b)$) is an isomorphism.

Here $K_2(F)$ is the 2nd K-group of F (introduced by Milnor) and $Br(F)$ is the Brauer group of F, equivalence classes of central simple algebras over F each uniquely represented by a finite division algebra with center F.

Commentary

Andrei and Sasha utilize the various aspects of foundational theory of Quillen K-theory, especially the Brown-Gersten spectral sequence involving K-cohomology, applying these techniques to Severi-Brauer varieties.

Perhaps the most famous of Andrei's theorems, this gives insights into generators of the Brauer group.

The proof provides a framework for the eventual proofs by Voevodsky of the Milnor and Bloch-Kato conjectures.

Significance: In some sense, the deepest result given for codimension 2 cycles.

mod-n K-theory

Theorem

(Suslin1983)
Let $F \to L$ be an extension of algebraically closed fields. Then

$$K_*(F, \mathbb{Z}/n) \stackrel{\sim}{\to} K_*(L, \mathbb{Z}/n)$$

is an isomorphism. In other words,

$$H_*(GL_\infty(F), \mathbb{Z}/n) \stackrel{\sim}{\to} H_*(GL_\infty(L), \mathbb{Z}/n).$$

This was conjectured by Lichtenbaum, a very special case of the Quillen-Lichtenbaum Conjecture.

Beautiful proof – a 5 page paper!

Suslin rigidity

Andrei's proof introduces and employs

- Specialization map $K_i(X_E, \mathbb{Z}/n) \to K_i(X_F, \mathbb{Z}/n)$ for X over a discrete valuation ring with fraction field E and algebraically closed residue field F.
- Consider smooth curve C with function field E and residue field F. Relate specializations associated to $X \times C$ to specializations for $X \times \mathbb{P}^1$ using properties of transfers.
- Suslin rigidity Let Θ be a contravariant functor on varieties over F with values in torsion abelian groups which admits well-behaved transfers with respect to finite flat maps $X \to Y$. Then for any two rational points x, y of X, $x^* = y^* : \Theta(X) \to \Theta(\operatorname{Spec} F)$.
  Hence, specialization is independent of closed points of C.
- Trick to consider both generic and closed points of C as closed points of C_E.

Milnor K-theory

Definition

$$K_*^M(F) \;\equiv\; T^*(F^*)/ < \{a \otimes (1-a), a \in F^*;\; 0 \neq a \neq 1\} > .$$

Theorem

(Suslin, 1980's)
$H_n(GL_n(F))/im\{H_n(GL_{n-1}(F))\} \;\simeq K_n^M(F),\;$ *F infinite.*

Moreover, the following composition is multiplication by $(n-1)!$:

$$K_n^M(F) \;\to\; K_n(F) \;\to\; H_n(GL_\infty(F)) \;\simeq$$

$$\simeq\; H_n(GL_n(F)) \;\to\; K_n^M(F).$$

Role of Milnor K-theory

Definition of $K_*^M(F)$ is very ad hoc; agrees with $K_*(F)$ for $* = 0, 1$.

Good properties are difficult to establish, unlike for $K_*(-)$.

On the other hand, connected with quadratic forms.

Suslin's results relate higher Milnor K-theory with higher algebraic K-theory

Suslin and Nesterenko establish the following interpretation of Milnor K-theory of a field with motivic cohomology:

$$H^n(F, \mathbb{Z}(n)) \;\simeq\; K_n^M(F).$$

Suslin complex

In 1987, Andrei introduced the Suslin complex $Sus_*(X)$ associated to a variety X. This is a simplicial abelian group whose topological analogue for a C.W. complex W has homotopy groups which are the integral homology W:

$$Sus_*(X) \; = \; n \mapsto Hom(\Delta^n, (\coprod_{d=0}^{\infty} S^d(X))^+).$$

Theorem

(Suslin, Voevodsky, 1997)
If X is a quasi-projective variety over $\mathbb{C}$, then the natural map

$$\pi_i(Sus_*(X), \mathbb{Z}/n) \; \to \; H_i(X(\mathbb{C})^{an}, \mathbb{Z}/n), \quad i \geq 0$$

is an isomorphism.

Why is this remarkable?

Motivation from the Dold-Thom theorem in algebraic topology. Andrei had the daring to introduce the analogous construction in algebraic geometry.

A typical variety X has few maps from Δ^n to X. Andrei recognized that symmetric powers of X do have many such maps.

One of the major achievements of etale cohomology is to give a natural algebraic formulation of $H_*(X(\mathbb{C})^{an}, \mathbb{Z}/n)$ (does not exist with $\mathbb{Q}$-coefficients.) Suslin-Voevodsky give an alternative to etale cohomology.

B-K if and only if B-L

Theorem

(Suslin, Voevodsky, 1998)
Let F be a field and ℓ a prime invertible in F. Then the Bloch-Kato Conjecture for F:

$$K_n^M(F)/\ell \;\simeq\; H^n(F, \mu_\ell^{\otimes n})$$

is equivalent to the Beilson-Lichtenuam Conjecture concerning motivic cohomology:

$$H^p(F, \mathbb{Z}/\ell(q)) \;\simeq\; H^p(F, \mu_\ell^{\otimes q}), \quad p \leq q.$$

They also derive a similar equivalence of conjectures for any smooth projective variety over F.

Remarks about applicability

Note that the first conjecture is the special case of the second with $p = q$.

The proof of both conjectures are intertwined using this equivalence, utilizes brilliant work of Voevodsky.

Andrei played the role of arbiter of developments in motivic cohomology. He spent a great deal of effort working through the details of this equivalence.

Motivic spectral sequence

Andrei provided the complete proof of the motivic spectral sequence following work of various others (including a 112 page joint paper with me) relating motivic cohomology to algebraic K-theory.

Theorem

finalized by Andrei (Suslin 2003); cf (F-Suslin, 2002)

Let X be a smooth quasi-projective variety over a field. Then there is a strongly convergent spectral sequence

$$E_2^{p,q} = H^{-p-q}(X, \mathbb{Z}(-q)) \ \Rightarrow \ K_{-p-q}(X).$$

Finite generation of the cohomology of finite group schemes

Andrei and I struggled for some time to prove the following theorem, which we both found pleasing.

Theorem

(F-Suslin, 1997) Let G be a finite group scheme over a field k. Then $H^*(G, k)$ is finitely generated. Moreover, if M is a finite dimensional G-module, then $H^*(G, M)$ is finitely generated over $H^*(G, k)$.

This is a first suggestion that one can find a common context for finite groups, restricted enveloping algebras of finite dimensional restricted Lie algebras, and other finite group schemes.

Consequences

Finite generation of cohomology enables use of algebraic geometry to study $H^*(G, k)$ and $Ext^*_G(M, M)$.

The heart of the proof is to construct explicit cohomology classes, which are readily seen to generate using spectral sequence arguments.

These explicit classes are constructed using strict polynomial functors. Andrei and coauthors (including myself) used this technique to make many cohomological calculation.

Rank varieties for infinitesimal group schemes

In joint work with Chris Bendel and myself, Andrei proved some remarkable theorems about the cohomology of infinitesimal group schemes. The arguments follow, to some extent, Quillen's description of the spectrum of the cohomology of a finite group, but they are technically more challenging.

Theorem

(Suslin-F-Bendel, 1997)
Let G be a connected affine scheme over a field k of positive characteristic. Consider the infinitesimal group scheme $G_{(r)}$ (kernel of $F^{(r)} : G \to G^{(r)}$) and the scheme $V_r(G)$ representing morphisms $\mathbb{G}_{a(r)} \to G$. Then there is a natural map

$$\psi : k[V_r(G)] \;\to\; H^*(G_{(r)}, k)$$

which induces a homeomorphism on prime ideal spectra.

Example

Let $G = GL_N$. Then $V_r(GL_N)$ is the scheme of r-tuples $(B_0, \ldots, B_{r-1})$ of p-nilpotent $r \times r$ matrices.

$k[V_r(GL_N]$ is generated by elements
$\{X^{i,j}(\ell) : \; 1 \leq i, j \leq N, 0 < r;$.
Explicit relations arising from the above description.

View this as a geometric approximation of the cohomology.

Use this to construct invariants of GL_N-modules.

Generic and maximal Jordan types

Andrei, with J. Pevtsova and myself, shows that one can get refined invariants of G-modules by considering Jordan types at generic points of $\operatorname{Proj} H^*(G,k)$.

Theorem

(F-Pevtsova-Suslin, 2007)
Let G be a finite group scheme, M a finite dimensional G-module, and $x \in \operatorname{Proj} H^*(G,k)$ correspond to a minimal homogeneous prime ideal of $H^*(G,k)$. Then this data naturally determines a natural partition of $m \equiv dim(M)$,

$$m \;=\; \sum_{i=1}^{p} a_i \cdot i, \quad a_i \geq 0.$$

This partition arises from the Jordan type of any representative of the p-nilpotent action of G on M at the generic point x.

Remark about Jordan types

The support variety of a G-module M is the "locus of points at which the module is not projective".

If the dimension of a G-module is not divisible by p, for example, the support variety is the same as for the trivial module.

The F-Pevtsova-Suslin theorem gives interesting non-trivial invariants for any module which is not projective, and these invariants are more detailed.

Other important contributions – only a list

Andrei has worked very successfully on many other problems. Here is a partial list.

- General development of motivic cohomology
- Investigation of division algebras and reduced norms
- Norm varieties and Rost motives
- Stability results for homology and K-theory
- Computations for K_3
- Excision in rational K-theory