Taking inspiration from the past for *Changing the Culture*: some few steps in the company of Euclid, Archimedes, Heron and al-Khwarizmi

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Changing The Culture 2013

The annual Changing the Culture Conference, organized and sponsored by the Pacific Institute for the Mathematical Sciences, brings together mathematicians, mathematics educators and school teachers from all levels to work together towards narrowing the gap between mathematicians and teachers of mathematics, and between those who do and enjoy mathematics and those who think they don't.

History of mathematics as a possible means in order to foster this Change of Culture

- between various groups of individuals
  (mathematicians, math educators, teachers)

- within classrooms

History of mathematics nowadays

as an increasing requirement of the school mathematics curriculum *(vg, in relation to culture)*

*hence new needs for **teacher preparation***

Issue: *How to provide support to teachers so to help them use history in school mathematics?*

Is the "teacher education system" willing / able to offer such support to teachers?

My comments are based on my experience as a mathematician involved in teacher education within a math department

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History of mathematics in the teaching of mathematics and in teacher education **A long tradition!**

Third International Congress of Mathematicians
Heidelberg (1904)

"[…] history of mathematics nowadays constitutes a discipline of undeniable importance,
[…] its benefit—from the directly mathematical viewpoint as well as from the pedagogical one—becomes ever more evident,
[…] it is, therefore, indispensable to accord it the proper position within public instruction."

Resolution recommending the introduction of historical components in education

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PLAN OF THE TALK

I- Introductory remarks *(done)*

II- History of mathematics and the school mathematics curriculum

III- Resources to support the teaching of history of mathematics to prospective teachers

IV- Examples of topics in the history of mathematics suitable for prospective teachers

V- Concluding remarks

II- History of mathematics and the school mathematics curriculum

Robust tendency: introduction in the school curricula of components explicitly related to matters of culture and history of mathematics

Examples from Québec — ***new ingredients*** of the school curriculum in mathematics

even at the primary level!

Québec Education

Program

Approved Version

(2001)

New Directions for success Together

Preschool Education
Elementary Education

Québec

Cultural References

• Numbers	1	2	3
– origin and creation of numbers	❶		
– development of systems for writing numbers	❶		
– number systems (*e.g. Arabic, Roman, Babylonian, Mayan*): characteristics, advantages and disadvantages		❷	❸
– social context (*e.g. price, date, telephone, address, age, quantity: mass, size*)	❶	❷	❸

• Operations	1	2	3
– own or conventional computation processes: development, limitations, advantages and disadvantages	❶	❷	❸
– technology: development (*e.g. sticks, strokes, abacus, calculator, software*), limitations, advantages and disadvantages	❶	❷	❸
– symbols (origin, development, need, mathematicians involved): +, −, >, <, =	❶		
– symbols (origin, development, need, mathematicians involved): ∞, ÷, ≠		❷	
– interdisciplinary or social context (*e.g. history, geography, science and technology*)	❶	❷	❸

Geometric figures	1	2	3
– interdisciplinary or social context (e.g. architecture, maps, arts, decoration)	❶	❷	❸
– symbols (origin, development, need, mathematicians involved): ∠, //, ⊥		❷	❸

• Measurement	1	2	3
– systems of measurement (historical aspect)	❶	❷	❸
– units of measure: development according to society's needs (*e.g. agrarian measurements, astronomy, standard measurement, precision*), instruments (rudimentary approach for measuring time, hourglass, clock)		❷	❸
– symbols (origin, development, need): m, dm, cm	❶		
– symbols (origin, development, need): m, dm, cm, mm		❷	
– symbols (origin, development, need): km, m, dm, cm, mm			❸
– symbols (origin, development, need): kg, g, L, mL			❸
– symbols (origin, development, need): h, min, s	❶	❷	❸
– symbols (origin, development, need): °C			❸
– symbols (origin, development, need, mathematicians involved): ( ), %			❸

In each cycle, students in a given class carry out at least one individual or group project or activity related to cultural references.

Symbols

	1	2	3
• 0 to 9, +, −, >, <, =	❶		
• 0 to 9, +, −, ×, ÷, >, <, =, ≠		❷	
• 0 to 9, +, −, ×, ÷, >, <, =, ≠, ( ), %			❸
• Calculator keys [keys: 0 to 9, +, −, ×, ÷, =, ON, OFF, AC, C, CE (all clear, clear, clear last entry)]	❶	❷	❸
• Certain commonly used calculator functions [memories (M+, M–, MR, MC), change of sign (+/–)]			❸
• Numbers written using digits	❶	❷	❸
• Writing fractions ($\frac{a}{b}$)	❶	❷	❸
• Writing decimals using a period as the decimal marker		❷	❸
• Exponential notation $■^2$, $■^3$			❸
• ∠, //, ⊥		❷	❸
• m, dm, cm	❶		

(2001)

(2005)

Québec Education Program

Secondary School Education, Cycle One

Reach for your Dreams

Québec

Mathematics

Cultural References

Mathematical knowledge is universal and used everyday to interpret and understand reality and to make decisions. It enables individuals to participate in many spheres of human endeavour and to appreciate the contribution of this subject. The historical evolution of mathematics and the invention of certain instruments have been directly or indirectly related to the needs of different societies.

Mathematics has a rich history, and many mathematicians, scientists, artists and philosophers have contributed to its advancement. In activities related to the history of mathematics, students may notice that concepts and processes are often attributed to a particular mathematician despite the fact that they emerged through the efforts of a number of mathematicians, both men and women, from different eras (e.g. the Pythagorean theorem was already known in Babylonian times). In studying the contribution of women to the development of mathematics, students will learn that a number of women had difficulty achieving acceptance in the mathematics community.[19] By investigating the origin of certain words, students can make concepts and processes more meaningful and discover that researchers from many nations contributed to the development of mathematics. An epistemological dimension should therefore be incorporated into learning activities to provide a window on the past, the present and the future.

Arithmetic and Algebra

The composer opens the cage door for arithmetic,
the draftsman gives geometry its freedom.
Jean Cocteau

The development of mathematics has been shaped by the influence and contributions of different civilizations and cultures. For example, the Indians and Arabs shaped the development of mathematics in the Western world with regard to numeration systems, algebra and trigonometry. By examining these different contributions, students will be able to see and somewhat better understand how the set of real numbers was developed over time. This would involve examining questions such as the revolutionary significance of the introduction of zero, the reluctance to accept negative numbers and the crisis resulting from the incommensurability of $\sqrt{2}$.

Proportional reasoning has considerable currency in everyday life, and it is used in various occupations related to construction, the arts, health, tourism, administration and other fields. In addition, it has been studied by many different mathematicians throughout history (e.g. Thales, Eudoxus, the Pythagorians, Euler) to explain or represent phenomena that also relate to the arts (e.g. harmony in music, aesthetics in architecture).

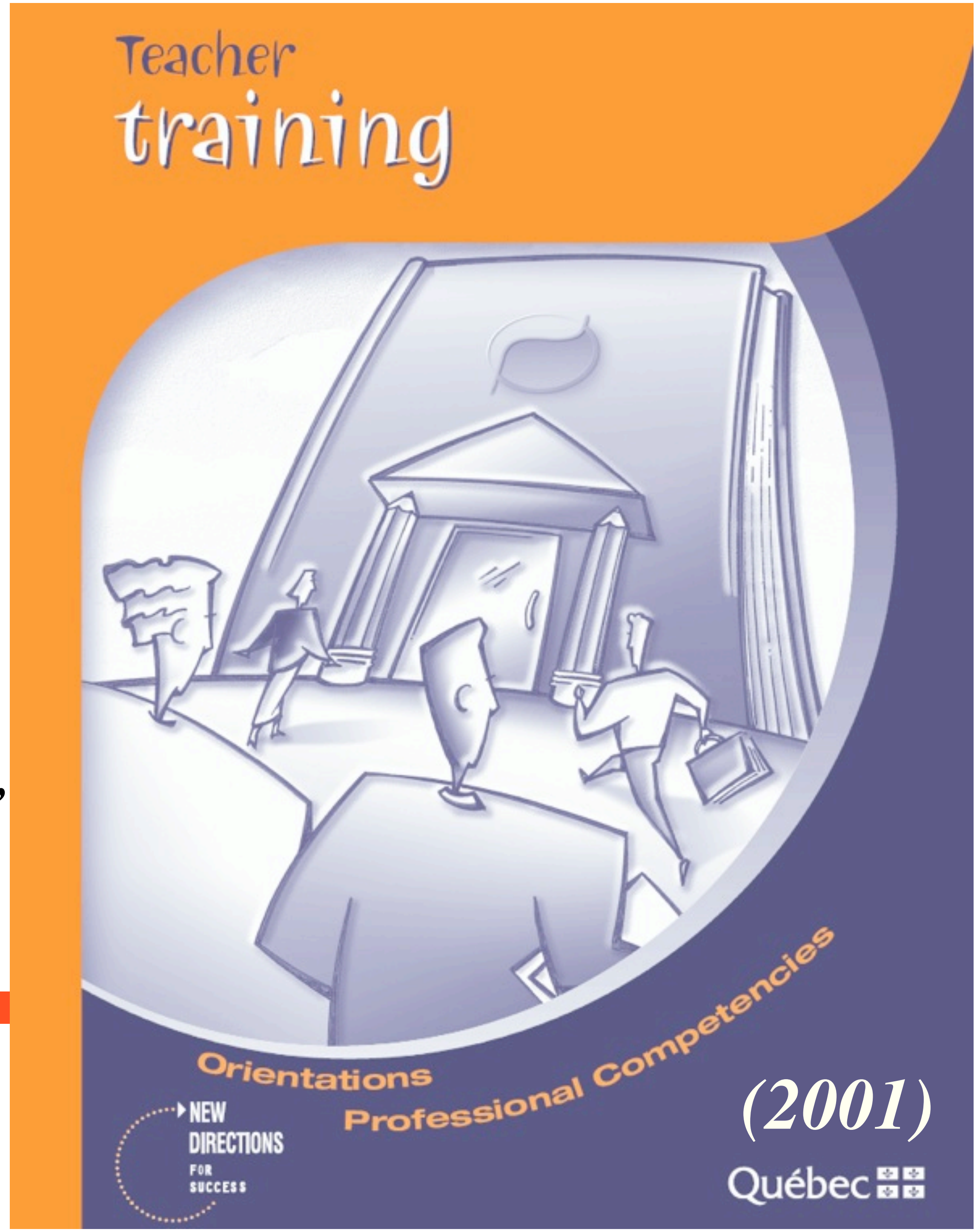
The problem of infinity has provided food for thought through the ages. A discussion of the concept of infinity will give students the opportunity to visualize and reflect on the infinitely small or the infinitely large in sequences or intervals, for example.

In studying algebra, students may explore its origins by examining the general rules developed by Arab mathematicians in their efforts to solve problems. For instance, the work of Al-Khawarizmi in the 9th century, which dealt with the decimal number system and the solution of first- and second-degree equations, contributed to the development of the algebraic processes used today. The concept of function appeared around the 1700s. The idea of functionality gained currency in our society out of a concern for interpreting reality, particularly with regard to the study of motion and the calculation of time. Students may discover connections between the concept of function and the fields of music, ballistics, navigation, cartography or astronomy.

The different types of notation established by certain mathematicians make it possible to manipulate expressions more efficiently. In learning mathematical language, students will discover that the standardization of symbols took place over many centuries. Diophantos was one of the first to use symbols. It was not until the 15th century, however, that dedicated efforts were undertaken to create symbols and standardize them, though not without difficulty. François Viète made a major contribution in this regard. Today's students cannot help but notice the widespread use of various types of symbols (e.g. acronyms, logos, the short form of words, letters, numbers) and their impact on everyday life.

19. Some even had to pass themselves off as men so their work would be given consideration. Sophie Germain (alias Antoine-Auguste Le Blanc) is one such example.

Ministère de l'Éducation, du Loisir et du Sport Québec



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(2001)

With regard to the cultural preparation of prospective teachers:

“To provide this training, we will not only have to determine a specific culture as an object, but also develop a particular relationship to culture among future teachers, a kind of shared awareness among teaching professionals.”

- “every concept or piece of knowledge has a history which, when known by the students, can either anchor or block their understanding”

- “culture is presented as a kind of *sensitivity* that allows us to define a relationship with the world, with ourselves and with others”

- “this sensitivity must be present in all the courses of the teacher education program”

Teacher preparation in Québec in connection with history of mathematics

- ***primary school teacher education:***
  no specific course devoted to culture / history matters

- ***secondary school teacher education:***
  no uniformity among universities

  in most Québec universities
  one *course on history of math* (often specific to teachers) *plus special attention* in other math courses specific to teachers

PLAN OF THE TALK

I- Introductory remarks *(done)*

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V- Concluding remarks

III- Resources to support the teaching of history of mathematics to prospective teachers

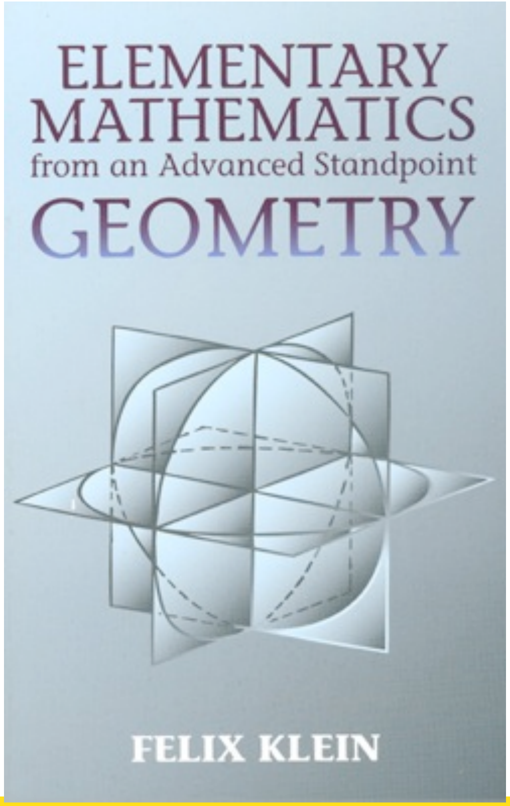
1-) One issue at stake: status of the field of history of mathematics within the university math department

- Level of acceptance by the math department of history of math as a “real” mathematical domain for which the department is responsible
- Level of support offered to those faculty members involved in the teaching of history of mathematics

2-) A second issue: the place and role of history of mathematics in the mathematical preparation of teachers

Importance of preparing teachers so that they are able to use history of math in their teaching

"I shall draw attention (…) to the *historical development of the science*, to the accomplishments of its great pioneers. I hope, by discussions of this sort, to further, as I like to say, your *general mathematical culture:* alongside of knowledge of details, (…) there should be a grasp of subject-matter and of historical relationship."



Felix Klein (1849 – 1925)

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Furinghetti, *Educational Studies in Mathematics* (2007)

Teacher education through the history of mathematics

“(…) an education program for prospective teachers in which the history of mathematics was introduced not per se, but **as a mediator of knowledge for teaching**. The aim was to make the participants reflect on the meaning of mathematical objects through experiencing historical moments of their construction. It was intended that this reflection would promote an appropriation of meaning for teaching mathematical objects that counteracts the passive reproduction of the style of mathematics teaching the prospective teachers have experienced as students.”

Resources on the role of history of mathematics in the mathematical preparation of teachers

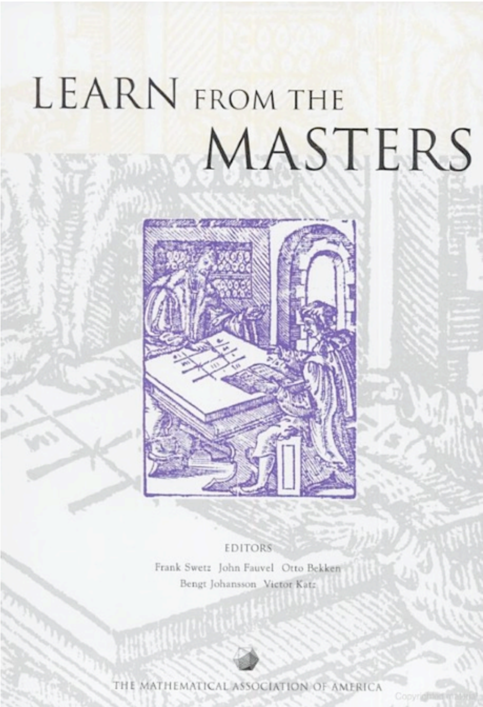
HISTORY and PEDAGOGY of MATHEMATICS * HPM

The International Study Group

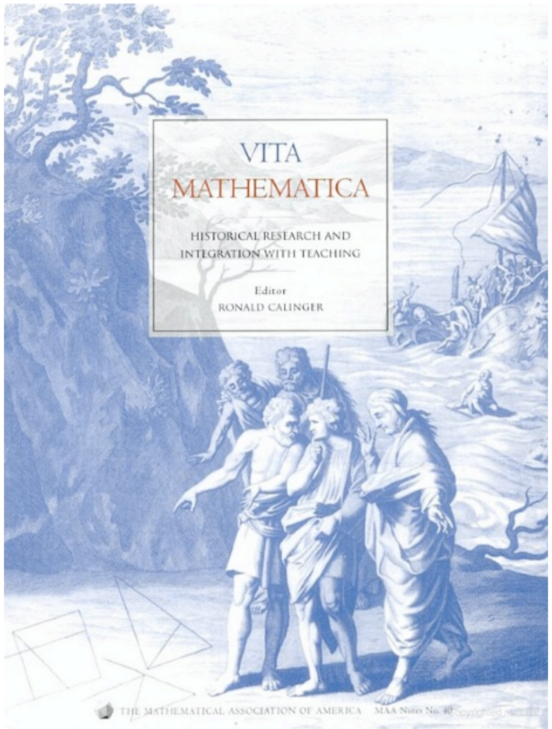
on the relations between
the HISTORY and PEDAGOGY of MATHEMATICS

(HPM Group)

An affiliate of the
International Commission on Mathematical Instruction (ICMI)



(1988)



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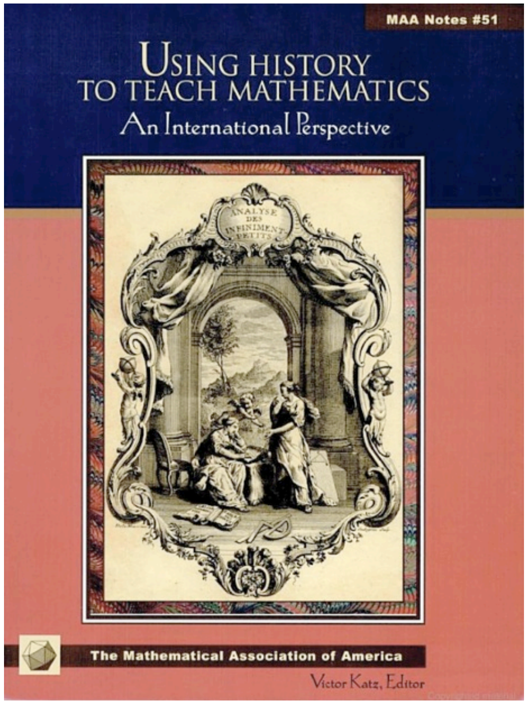
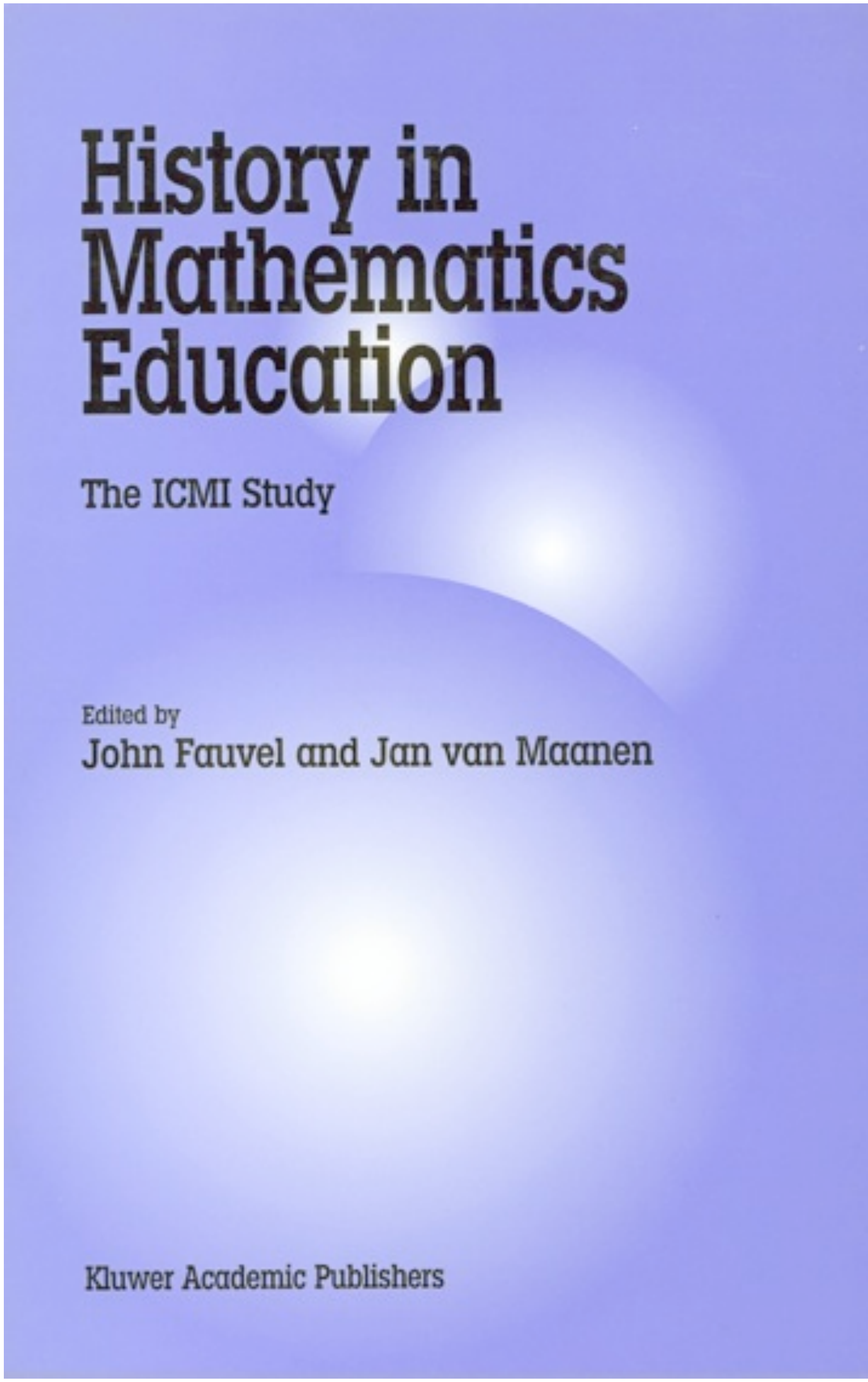


(1996)

Working Groups on history at ICME's



ICMI Study 10
(Luminy, France, 1998)

International Commission on Mathematical Instruction

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Questions at the basis of the Study

- *Does* history of mathematics have a role to play in mathematics education? ***Yes!***
- *Why* should history of math be integrated in mathematics education?
    - *objections:* lack of classroom time, resources, teacher's expertise, robust assessment tradition
    - ***claim****:* history of math as a means in the construction of math knowledge and offering a *refreshed perspective* on various math topics

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Discussion / Questions (Act 1)

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Discussion / Questions   (Act 1)

- History of math as a means for "Changing the Culture"
- Place / role of history of math in math education
- Impact on the preparation of teachers (pre- / in-service)
- Ministerial expectations: situation in Québec
  vs BC / Canada
- …

IV- Examples of topics in the history of mathematics suitable for prospective teachers

Choosing "good historical topics" for teachers: a non-trivial task…

Some examples from my teaching to pre-service secondary school teachers

I invite my students to stay alert to a triple perspective:

- history of mathematics
- didactics / pedagogy
- epistemology *(part of "math culture")*

The latter is especially crucial for "Changing the Culture"

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My examples belong to the “classics” of the math literature

Difficulties in the use of original sources

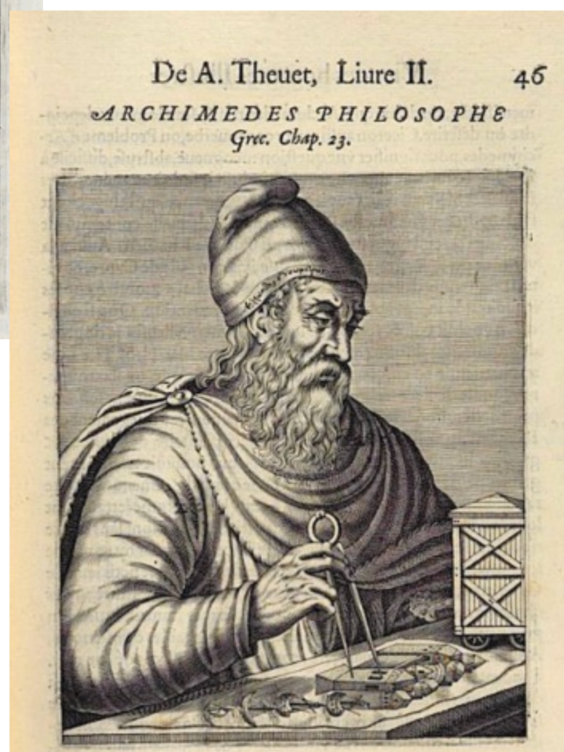
It is far from trivial to find material accessible to teachers / students (vg linguistic barriers, style of writing)

Reading “old” texts can be time-consuming

Need to find material useful in connection to the program / curriculum under consideration

But this can be most rewarding

a refreshed perspective on given mathematical themes



Use of original sources

IV-a Some steps in the company of Euclid

Portrait by Justus of Ghent
(Joos van Wassenhove)
c. 1474

IV-a Some steps in the company of Euclid

Preface

Paul Erdős liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, following the dictum of G. H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book. A few years ago, we suggested to him to write up a first (and very modest) approximation to The Book. He was enthusiastic about the idea and, characteristically, went to work immediately, filling page after page with his suggestions. Our book was supposed to appear in March 1998 as a present to Erdős' 85th birthday. With Paul's unfortunate death in the summer of 1996, he is not listed as a co-author. Instead this book is dedicated to his memory.

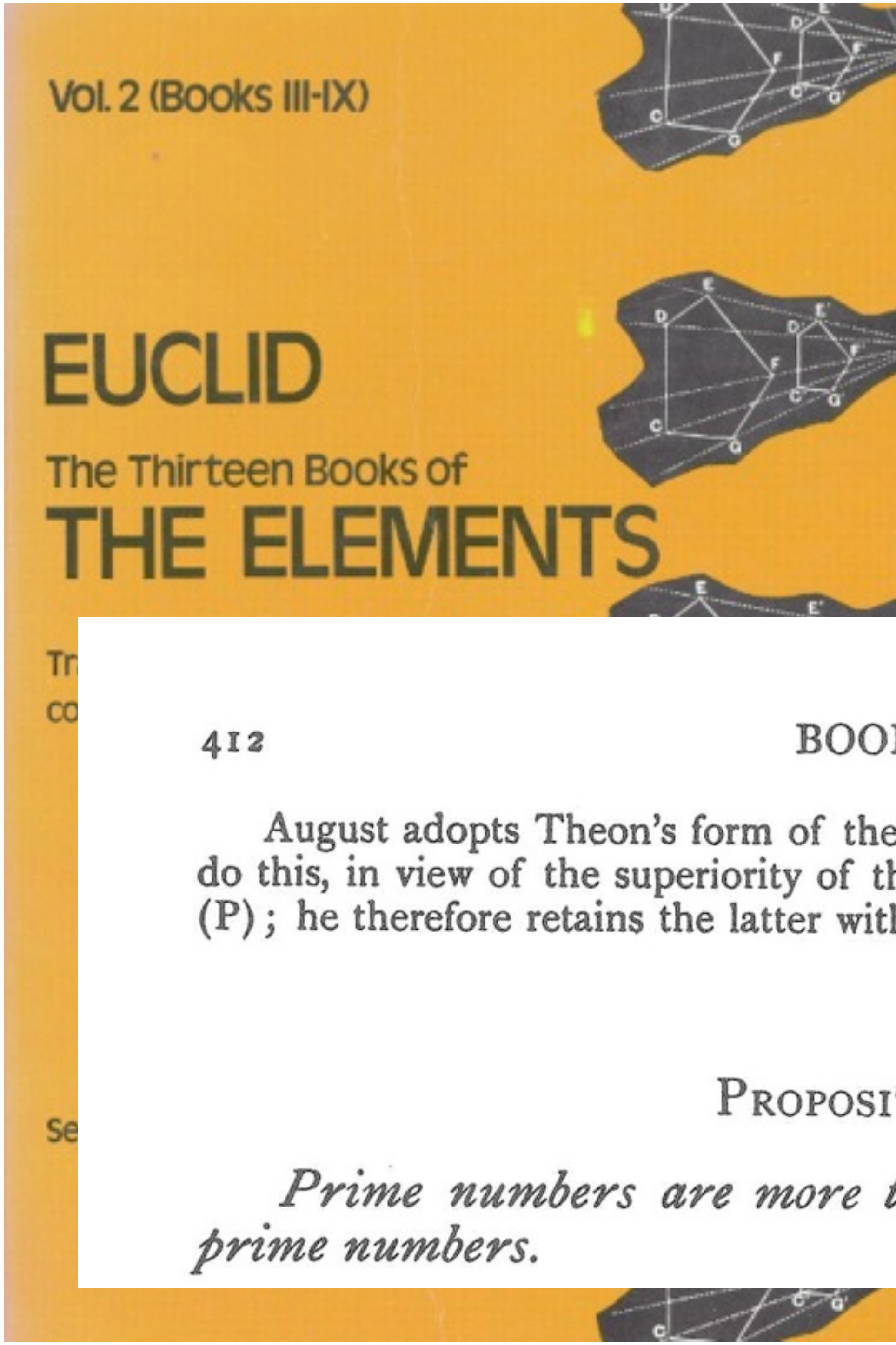

Paul Erdős

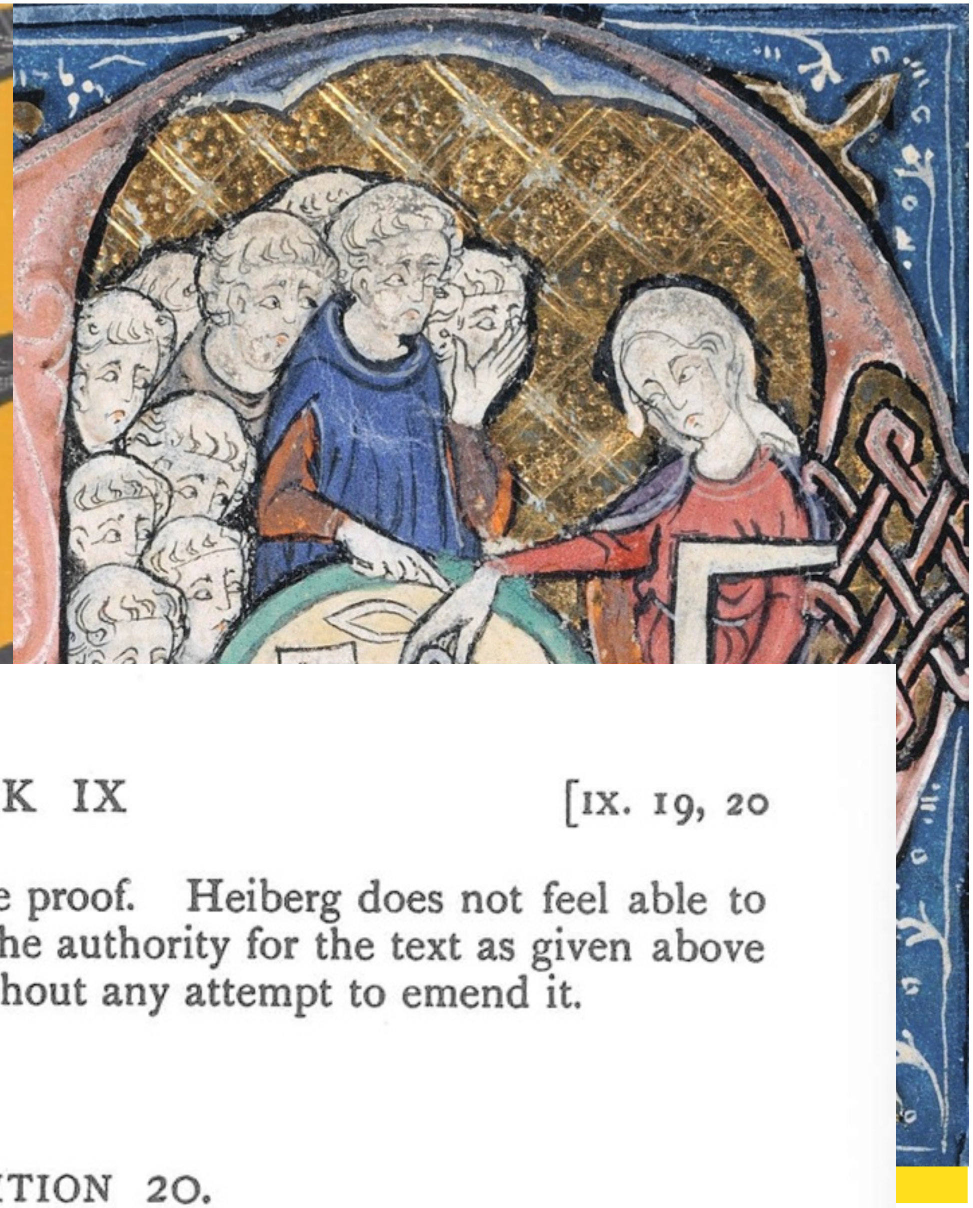
Six proofs of the infinity of primes

Chapter 1

It is only natural that we start these notes with probably the oldest Book Proof, usually attributed to Euclid (*Elements* IX, 20). It shows that the sequence of primes does not end.

■ **Euclid's Proof.** For any finite set $\{p_1, \ldots, p_r\}$ of primes, consider the number $n = p_1 p_2 \cdots p_r + 1$. This n has a prime divisor p. But p is not one of the p_i: otherwise p would be a divisor of n and of the product $p_1 p_2 \cdots p_r$, and thus also of the difference $n - p_1 p_2 \cdots p_r = 1$, which is impossible. So a finite set $\{p_1, \ldots, p_r\}$ cannot be the collection of *all* prime numbers. □





August adopts Theon's form of the proof. Heiberg does not feel able to do this, in view of the superiority of the authority for the text as given above (P); he therefore retains the latter without any attempt to emend it.

PROPOSITION 20.

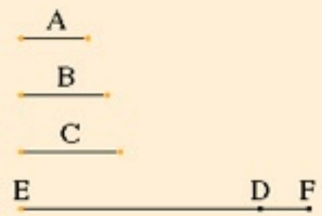
Prime numbers are more than any assigned multitude of prime numbers.

Euclid's Elements
Book IX
Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

Let *A*, *B*, and *C* be the assigned prime numbers.

I say that there are more prime numbers than *A*, *B*, and *C*.



Take the least number *DE* measured by *A*, *B*, and *C*. Add the unit *DF* to *DE*.

Then *EF* is either prime or not.

First, let it be prime. Then the prime numbers *A*, *B*, *C*, and *EF* have been found which are more than *A*, *B*, and *C*.

Next, let *EF* not be prime. Therefore it is measured by some prime number. Let it be measured by the prime number *G*. VII.31

I say that *G* is not the same with any of the numbers *A*, *B*, and *C*.

If possible, let it be so.

Now *A*, *B*, and *C* measure *DE*, therefore *G* also measures *DE*. But it also measures *EF*. Therefore *G*, being a number, measures the remainder, the unit *DF*, which is absurd.

Therefore *G* is not the same with any one of the numbers *A*, *B*, and *C*. And by hypothesis it is prime. Therefore the prime numbers *A*, *B*, *C*, and *G* have been found which are more than the assigned multitude of *A*, *B*, and *C*.

Therefore, *prime numbers are more than any assigned multitude of prime numbers*.

http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

Q.E.D.

Six proofs of the infinity of primes

Chapter 1

It is only natural that we start these notes with probably the oldest Book Proof, usually attributed to Euclid (*Elements* IX, 20). It shows that the sequence of primes does not end.

■ **Euclid's Proof.** For any finite set $\{p_1, \ldots, p_r\}$ of primes, consider the number $n = p_1 p_2 \cdots p_r + 1$. This n has a prime divisor p. But p is

■ **Second Proof.** Let us first look at the *Fermat numbers* $F_n = 2^{2^n}+1$ for $n = 0, 1, 2, \ldots$. We will show that any two Fermat numbers are relatively prime; hence there must be infinitely many primes. To this end, we verify the recursion

$$\prod_{k=0}^{n-1} F_k = F_n - 2 \qquad (n \geq 1),$$

from which our assertion follows immediately. Indeed, if m is a divisor of, say, F_k and F_n $(k < n)$, then m divides 2, and hence $m = 1$ or 2. But $m = 2$ is impossible since all Fermat numbers are odd.

To prove the recursion we use induction on n. For $n = 1$ we have $F_0 = 3$ and $F_1 - 2 = 3$. With induction we now conclude

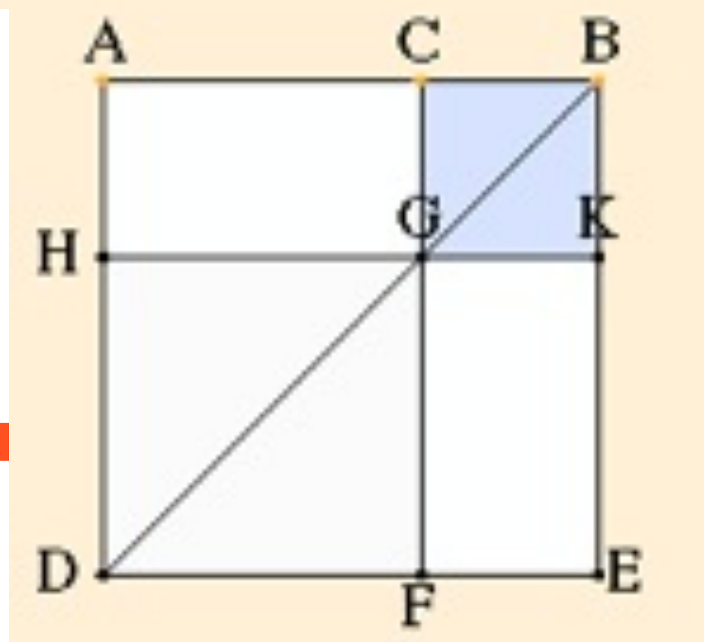
$$\prod_{k=0}^{n} F_k = \Big(\prod_{k=0}^{n-1} F_k\Big) F_n = (F_n - 2)F_n =$$
$$= (2^{2^n}-1)(2^{2^n}+1) = 2^{2^{n+1}} - 1 = F_{n+1} - 2. \qquad \square$$

$$\begin{aligned} F_0 &= 3 \\ F_1 &= 5 \\ F_2 &= 17 \\ F_3 &= 257 \\ F_4 &= 65537 \\ F_5 &= 641 \cdot 6700417 \end{aligned}$$

The first few Fermat numbers

Euclid's Elements
Book II
Proposition 4

If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.



Euclid's *Book II*:

"Geometric Algebra"

(H. Zeuthen)

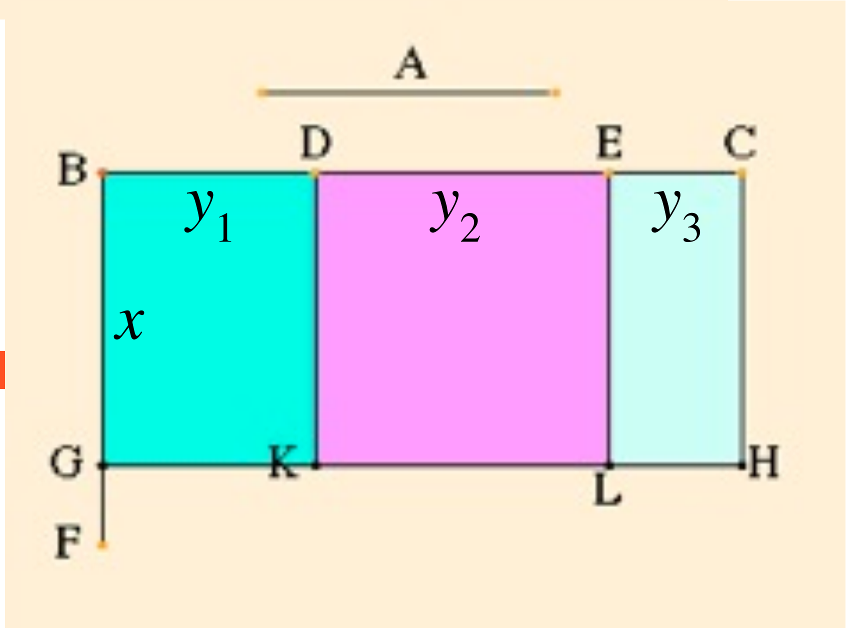
Possible challenge of reading the text

Euclid's Elements
Book II
Proposition 1

If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

$$x(y_1 + y_2 + y_3)$$

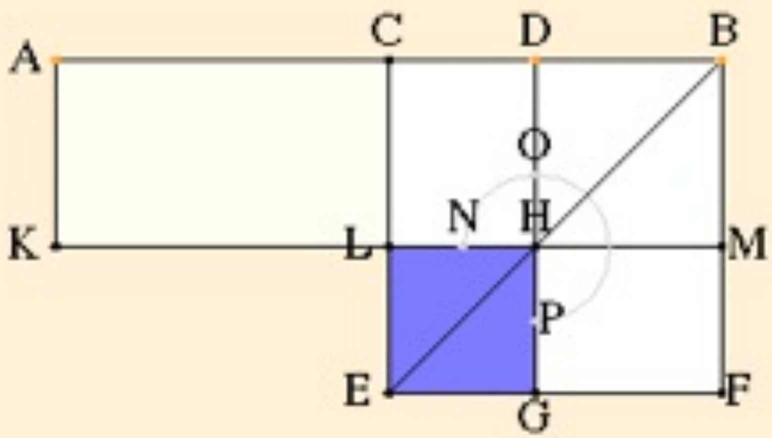
$$= xy_1 + xy_2 + xy_3$$



distributivity x / +

Book II
Proposition 5

If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.



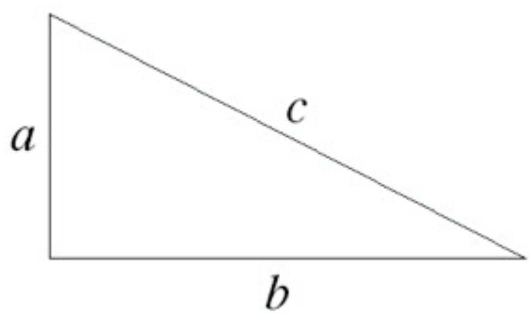
Let $u = AD$
$v = DB = AK$

$$uv + \left(\frac{u-v}{2}\right)^2 = \left(\frac{u+v}{2}\right)^2$$

basis for solution of quadratic equations by Mesopotamians

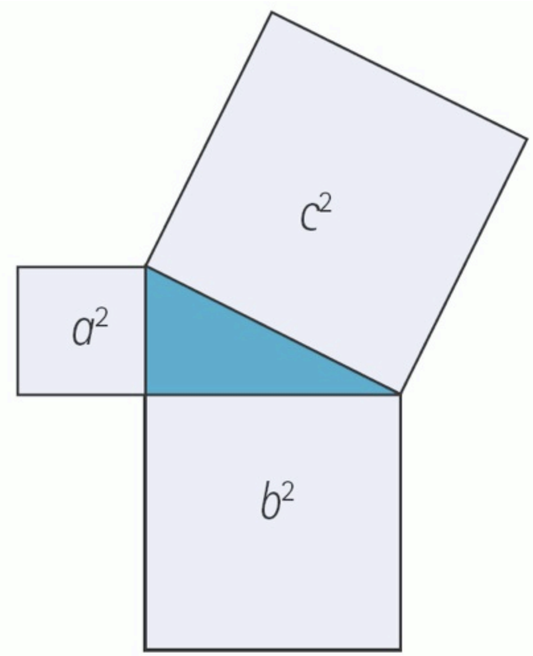
Euclid's Elements
Book I
Proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.



$$a^2 + b^2 = c^2$$

*squares **OF** the sides (**algebra**)*

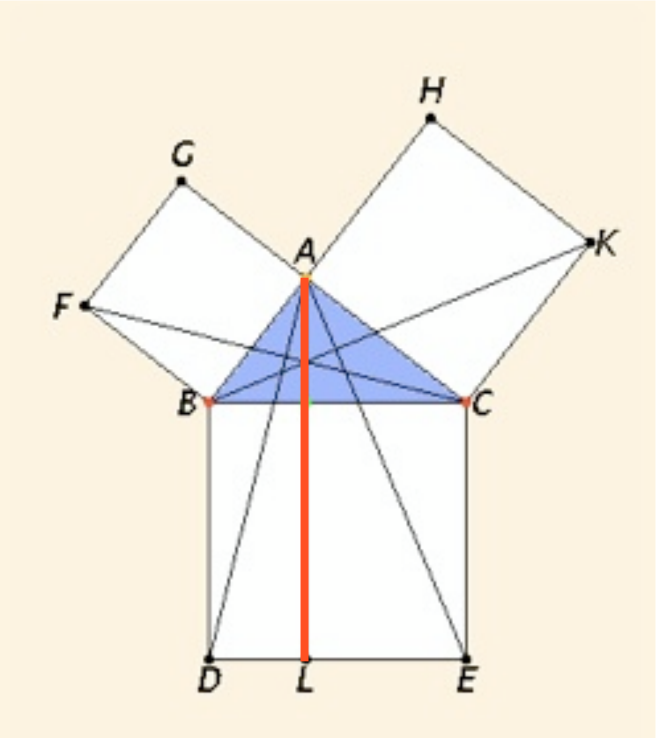


*squares **ON** the sides (**geometry**)*

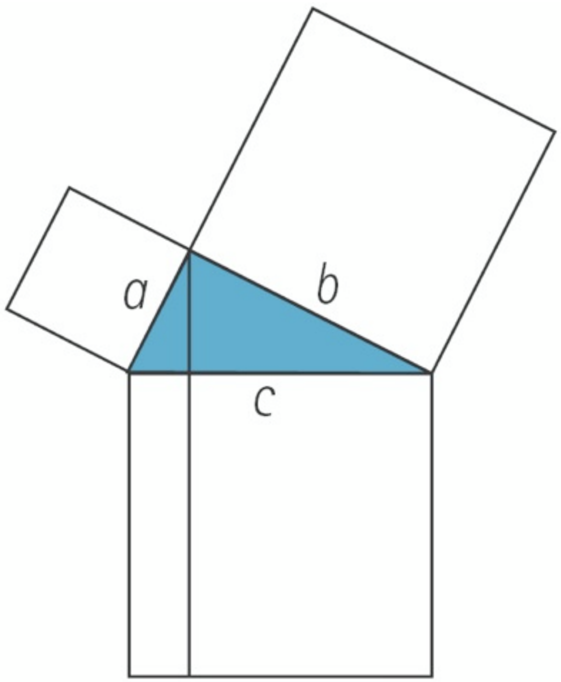
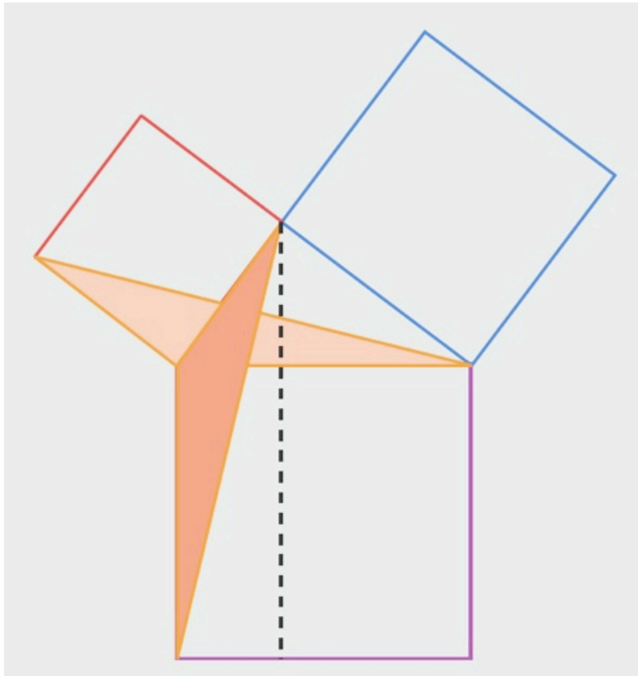
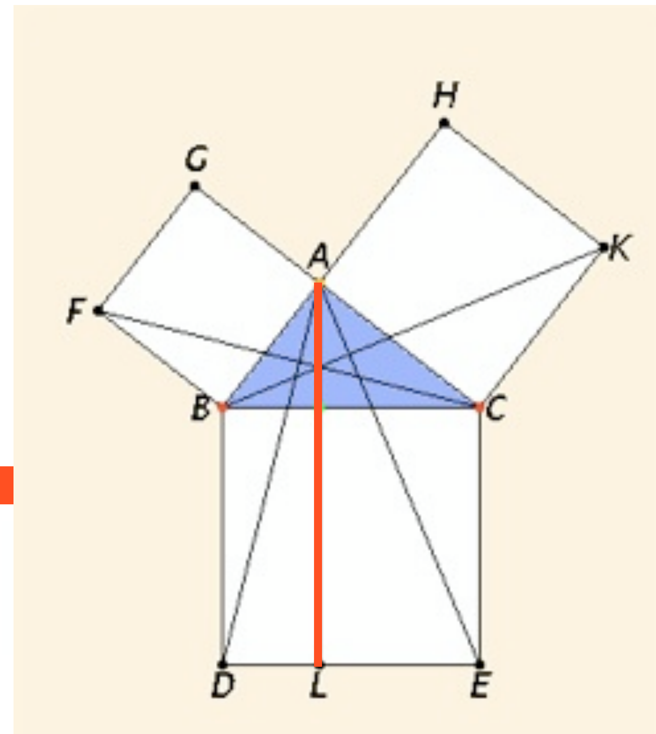
Euclid's Elements
Book I
Proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Euclid's text is really accessible… and the proof, clear and beautiful!!!



Proving the Pythagorean Theorem


a
b
c


H
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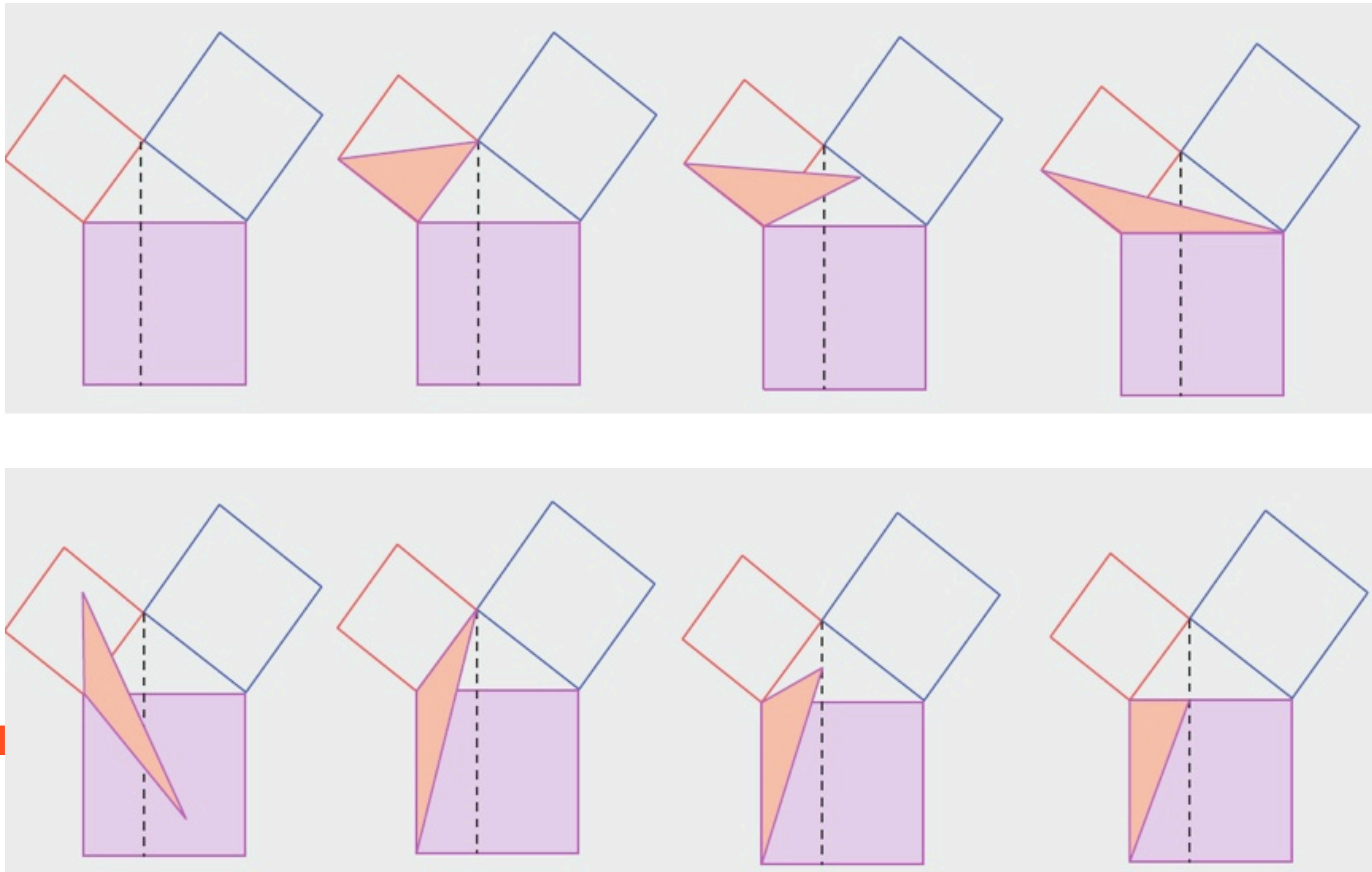
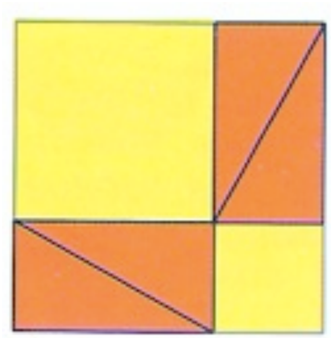
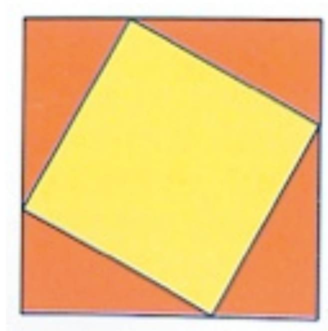
Animated proof – *Transforming a^2 and b^2 into c^2*





Proof by algebra

$$(a+b)^2 = a^2 + b^2 + 2ab$$



$$= c^2 + 4 \cdot \frac{ab}{2}$$

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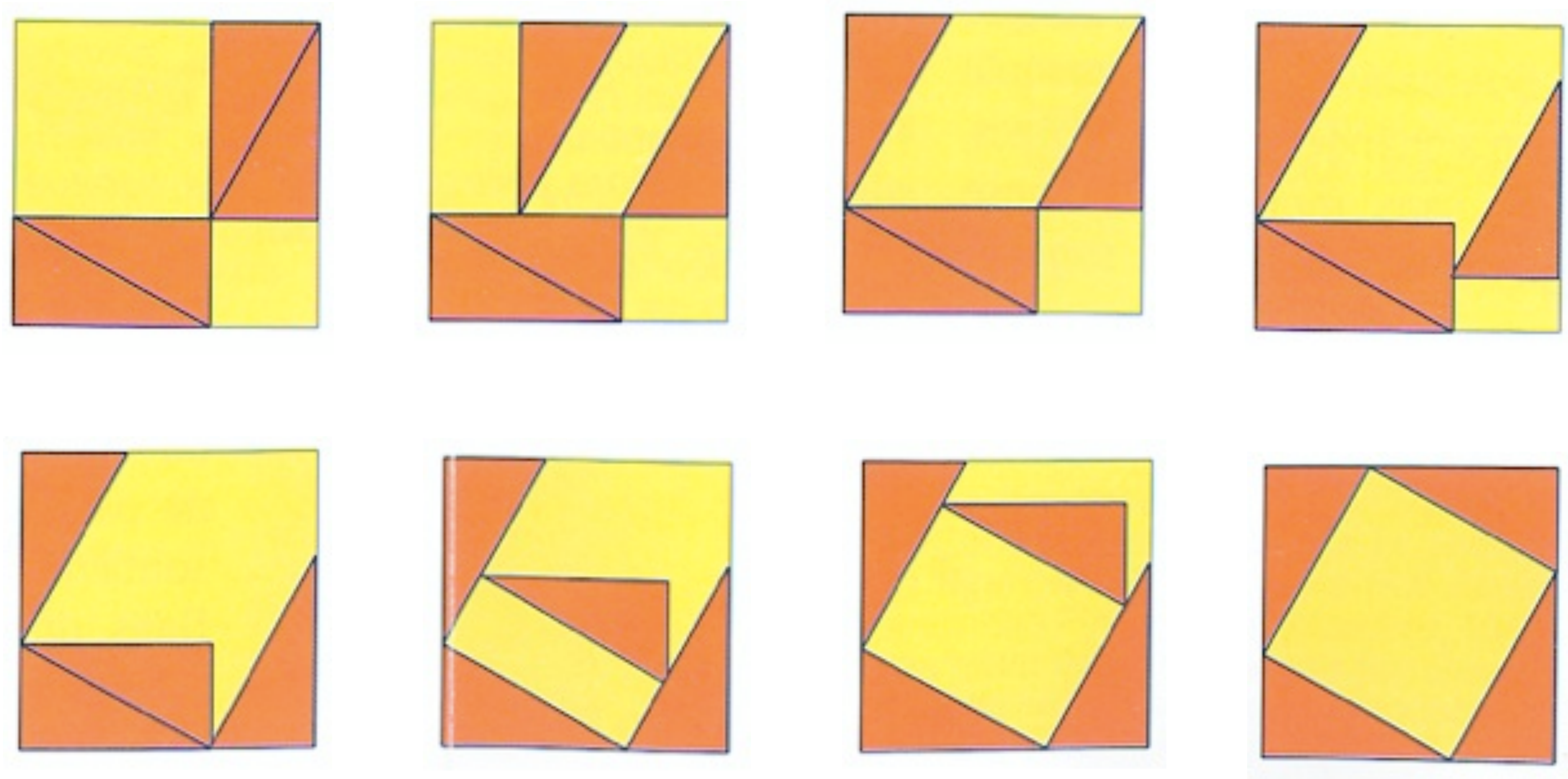

One can "see" the squares on sides a and b "becoming" the square on the hypotenuse c

Visual proof

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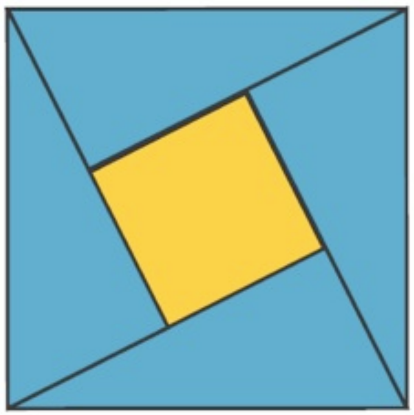
From Bhaskara…

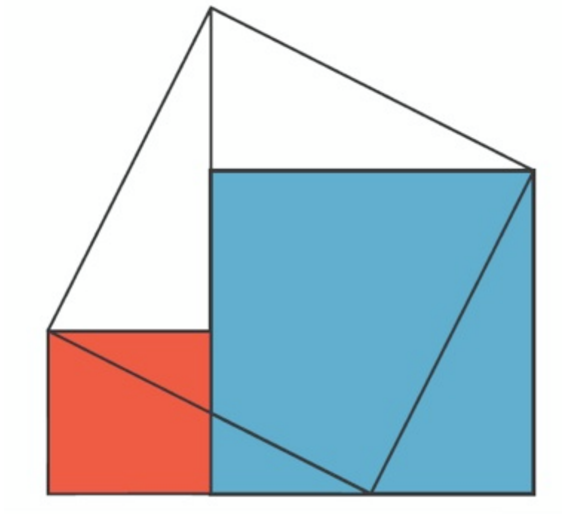
(12th century)



Behold!

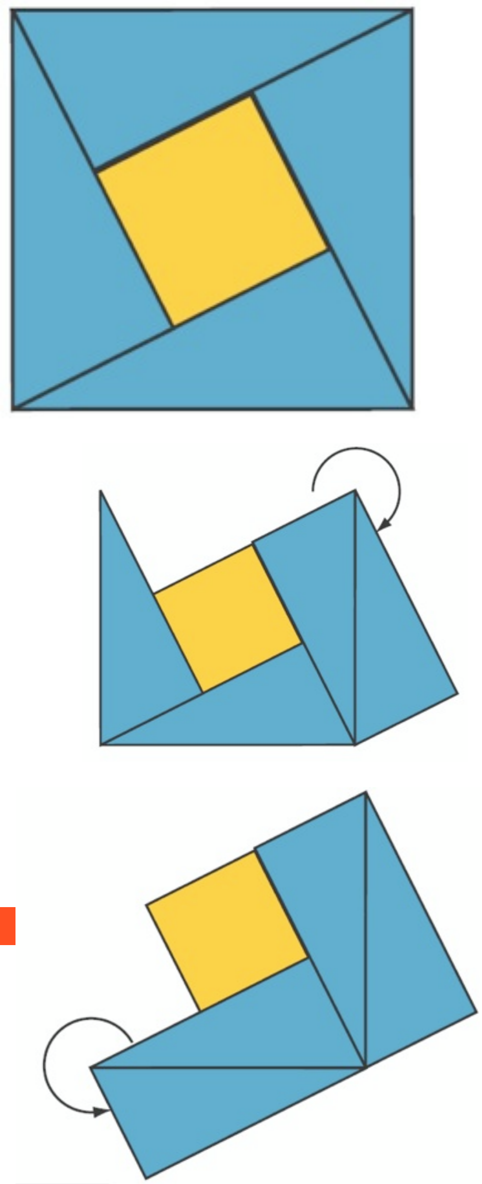
… to Ibn Qurra

(9th century)



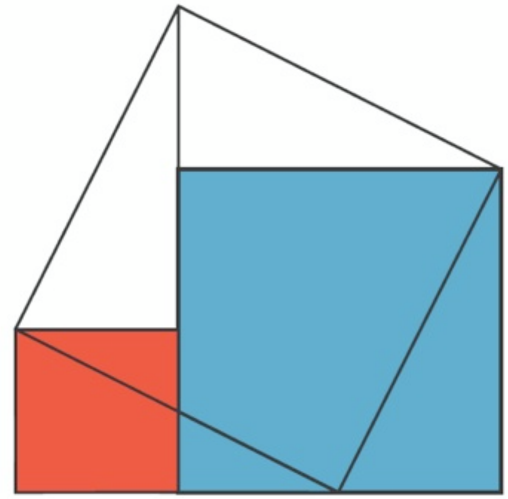
From Bhaskara

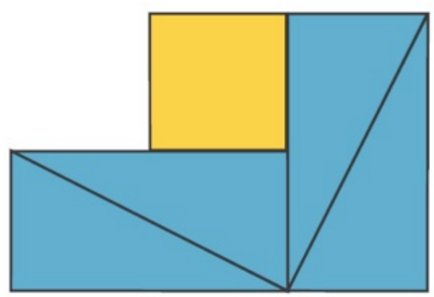
(12th century)

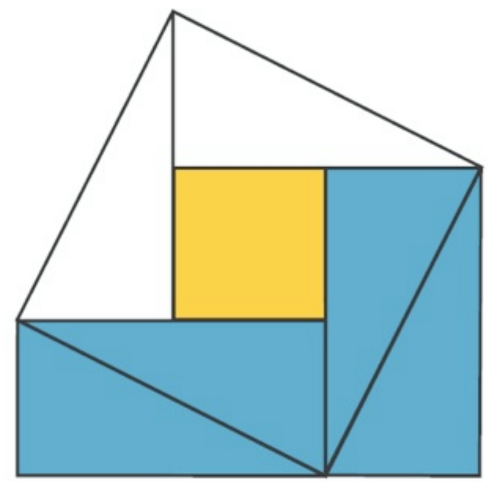


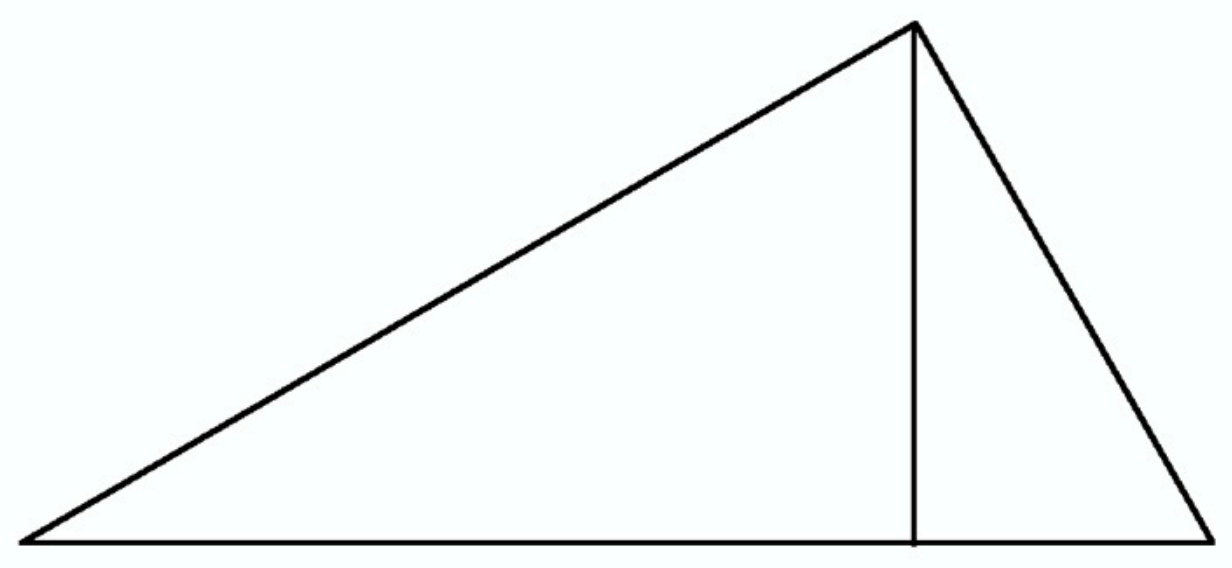
… to Ibn Qurra

(9th century)

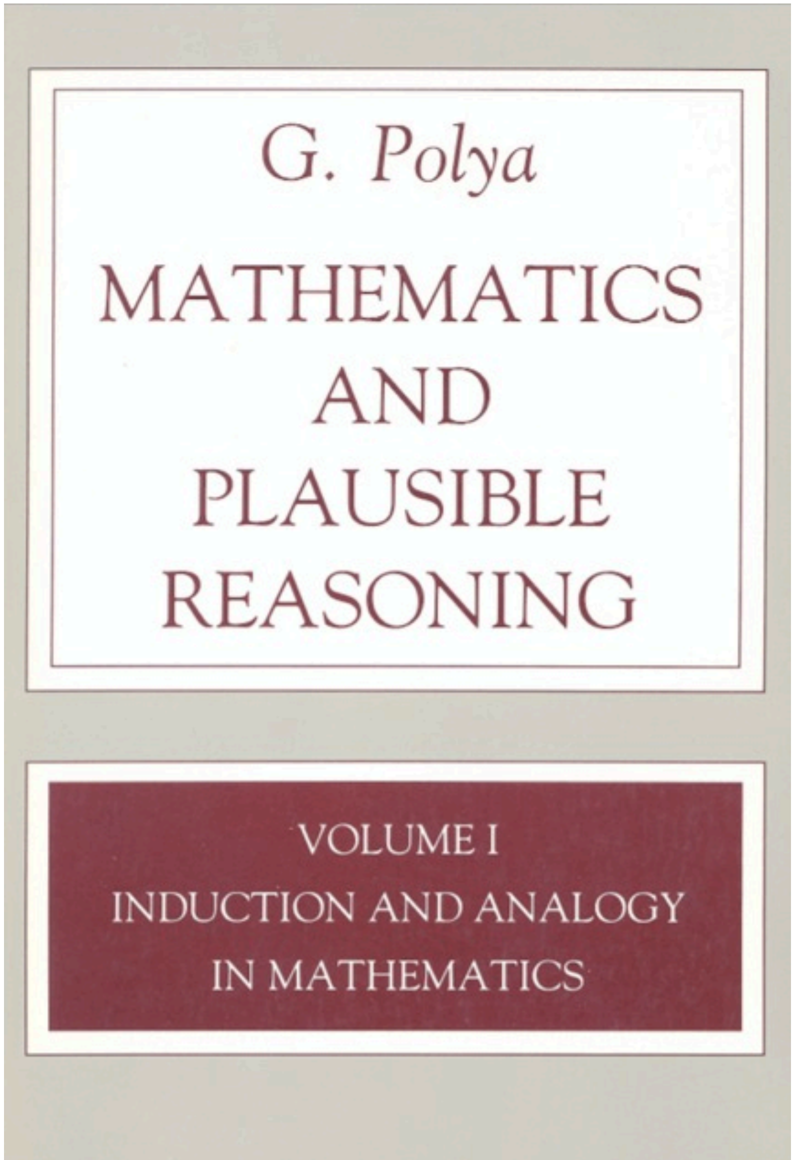


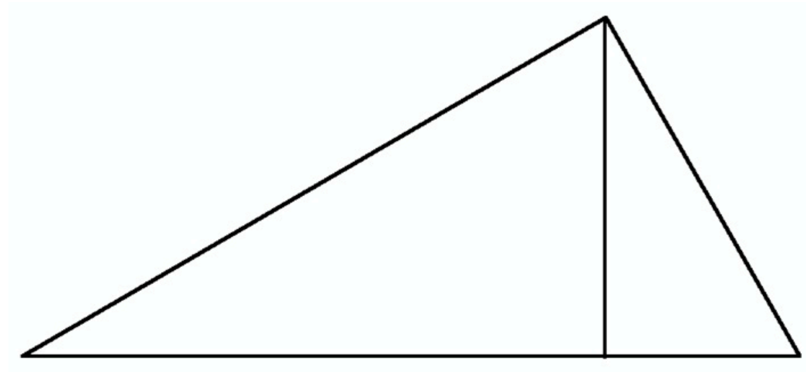




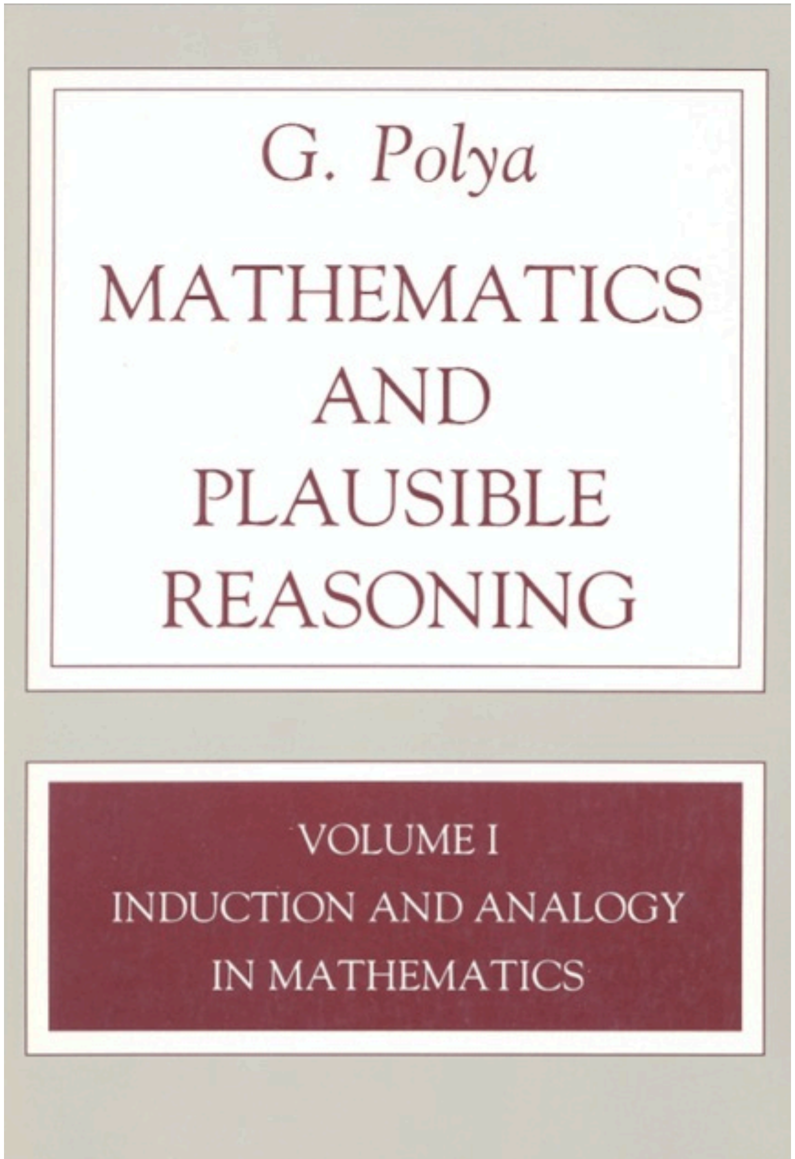


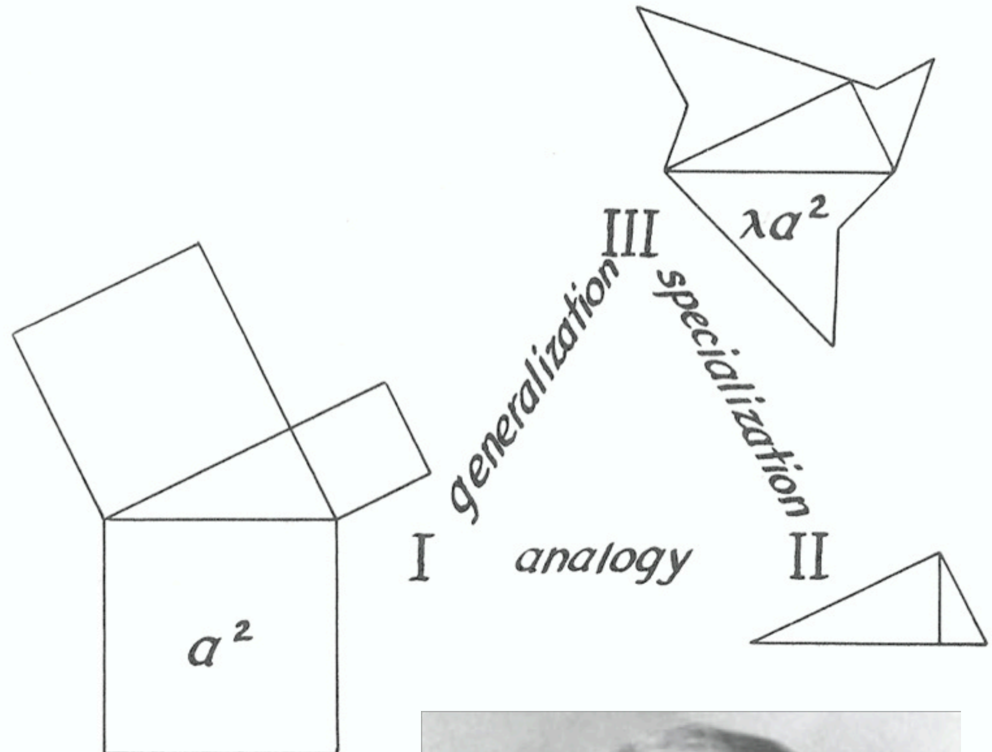
- **Fibonacci,** ***De practica geometrie*** **(1223)**
- **H.A. Naber – Dutch mathematics teacher (1908)**



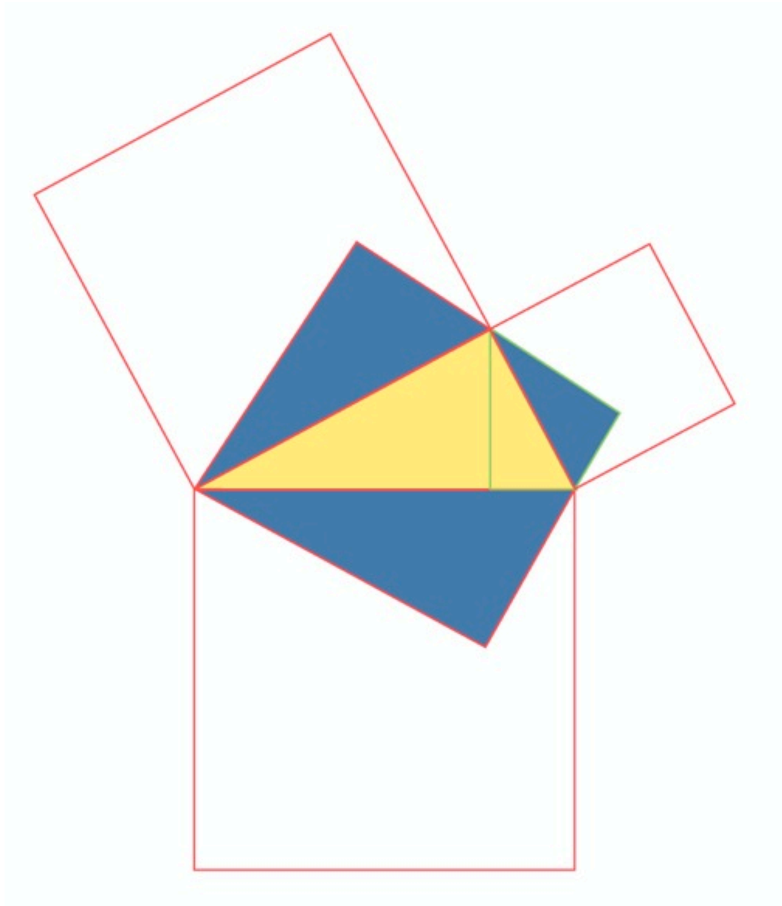


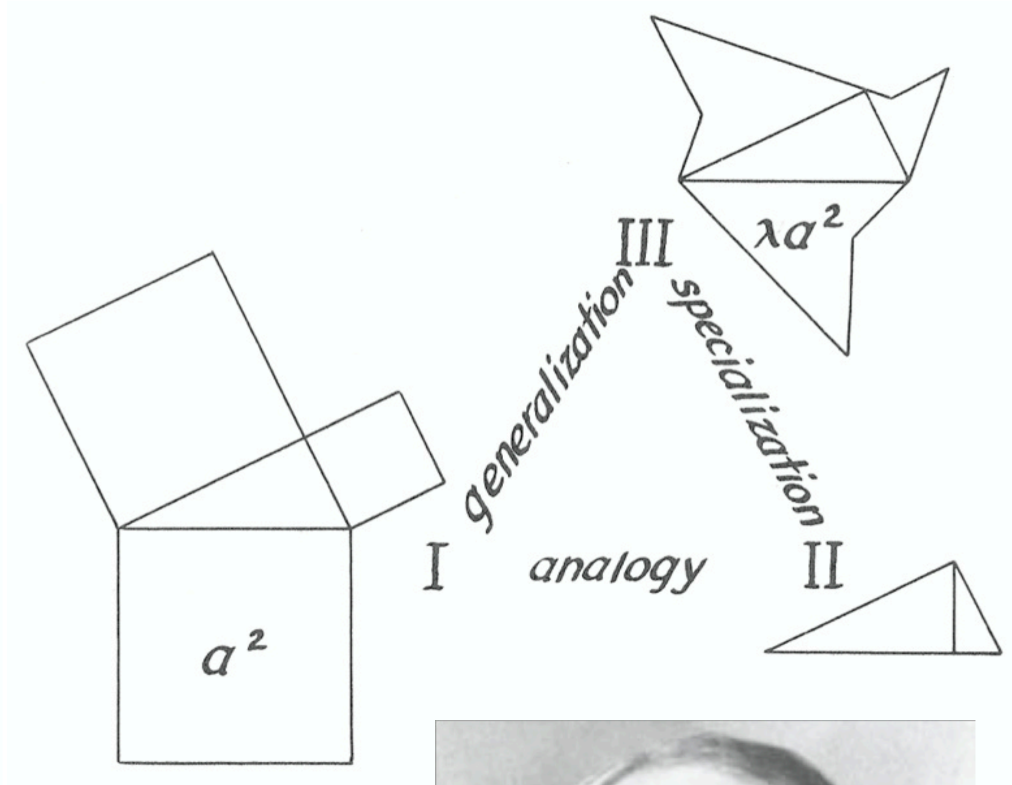
George Pólya
(1887 – 1985)





George Pólya
(1887 – 1985)

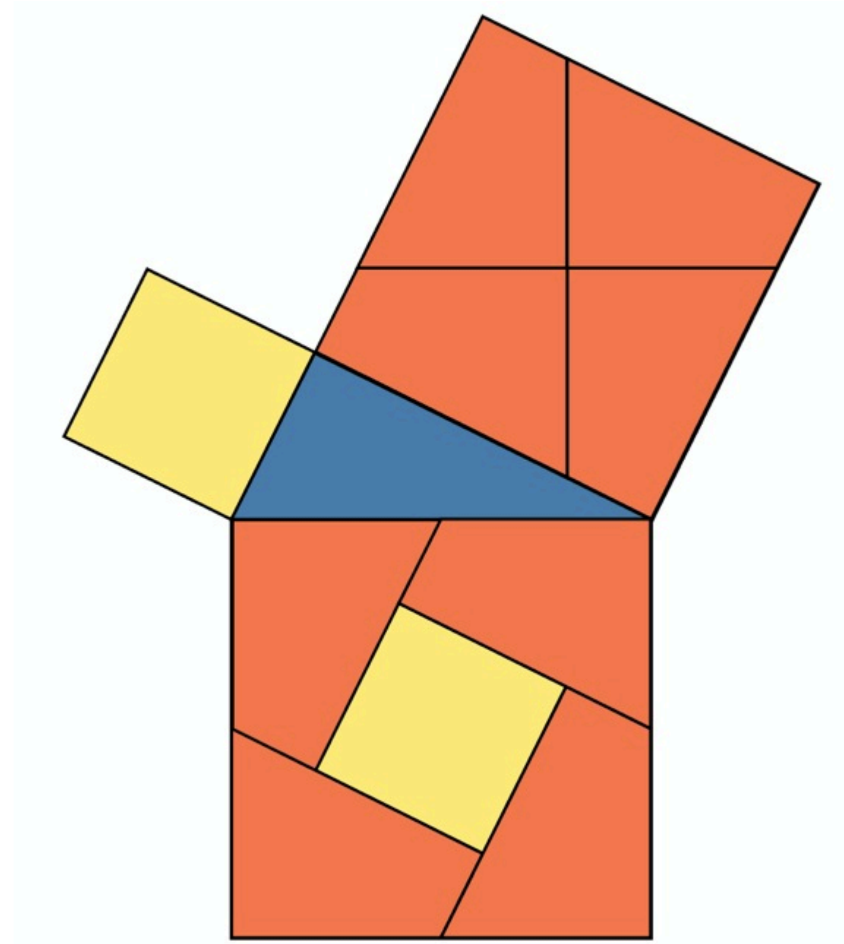




George Pólya
(1887 – 1985)

A particularly neat visual proof

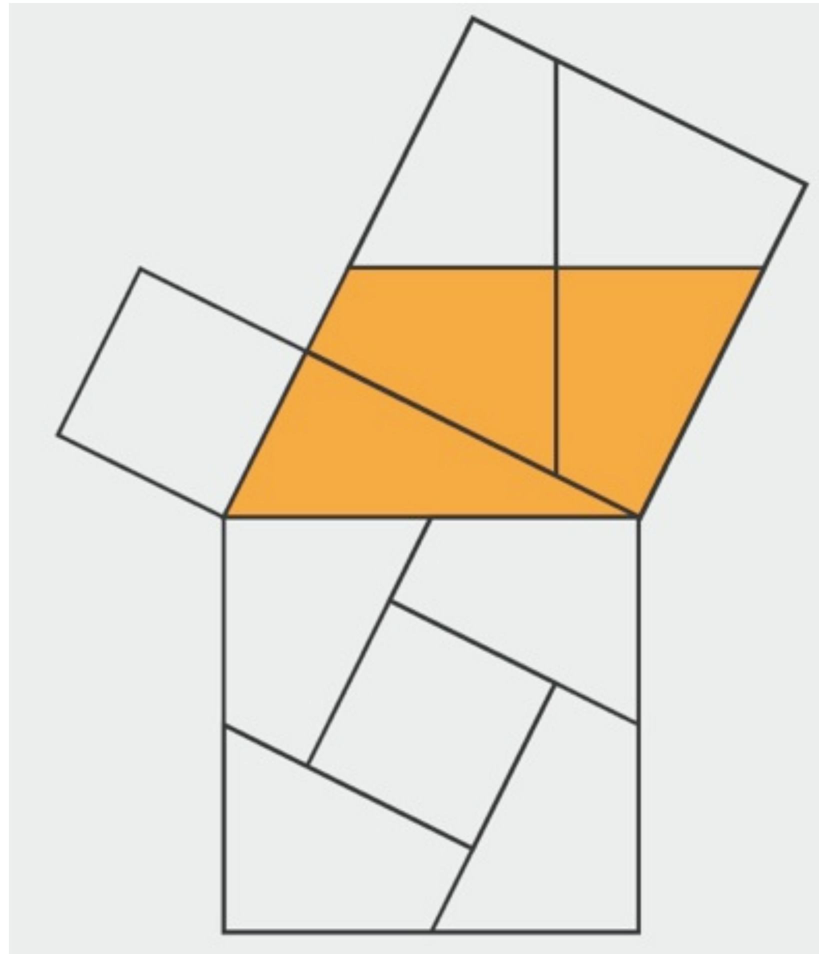
Henry Perigal
(1801-1898)



This shows directly the two squares on the sides becoming the square on the hypotenuse

A particularly neat visual proof

Henry Perigal
(1801-1898)



This shows directly the two squares on the sides becoming the square on the hypotenuse

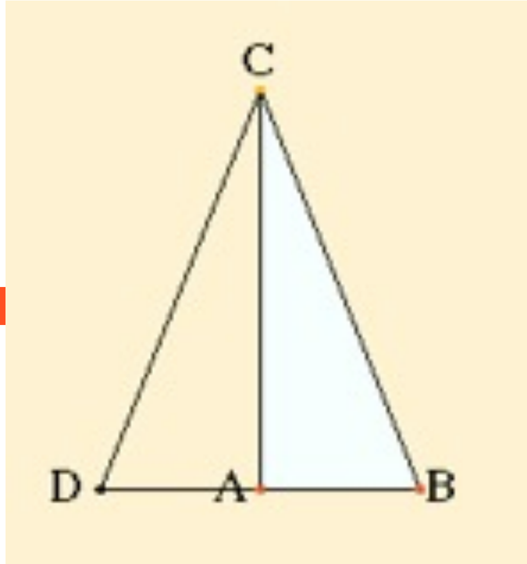
Euclid's Elements
Book I
Proposition 48

If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

But the square on *DC* equals the sum of the squares on *DA* and *AC*, for the angle *DAC* is right, and the square on *BC* equals the sum of the squares on *BA* and *AC*, for this is the hypothesis, therefore the square on *DC* equals the square on *BC*, so that the side *DC* also equals *BC*.

I.47

C.N.1



A 3-4-5 triangle is right-angled!

Proving the RECIPROCAL of the Pythagorean Theorem (I.47)

IV-a Some steps in the company of Euclid

Euclid
Max Ernst
1945

Another context linking geometry and algebra

IV-b Some algebra in the company of al-Khwarizmi

al-Khwarizmi
(c. 790 – c. 850)

Kitab al-jabr wa'l-muqabala

Roots and square equal a number. For instance, a square and ten roots are equal to thirty-nine dirhems. That is to say: what must be the square which combined with ten of its roots will give a sum total of thirty-nine?

$$x^2 + 10x = 39$$

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The manner of solving this type of equation is to take one-half of the number of roots just mentioned, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine, giving sixty-four. Now take the root of this, which is eight, and subtract from it half the number of roots, which is five. The remainder is three, which is the root of the square. Nine therefore gives the square.

$$x^2 + bx = c$$

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}$$

$$x^2 + 10x = 39$$

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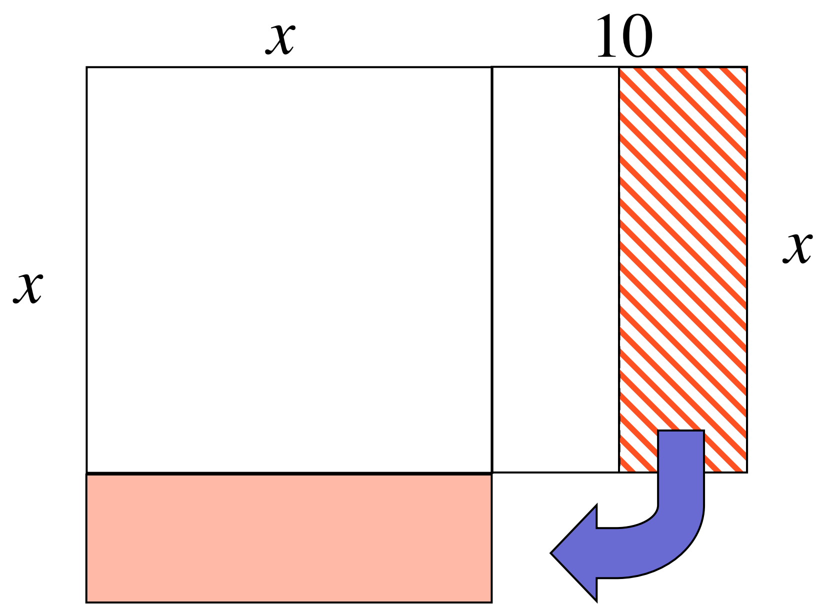
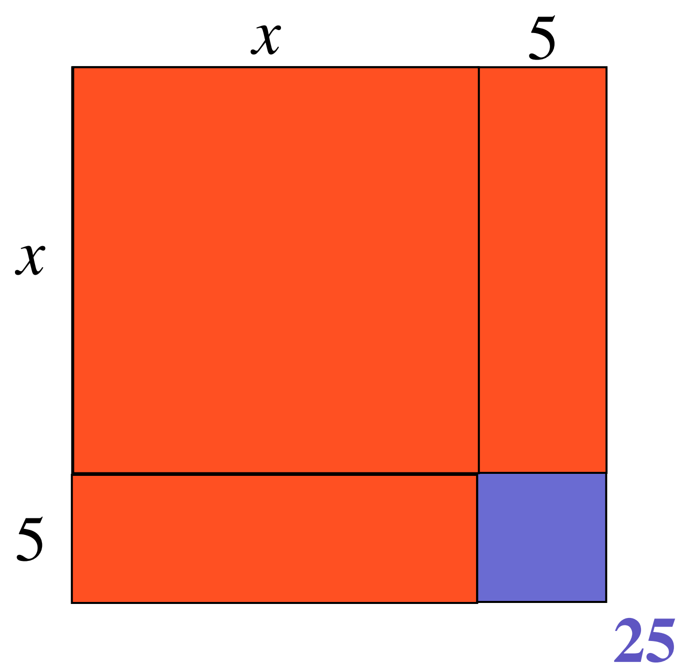
Geometric justification! (al-Khwarizmi)



$$x^2 + 10x = 39$$

Geometric justification! (al-Khwarizmi)



39 + 25

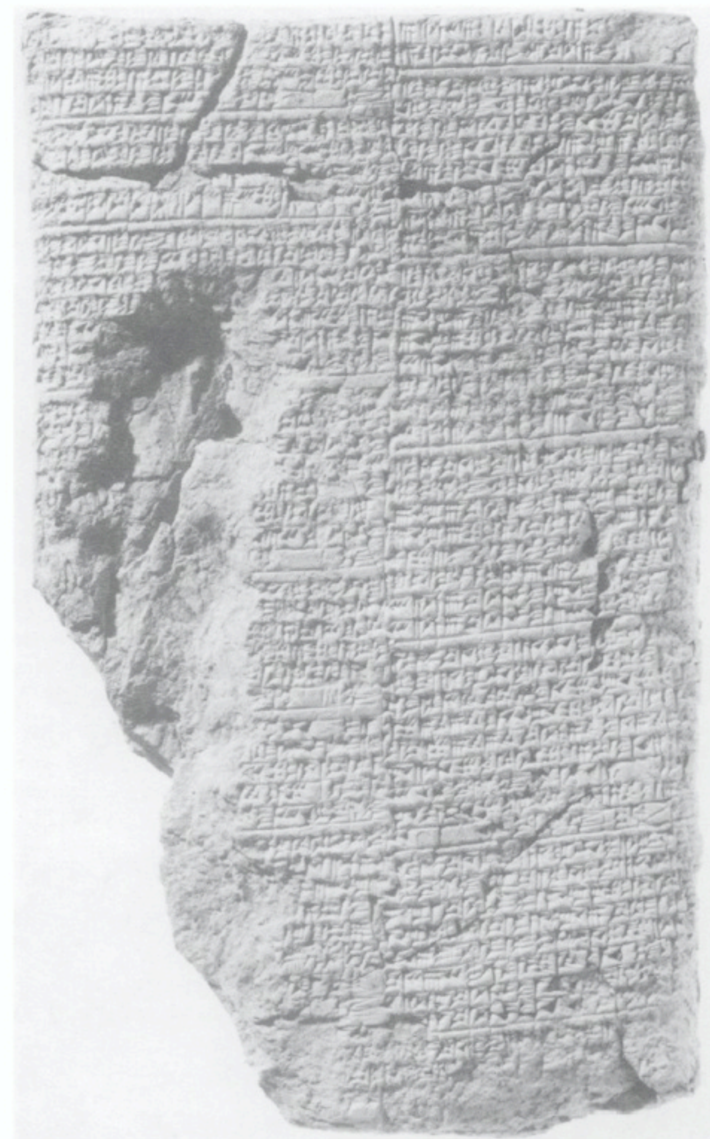
$$x + 5 = 8$$

$$x^2 + 10x = 39$$

Note 1: Similar techniques used in Mesopotamian ‘geometrical algebra’

Old Babylonian Tablet BM 13901 (c. -1800)

A compendium of 24 model solutions of problems in quadratic geometrical algebra

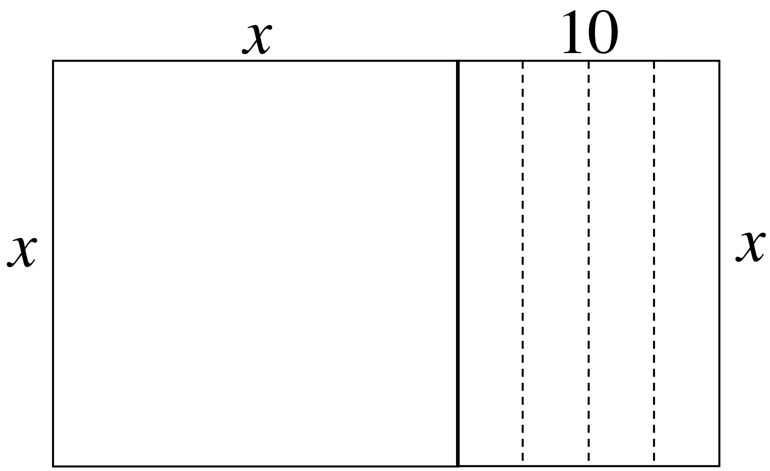


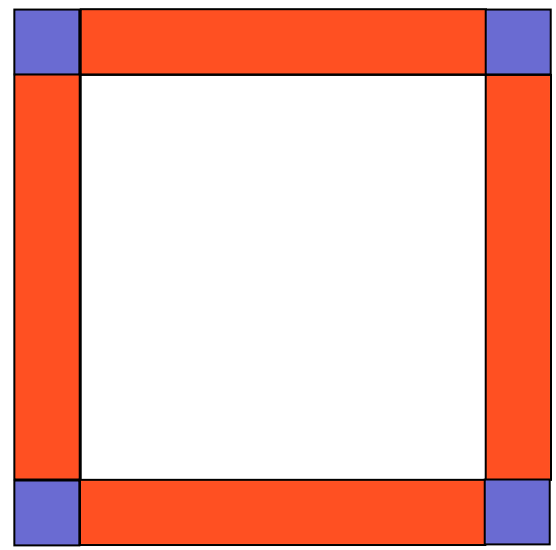
Problem #1

The sum of the area and the side of a square gives 3/4.

$$s^2 + s = \frac{3}{4}$$

Note 2: Another geometric dissection proposed by al-Khwarizmi





$$x^2 + 10x = 39$$

IV-c Heron's method for square roots

Heron of Alexandria
(c. 10 – c. 75)

Method of approximating the square root of a non-square number

Area of triangle a,b,c

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$

Triangle 7,8,9

$$\sqrt{12 \cdot 5 \cdot 4 \cdot 3} = \sqrt{720}$$

(From a 1688 German translation of Heron's *Pneumatics*)

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'Since', says Heron,[1] '720 has not its side rational, we can obtain its side within a very small difference as follows. Since the next succeeding square number is 729, which has 27 for its side, divide 720 by 27. This gives $26\frac{2}{3}$. Add 27 to this, making $53\frac{2}{3}$, and take half of this or $26\frac{1}{2}\frac{1}{3}$. The side of 720 will therefore be very nearly $26\frac{1}{2}\frac{1}{3}$. In fact, if we multiply $26\frac{1}{2}\frac{1}{3}$ by itself, the product is $720\frac{1}{36}$, so that the difference (in the square) is $\frac{1}{36}$.

'If we desire to make the difference still smaller than $\frac{1}{36}$, we shall take $720\frac{1}{36}$ instead of 729 [or rather we should take $26\frac{1}{2}\frac{1}{3}$ instead of 27], and by proceeding in the same way we shall find that the resulting difference is much less than $\frac{1}{36}$.'

In other words, if we have a non-square number A, and a^2 is the nearest square number to it, so that $A = a^2 \pm b$, then we have, as the first approximation to $\sqrt{A}$,

$$\alpha_1 = \tfrac{1}{2}\left(a + \frac{A}{a}\right); \qquad (1)$$

for a second approximation we take

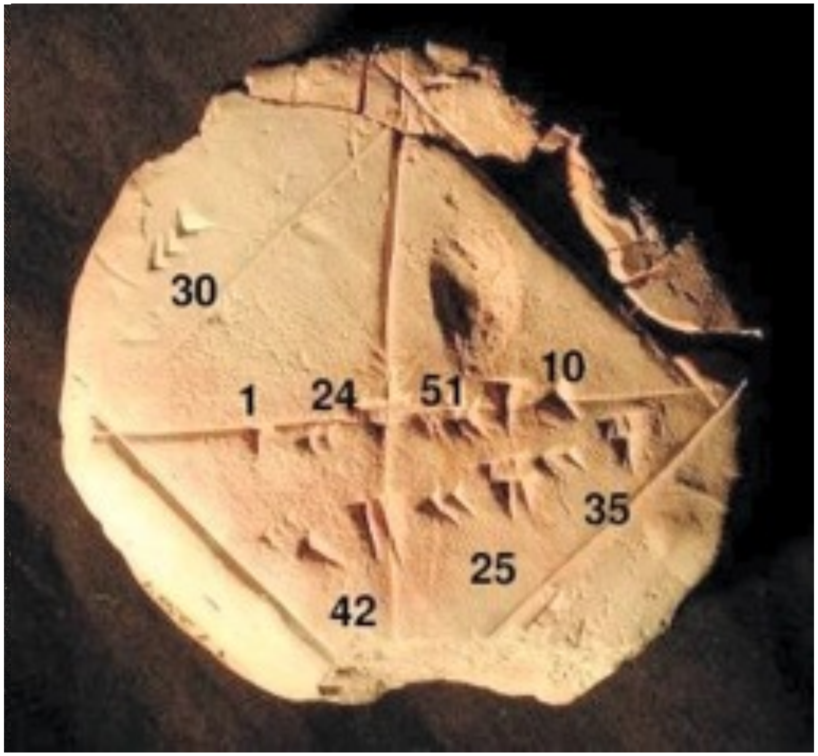
$$\alpha_2 = \tfrac{1}{2}\left(\alpha_1 + \frac{A}{\alpha_1}\right), \qquad (2)$$

and so on.

[1] *Metrica*, i. 8, pp. 18. 22–20. 5.

Heath,
History of Greek Mathematics, vol. II

Square roots in Mesopotamia



$$\sqrt{2} \approx 1;24,51,10$$

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

YBC 7289

(c. – 1800)

- *30*
- *1 ; 24 , 51 , 10*
- *42 ; 25 , 35*

Remarkably good approximation — still used almost 2000 years later by ***Claudius Ptolemae*** (c. 100 – c.178) in computing his table of chords

???

(color photographs:
Bill Casselman's website, UBC, Vancouver)

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Heron's Method Successive approximations to $\sqrt{k}$

$$\sqrt{2} = ?$$

$$a_1 = 1$$

$$a_2 = \tfrac{1}{2}\left(1 + \tfrac{2}{1}\right) = \tfrac{3}{2} = 1\tfrac{1}{2} = 1;30$$

$$a_3 = \tfrac{1}{2}\left(\tfrac{3}{2} + \tfrac{2}{3/2}\right) = \tfrac{17}{12} = 1\tfrac{5}{12} = 1;25$$

$$a_4 = \tfrac{1}{2}\left(\tfrac{17}{12} + \tfrac{2}{17/12}\right) = \tfrac{577}{408} = 1\tfrac{169}{408} \approx 1;24,51,10$$

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1,\underline{41421}296$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{k}{a_n}\right)$$

Was this known to Mesopotamians???

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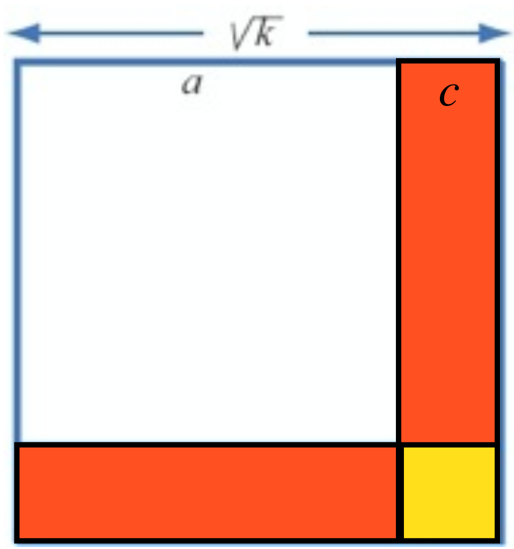
Approximating $\sqrt{k}$ using ***default*** value a

Let c be such that $a + c = \sqrt{k}$ and set $k = a^2 + b$



$$\sqrt{a^2 + b} \approx a + \frac{b}{2a}$$

formula found in Mesopotamian texts – and throughout the ages

$b =$ area of gnomon $= 2ac +$ ~~c^2~~

$$c \approx \frac{b}{2a}$$

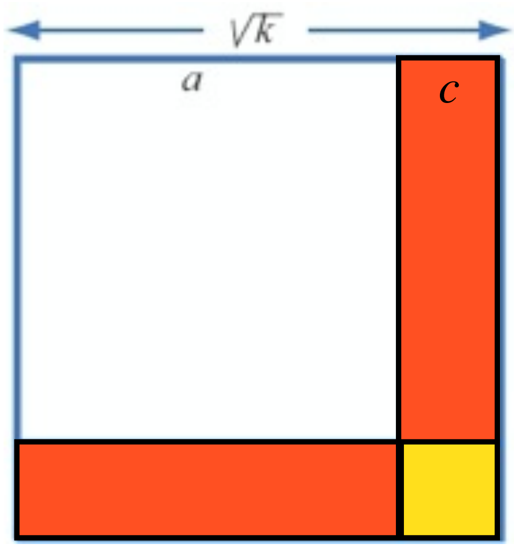
A method "for which there is some textual evidence" in Mesopotamian math

Katz, *A History of Mathematics*

Possible / plausible geometrical justification!

Approximating $\sqrt{k}$ using ***default*** value a

Let c be such that $a + c = \sqrt{k}$ and set $k = a^2 + b$



$$\sqrt{a^2 + b} \approx a + \frac{b}{2a}$$

formula found in Mesopotamian texts — and throughout the ages

b = area of gnomon $= 2ac + \cancel{c^2}$

$$c \approx \frac{b}{2a}$$

A method "for which there is some textual evidence" in Mesopotamian math

Katz, *A History of Mathematics*

Approximation using ***excess*** value a

$$\sqrt{a^2 - b} \approx a - \frac{b}{2a}$$

Approximation a' to $a + c = \sqrt{k}$

$$a' = a + \frac{b}{2a} = a + \frac{k - a^2}{2a} = \frac{a^2 + k}{2a} = \frac{1}{2}\left(a + \frac{k}{a}\right)$$

(Heron!!!)

With a drop of anachronism…

Approximation a' to $a + c = \sqrt{k}$

$$a' = a + \frac{b}{2a} = a + \frac{k - a^2}{2a} = \frac{a^2 + k}{2a} = \frac{1}{2}\left(a + \frac{k}{a}\right)$$

(Heron!!!)

Note: We do not know what is the reasoning that lead Heron to his formula. *Folklore???*

In most countries, the Heron method did not become part of the school culture – why?

Quadratic convergence *(Newton-Raphson)*

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Another method for the extraction of square roots

Nine Chapters on the Mathematical Art

Chinese "Classic" from the Han Dynasty (-206 – 220)

Liu Hui
(c. 200 – c. 280)

(portrait by Jiang Zhaohe 1904 – 1986)

"Commentaries" (263)

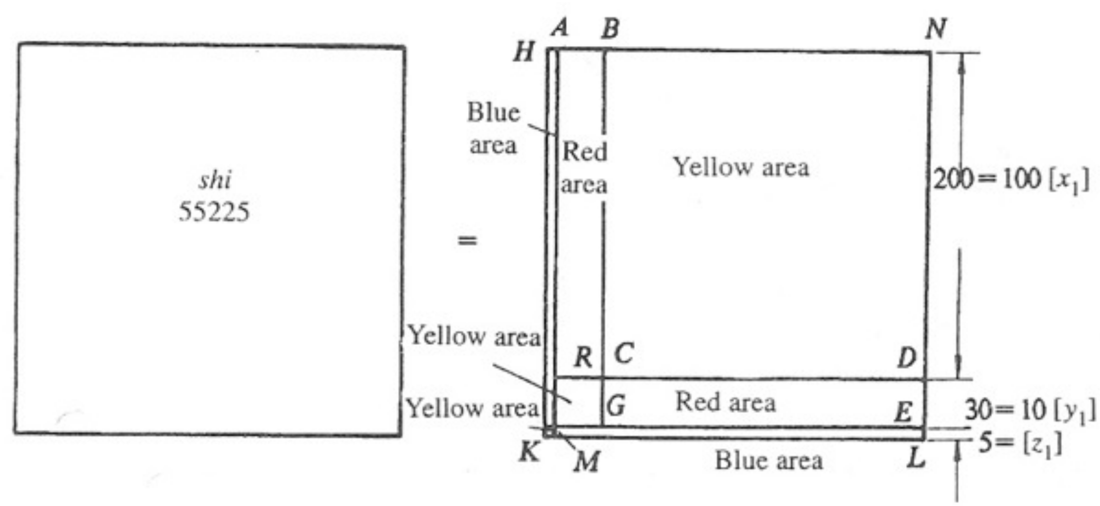
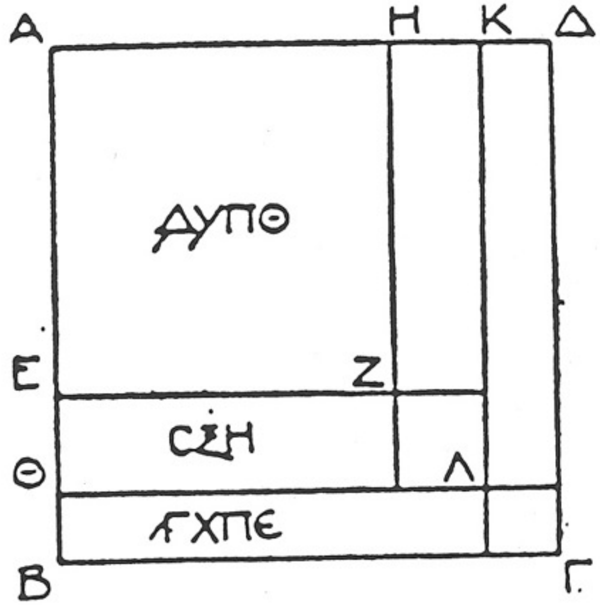


FIG. 4.7. The geometric meaning of the Rule for Extracting the Square Root.

Theon of Alexandria

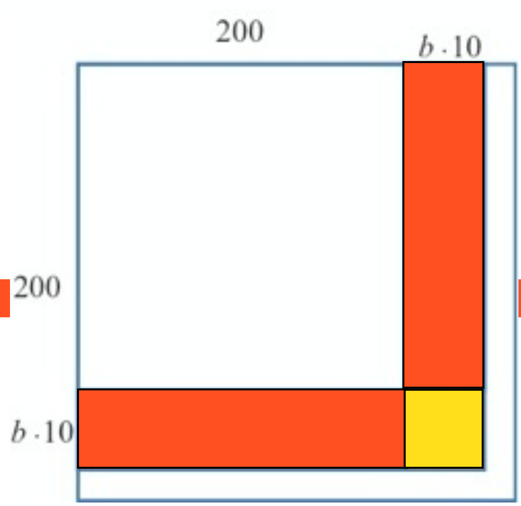
Commentaries on Ptolemae's Almagest (370)



5 ~~52~~ ~~25~~	235
4	~~2 × 2~~
1 52	~~43 × 3~~
1 29	465 × 5
23 25	
23 25	
0	

5 ~~52~~ ~~25~~	235
4 00 00	~~200 × 200~~
1 52 25	
1 29 00	~~430 × 30~~
23 25	
23 25	465 × 5
0	

$$430 \times 30 = (400 + 30) \times 30$$
$$= (2 \cdot 200 + 30) \times 30$$
$$= 2(200 \times 30) + 30^2$$



$b = 3$

$$\sqrt{55225} = ?$$

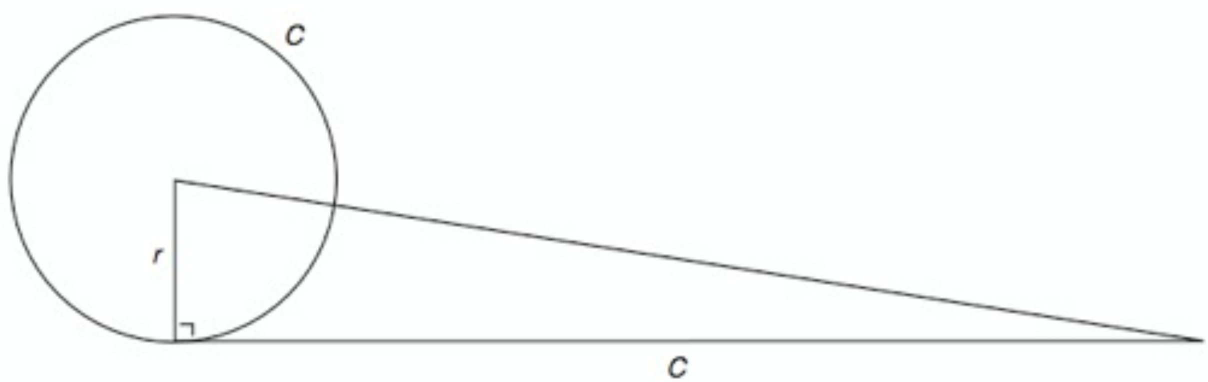
IV-d Using original texts from Archimedes

(-287 – -212)

On the measurement of the circle

Proposition 1:

Any circle is equivalent to the right triangle with one of the sides of the right angle equal to the radius and the other to the circumference of the circle.



$$A = \frac{1}{2} rC$$

Proof by exhaustion and double "reductio ad absurdum"

Case 1: $A > \frac{1}{2} rC$

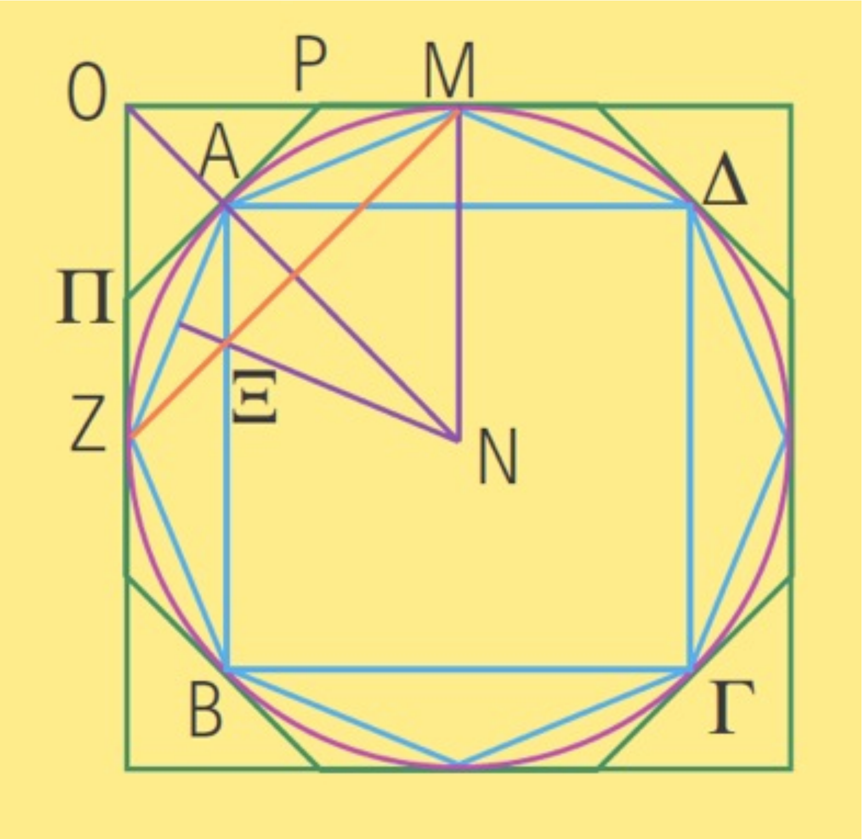
Inscribe regular polygons getting closer to the circle

Consider an inscribed polygon such that $A - A_{inscr} < A - \frac{1}{2} rC$

Thus $A_{inscr} > \frac{1}{2} rC$

But $A_{inscr} = \frac{1}{2} ap < \frac{1}{2} rC$

Case 2: $A < \frac{1}{2} rC$



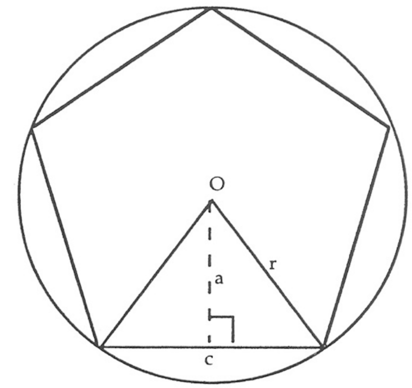
$$A = \frac{1}{2} rC$$

Text is not really difficult

How did Archimedes identify that specific relationship for the area of the circle??? *We do not know!*

- *A "limit" process???*

$$\frac{1}{2}ap \longrightarrow \frac{1}{2}rC$$



area of n-gon

$n \longrightarrow \infty$

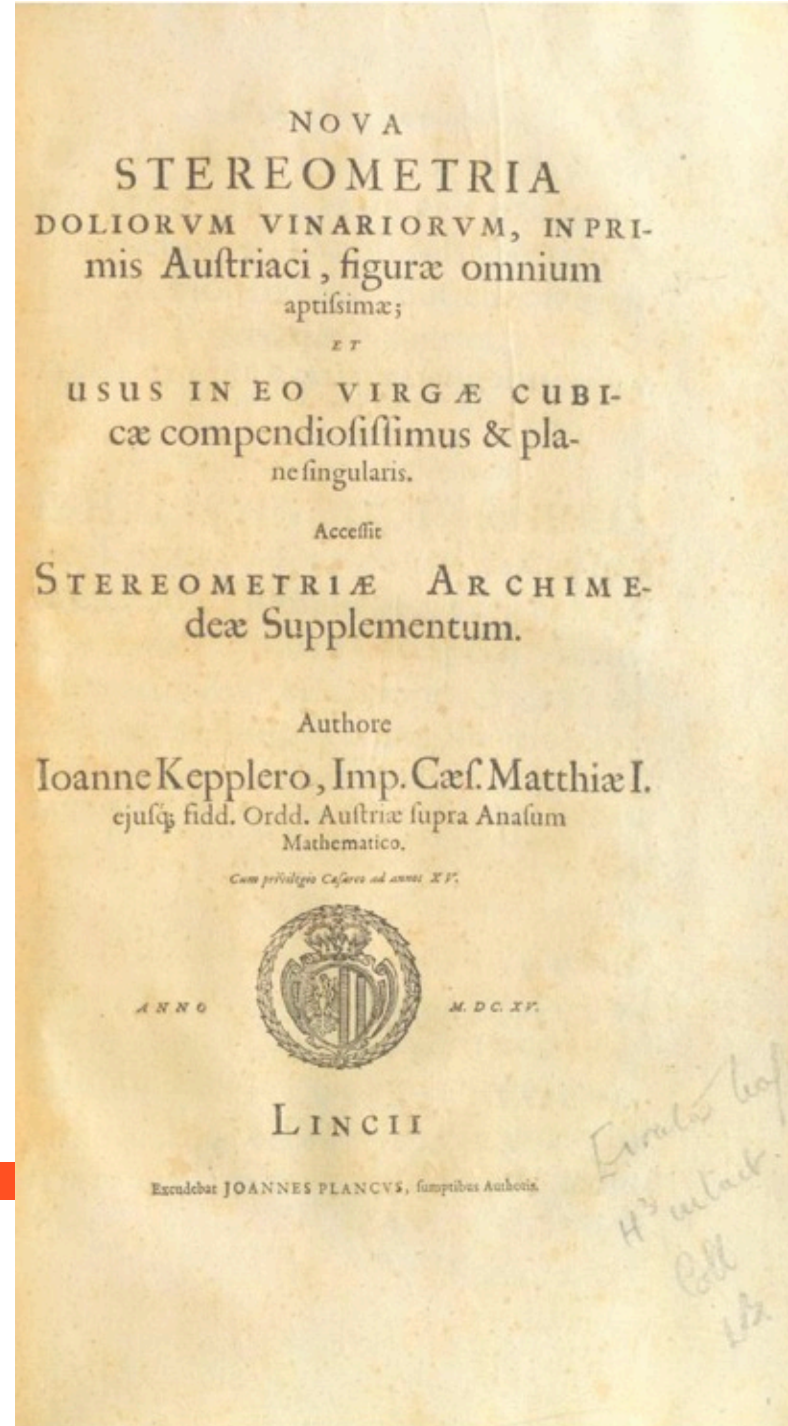
- *An infinitesimal vision???*

$$A = \frac{1}{2}rC$$

An infinitesimal vision

Johannes Kepler
(1571—1630)

$$A = \frac{1}{2} r C$$



NOVA
STEREOMETRIA
DOLIORVM VINARIORVM, IN PRI-
mis Auſtriaci, figuræ omnium
aptiſsimæ;
ET
USUS IN EO VIRGÆ CUBI-
cæ compendioſiſſimus & pla-
ne ſingularis.
Acceſſit
STEREOMETRIÆ ARCHIME-
deæ Supplementum.

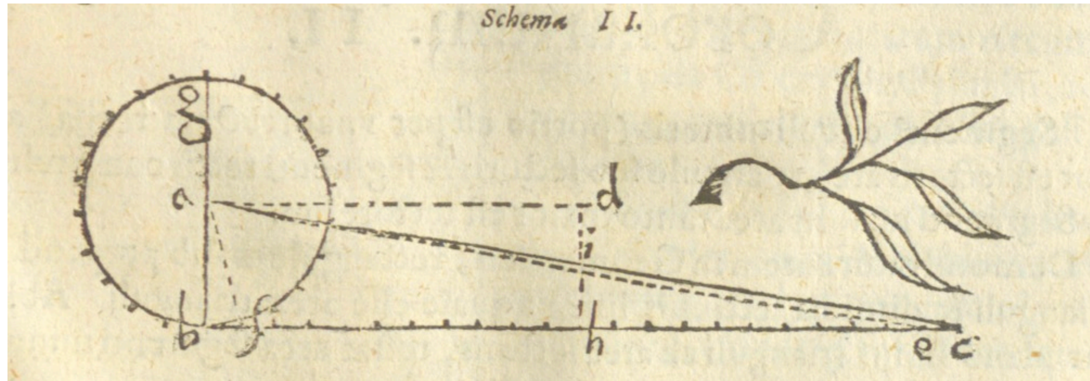
Authore
Ioanne Kepplero, Imp. Cæſ. Matthiæ I.
ejuſq; fidd. Ordd. Auſtriæ ſupra Anaſum
Mathematico.
Cum privilegio Cæſareo ad annos XV.

ANNO M. DC. XV.

LINCII

Excudebat JOANNES PLANCVS, ſumptibus Authoris.

An infinitesimal vision



"The circumference of the circle BG has as many parts as points, think of an infinity; let any be considered as the basis of a certain isosceles triangle with equal legs AB, so that there is an infinite number of triangles inside the area of the circle, all meeting in the centre A by the vertices."

$$A = \frac{1}{2} rC$$

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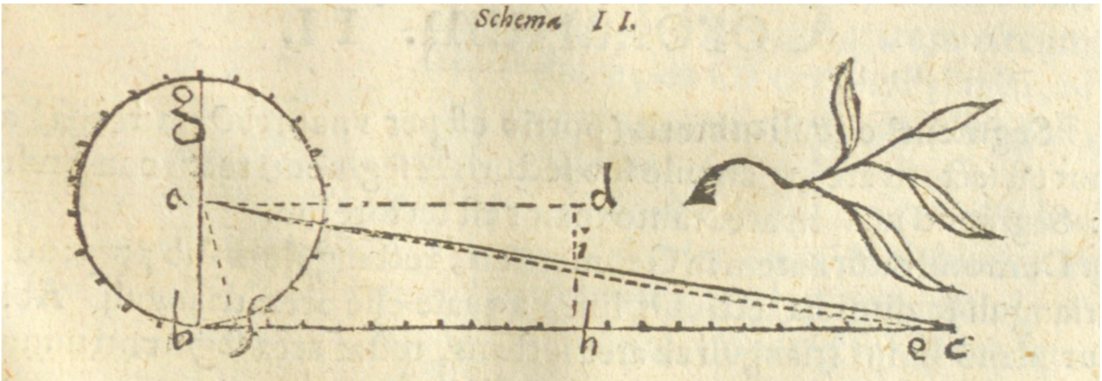
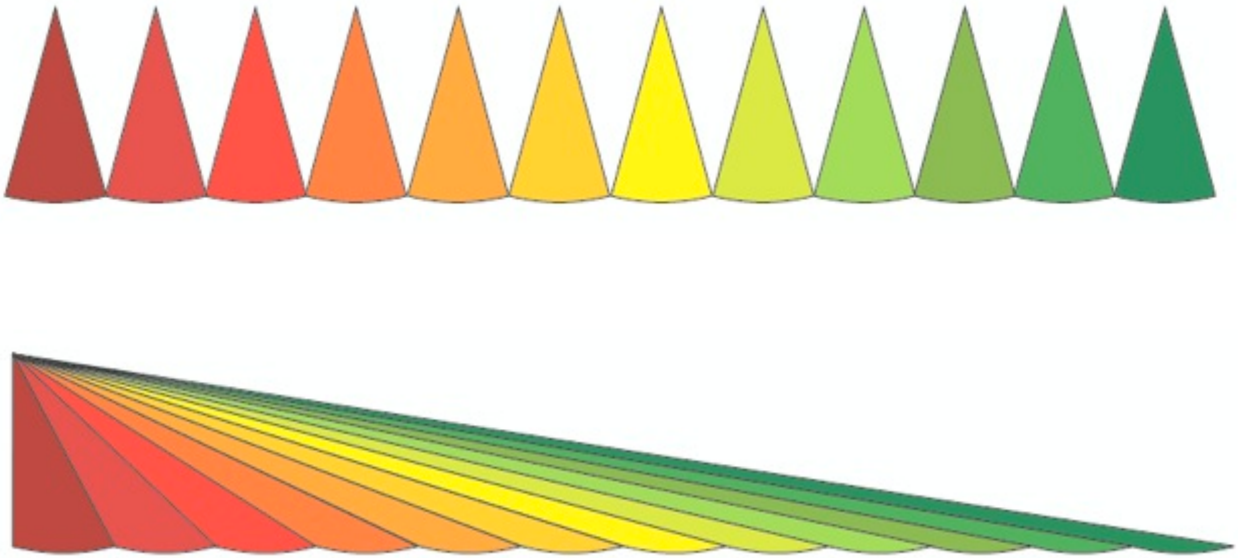
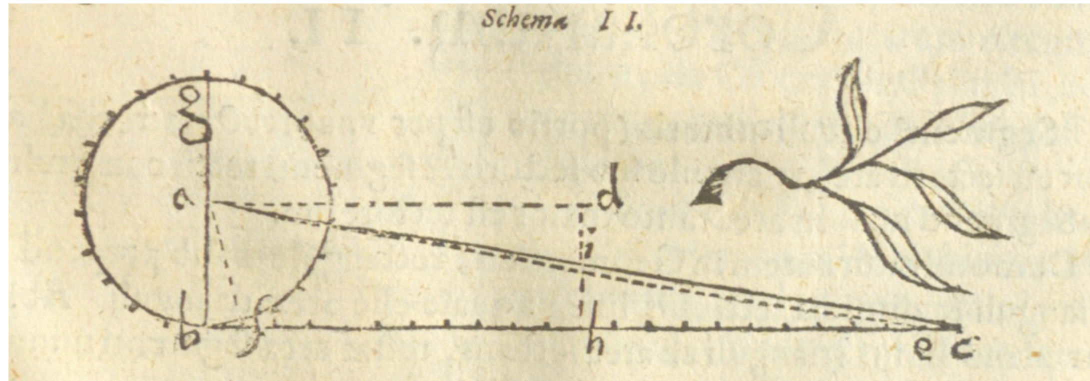
An infinitesimal vision

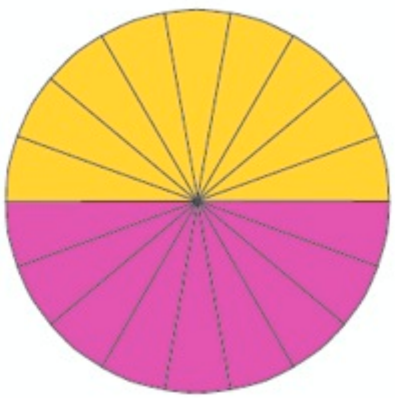


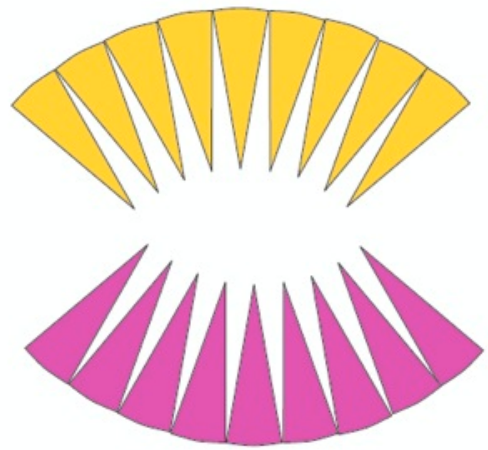


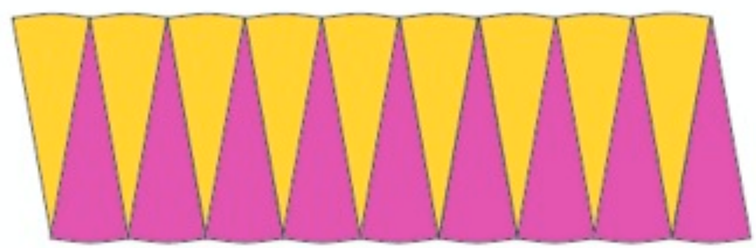
$$A = \frac{1}{2} rC$$

An infinitesimal vision





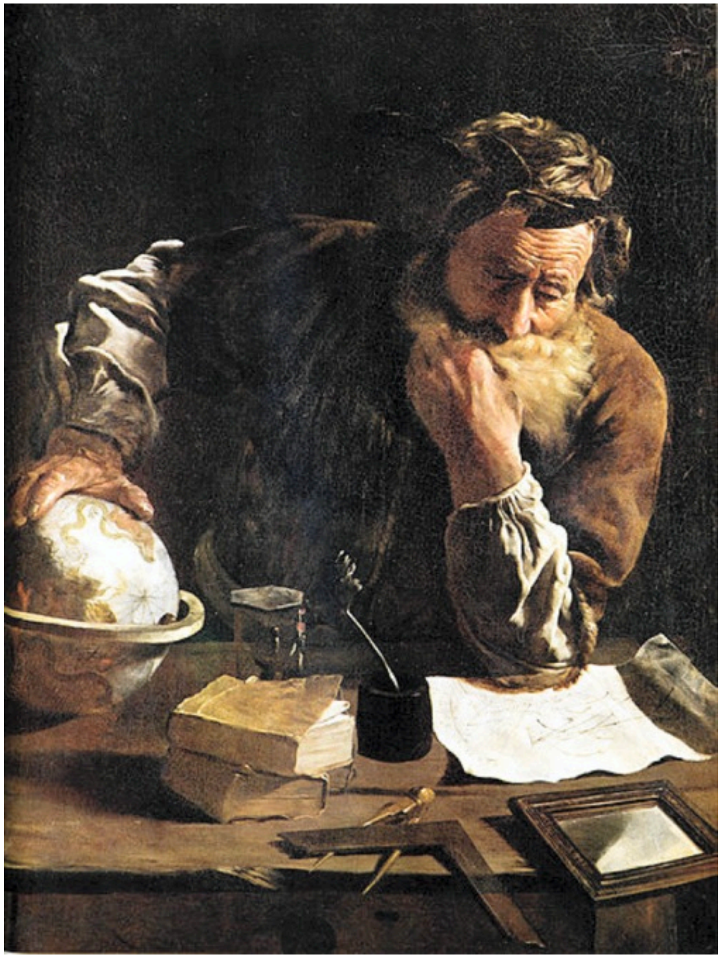




$$A = \frac{1}{2} rC$$

$$A = r \times \frac{C}{2}$$

On the measurement of the circle



Proposition 3:

The perimeter of any circle is equal to the triple of the diameter, augmented by less than the seventh part, but by more than ten seventy-first parts, of the diameter.

$$3\frac{10}{71}\times d < C < 3\frac{10}{70}\times d$$

Painting by Domenico Fetti (1620)

This proof is a bit difficult!

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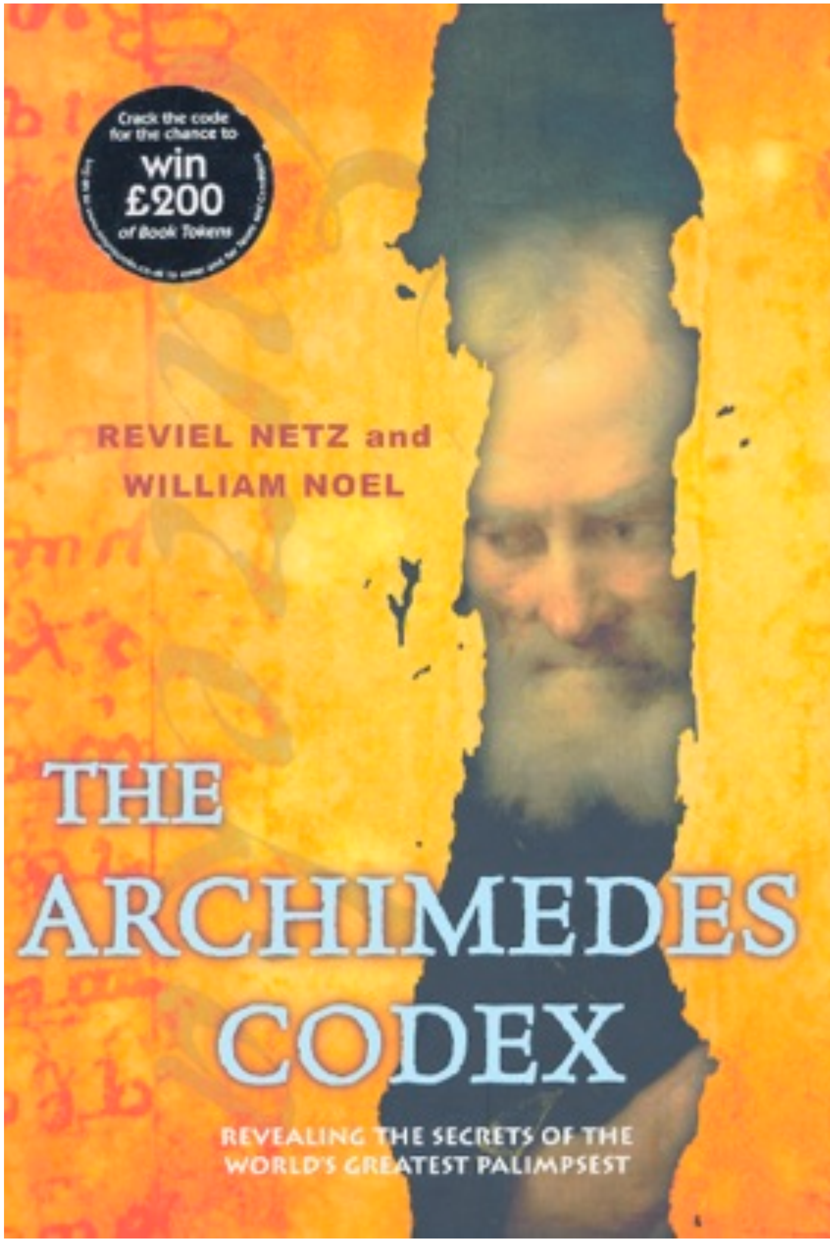
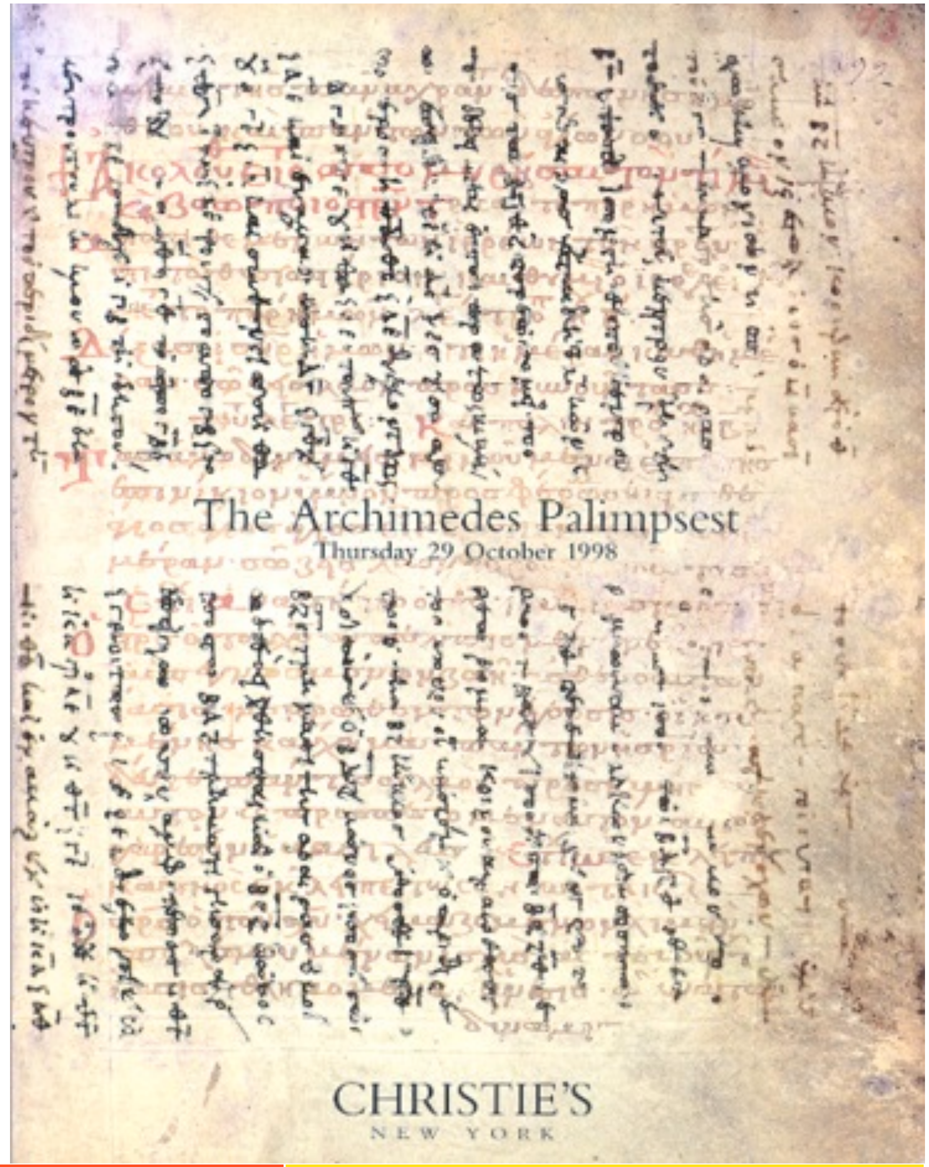
The Method

It was known that Archimedes had written a text on a certain "discovery method", but its content was unknown and there was no trace of it.

Heiberg (1906)

Investigation of problems of mathematics by means of mechanics (statics)





www.archimedespalimpsest.org

UNIVERSITÉ LAVAL
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THE METHOD OF ARCHIMEDES TREATING OF MECHANICAL PROBLEMS—TO ERATOSTHENES

"Archimedes to Eratosthenes greeting.

I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows.

1. If in a right prism with a parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms*, and its sides [i.e. four generators] on the remaining planes (faces) of the prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn, the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is, and the surface being that which is between the said planes; and the segment cut off from the cylinder is one sixth part of the whole prism.

2. If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms† and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube.

Now these theorems differ in character from those communicated before; for we compared the figures then in question,

* The parallelograms are apparently *squares*. † i.e. squares.

conoids and spheroids and segments of them, in respect of size, with figures of cones and cylinders: but none of those figures have yet been found to be equal to a solid figure bounded by planes; whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are bounded by planes. The proofs then of these theorems I have written in this book and now send to you. Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure* though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it† and I do not want to be thought to have uttered vain words, but

* περὶ τοῦ εἰρημένου σχήματος, in the singular. Possibly Archimedes may have thought of the case of the pyramid as being the more fundamental and as really involving that of the cone. Or perhaps "figure" may be intended for "type of figure."

† Cf. Preface to *Quadrature of Parabola*.

"Archimedes to Eratosthenes greeting. (…)
Seeing moreover in you (…) a man of considerable eminence in philosophy, and an admirer [of mathematical enquiry], I thought fit to write out for you and explain in detail (…) the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. (…) [C]ertain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge."

Archimedes, prefatory part of
The Method

equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

First then I will set out the very first theorem which became known to me by means of mechanics, namely that

Any segment of a section of a right-angled cone (i.e. a parabola) is four-thirds of the triangle which has the same base and equal height,

and after this I will give each of the other theorems investigated by the same method. Then, at the end of the book, I will give the geometrical [proofs of the propositions]...

[I premise the following propositions which I shall use in the course of the work.]

1. If from [one magnitude another magnitude be subtracted which has not the same centre of gravity, the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magnitude and of the subtracted part in the direction of the centre of gravity of the whole] and cutting off from it a length which has to the distance between the said centres of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder. [*On the Equilibrium of Planes*, I. 8]

2. If the centres of gravity of any number of magnitudes whatever be on the same straight line, the centre of gravity of the magnitude made up of all of them will be on the same straight line. [Cf. *Ibid.* I. 5]

3. The centre of gravity of any straight line is the point of bisection of the straight line. [Cf. *Ibid.* I. 4]

4. The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the (opposite) sides cut one another. [*Ibid.* I. 13, 14]

5. The centre of gravity of any parallelogram is the point in which the diagonals meet. [*Ibid.* I. 10]

6. The centre of gravity of a circle is the point which is also the centre [of the circle].

7. The centre of gravity of any cylinder is the point of bisection of the axis.

8. The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base].

[All these propositions have already been] proved*. [Besides these I require also the following proposition, which is easily proved:

If in two series of magnitudes those of the first series are, in order, proportional to those of the second series and further] the magnitudes [of the first series], either all or some of them, are in any ratio whatever [to those of a third series], and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes [of a fourth series], then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series. [*On Conoids and Spheroids*, Prop. 1.]"

Proposition 1.

Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC.

Then shall the segment ABC be $\frac{4}{3}$ of the triangle ABC.

From A draw AKF parallel to DE, and let the tangent to the parabola at C meet DBE in E and AKF in F. Produce CB to meet AF in K, and again produce CK to H, making KH equal to CK.

* The problem of finding the centre of gravity of a cone is not solved in any extant work of Archimedes. It may have been solved either in a separate treatise, such as the περὶ ζυγῶν, which is lost, or perhaps in a larger mechanical work of which the extant books *On the Equilibrium of Planes* formed only a part.

Consider CH as the bar of a balance, K being its middle point.

Let MO be any straight line parallel to ED, and let it meet CF, CK, AC in M, N, O and the curve in P.

Now, since CE is a tangent to the parabola and CD the semi-ordinate,

$$EB = BD;$$

"for this is proved in the Elements [of Conics]*."

Since FA, MO are parallel to ED, it follows that

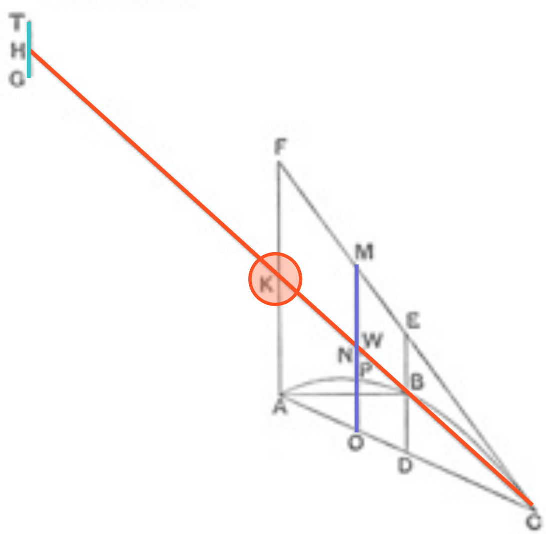
$$FK = KA, \quad MN = NO.$$

Now, by the property of the parabola, "proved in a lemma,"

$MO : OP = CA : AO$ [Cf. *Quadrature of Parabola*, Prop. 5]

$= CK : KN$ [Eucl. VI. 2]

$= HK : KN.$



Take a straight line TG equal to OP, and place it with its centre of gravity at H, so that $TH = HG$; then, since N is the centre of gravity of the straight line MO,

and $$MO : TG = HK : KN,$$

* i.e. the works on conics by Aristaeus and Euclid. Cf. the similar expression in *On Conoids and Spheroids*, Prop. 3, and *Quadrature of Parabola*, Prop. 3.

it follows that TG at H and MO at N will be in equilibrium about K. [*On the Equilibrium of Planes*, I. 6, 7]

Similarly, for all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC, AC with its middle point on KC and (2) a length equal to the intercept between the curve and AC placed with its centre of gravity at H will be in equilibrium about K.

Therefore K is the centre of gravity of the whole system consisting (1) of all the straight lines as MO intercepted between FC, AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straight lines as PO intercepted between the curve and AC.

And, since the triangle CFA is made up of all the parallel lines like MO,

and the segment CBA is made up of all the straight lines like PO within the curve,

it follows that the triangle, placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its centre of gravity at H.

Divide KC at W so that $CK = 3KW$;

then W is the centre of gravity of the triangle ACF; "for this is proved in the books on equilibrium" (ἐν τοῖς ἰσορροπικοῖς).

[Cf. *On the Equilibrium of Planes* I. 15]

Therefore $\triangle ACF : (\text{segment } ABC) = HK : KW$

$= 3 : 1.$

Therefore $\quad$ segment $ABC = \frac{1}{3} \triangle ACF.$

But $\quad \triangle ACF = 4 \triangle ABC.$

Therefore $\quad$ segment $ABC = \frac{4}{3} \triangle ABC.$

"Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time

"Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published."

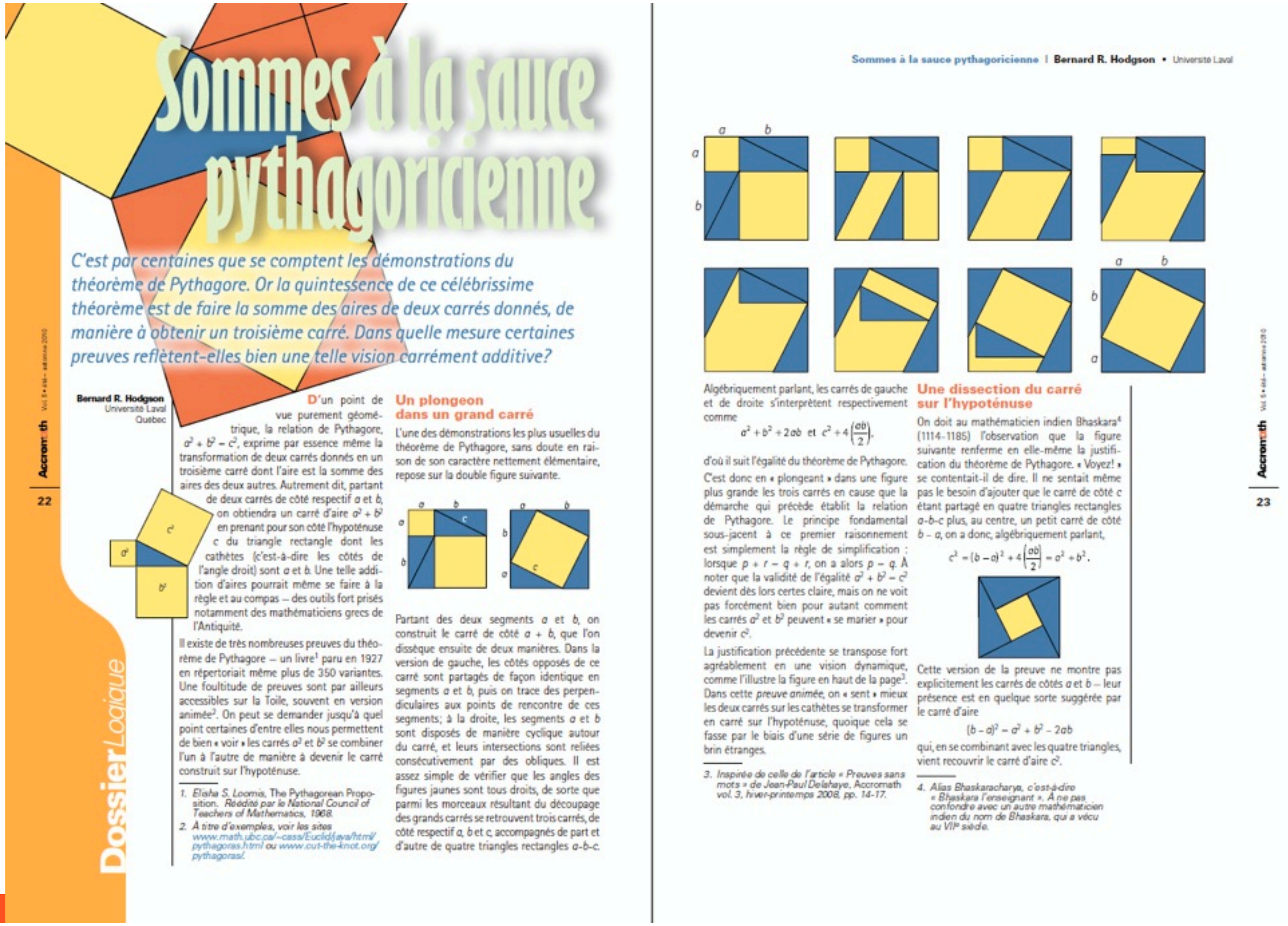
Archimedes, *The Method*

Theorems being *investigated*
vs
theorems being *proved*

Use of means sufficient to suggest the ***truth*** of theorems, although not furnishing scientific ***proofs*** of them

(Heath, Introd. to *The Method*, p. 7)

An act (a “social act”) taking place in a given cultural and scientific framework — an argument accepted as valid by a group of people



Sommes à la sauce pythagoricienne

C'est par centaines que se comptent les démonstrations du théorème de Pythagore. Or la quintessence de ce célébrissime théorème est de faire la somme des aires de deux carrés donnés, de manière à obtenir un troisième carré. Dans quelle mesure certaines preuves reflètent-elles bien une telle vision carrément additive?

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D'un point de vue purement géométrique, la relation de Pythagore, $a^2 + b^2 = c^2$, exprime par essence même la transformation de deux carrés donnés en un troisième carré dont l'aire est la somme des aires des deux autres. Autrement dit, partant de deux carrés de côté respectif a et b, on obtiendra un carré d'aire $a^2 + b^2$ en prenant pour son côté l'hypoténuse c du triangle rectangle dont les cathètes (c'est-à-dire les côtés de l'angle droit) sont a et b. Une telle addition d'aires pourrait même se faire à la règle et au compas – des outils fort prisés notamment des mathématiciens grecs de l'Antiquité.

Il existe de très nombreuses preuves du théorème de Pythagore – un livre[1] paru en 1927 en répertoriait même plus de 350 variantes. Une foultitude de preuves sont par ailleurs accessibles sur la Toile, souvent en version animée[2]. On peut se demander jusqu'à quel point certaines d'entre elles nous permettent de bien « voir » les carrés a^2 et b^2 se combiner l'un à l'autre de manière à devenir le carré construit sur l'hypoténuse.

1. *Elisha S. Loomis,* The Pythagorean Proposition. *Réédité par le National Council of Teachers of Mathematics, 1968.*
2. *À titre d'exemples, voir les sites www.math.ubc.ca/~cass/Euclid/java/html/pythagoras.html ou www.cut-the-knot.org/pythagoras/.*

Un plongeon dans un grand carré

L'une des démonstrations les plus usuelles du théorème de Pythagore, sans doute en raison de son caractère nettement élémentaire, repose sur la double figure suivante.

Partant des deux segments a et b, on construit le carré de côté $a + b$, que l'on dissèque ensuite de deux manières. Dans la version de gauche, les côtés opposés de ce carré sont partagés de façon identique en segments a et b, puis on trace des perpendiculaires aux points de rencontre de ces segments; à la droite, les segments a et b sont disposés de manière cyclique autour du carré, et leurs intersections sont reliées consécutivement par des obliques. Il est assez simple de vérifier que les angles des figures jaunes sont tous droits, de sorte que parmi les morceaux résultant du découpage des grands carrés se retrouvent trois carrés, de côté respectif a, b et c, accompagnés de part et d'autre de quatre triangles rectangles a-b-c.

Algébriquement parlant, les carrés de gauche et de droite s'interprètent respectivement comme

$$a^2 + b^2 + 2ab \text{ et } c^2 + 4\left(\frac{ab}{2}\right),$$

d'où il suit l'égalité du théorème de Pythagore.

C'est donc en « plongeant » dans une figure plus grande les trois carrés en cause que la démarche qui précède établit la relation de Pythagore. Le principe fondamental sous-jacent à ce premier raisonnement est simplement la règle de simplification : lorsque $p + r = q + r$, on a alors $p = q$. À noter que la validité de l'égalité $a^2 + b^2 = c^2$ devient dès lors certes claire, mais on ne voit pas forcément bien pour autant comment les carrés a^2 et b^2 peuvent « se marier » pour devenir c^2.

La justification précédente se transpose fort agréablement en une vision dynamique, comme l'illustre la figure en haut de la page[3]. Dans cette *preuve animée*, on « sent » mieux les deux carrés sur les cathètes se transformer en carré sur l'hypoténuse, quoique cela se fasse par le biais d'une série de figures un brin étranges.

3. *Inspirée de celle de l'article « Preuves sans mots » de Jean-Paul Delahaye, Accromath vol. 3, hiver-printemps 2008, pp. 14-17.*

Une dissection du carré sur l'hypoténuse

On doit au mathématicien indien Bhaskara[4] (1114-1185) l'observation que la figure suivante renferme en elle-même la justification du théorème de Pythagore. « Voyez! » se contentait-il de dire. Il ne sentait même pas le besoin d'ajouter que le carré de côté c étant partagé en quatre triangles rectangles a-b-c plus, au centre, un petit carré de côté $b - a$, on a donc, algébriquement parlant,

$$c^2 = (b-a)^2 + 4\left(\frac{ab}{2}\right) = a^2 + b^2.$$

Cette version de la preuve ne montre pas explicitement les carrés de côtés a et b – leur présence est en quelque sorte suggérée par le carré d'aire

$$(b-a)^2 = a^2 + b^2 - 2ab$$

qui, en se combinant avec les quatre triangles, vient recouvrir le carré d'aire c^2.

4. *Alias Bhaskaracharya, c'est-à-dire « Bhaskara l'enseignant ». À ne pas confondre avec un autre mathématicien indien du nom de Bhaskara, qui a vécu au VII[e] siècle.*

Accromath vol. 5 (Été-Aut. 2010)

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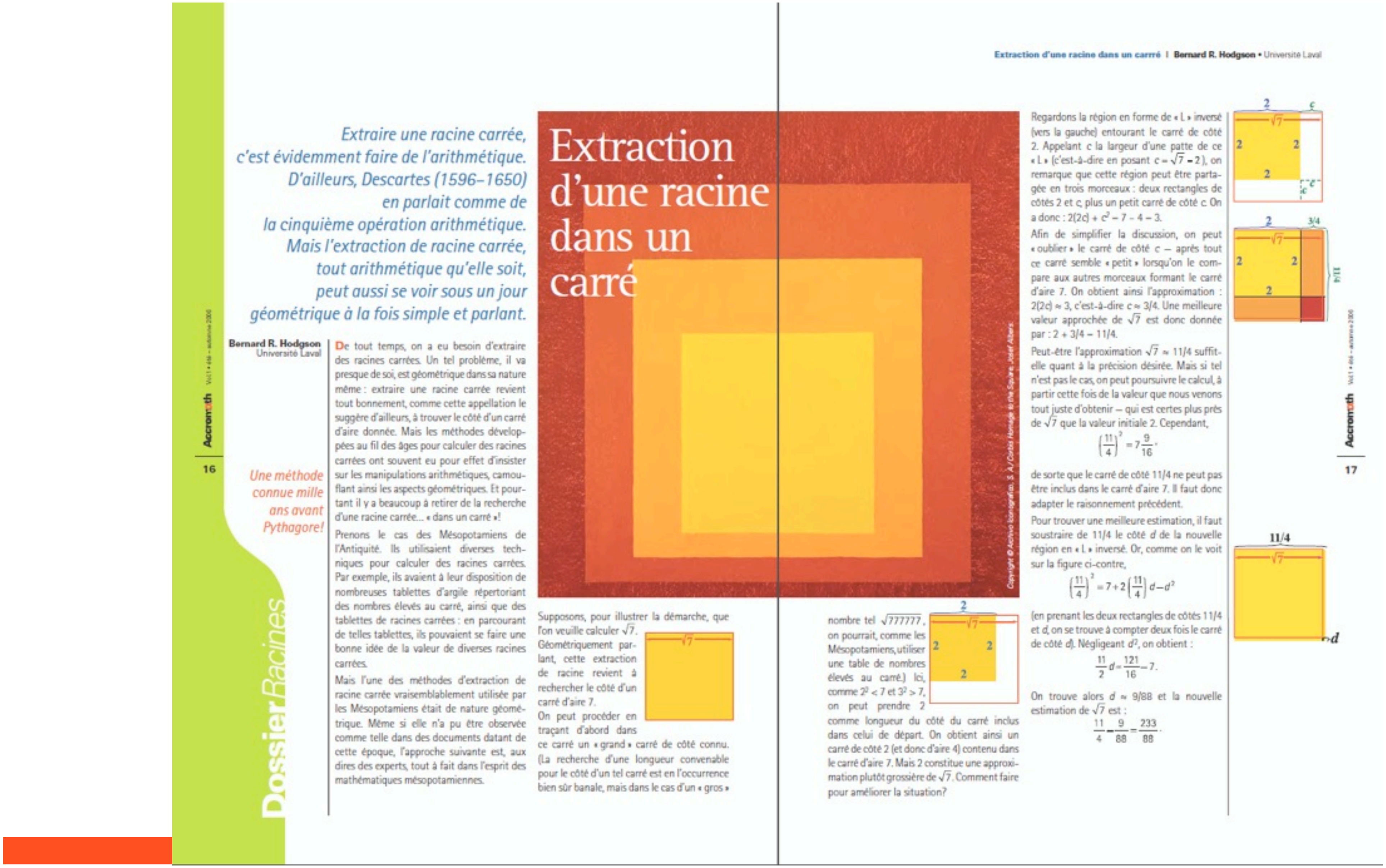

Extraire une racine carrée, c'est évidemment faire de l'arithmétique. D'ailleurs, Descartes (1596–1650) en parlait comme de la cinquième opération arithmétique. Mais l'extraction de racine carrée, tout arithmétique qu'elle soit, peut aussi se voir sous un jour géométrique à la fois simple et parlant.

Extraction d'une racine dans un carré

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Une méthode connue mille ans avant Pythagore!

De tout temps, on a eu besoin d'extraire des racines carrées. Un tel problème, il va presque de soi, est géométrique dans sa nature même : extraire une racine carrée revient tout bonnement, comme cette appellation le suggère d'ailleurs, à trouver le côté d'un carré d'aire donnée. Mais les méthodes développées au fil des âges pour calculer des racines carrées ont souvent eu pour effet d'insister sur les manipulations arithmétiques, camouflant ainsi les aspects géométriques. Et pourtant il y a beaucoup à retirer de la recherche d'une racine carrée… « dans un carré »!

Prenons le cas des Mésopotamiens de l'Antiquité. Ils utilisaient diverses techniques pour calculer des racines carrées. Par exemple, ils avaient à leur disposition de nombreuses tablettes d'argile répertoriant des nombres élevés au carré, ainsi que des tablettes de racines carrées : en parcourant de telles tablettes, ils pouvaient se faire une bonne idée de la valeur de diverses racines carrées.

Mais l'une des méthodes d'extraction de racine carrée vraisemblablement utilisée par les Mésopotamiens était de nature géométrique. Même si elle n'a pu être observée comme telle dans des documents datant de cette époque, l'approche suivante est, aux dires des experts, tout à fait dans l'esprit des mathématiques mésopotamiennes.

Supposons, pour illustrer la démarche, que l'on veuille calculer $\sqrt{7}$. Géométriquement parlant, cette extraction de racine revient à rechercher le côté d'un carré d'aire 7. On peut procéder en traçant d'abord dans ce carré un « grand » carré de côté connu. (La recherche d'une longueur convenable pour le côté d'un tel carré est en l'occurrence bien sûr banale, mais dans le cas d'un « gros » nombre tel $\sqrt{777777}$, on pourrait, comme les Mésopotamiens, utiliser une table de nombres élevés au carré.) Ici, comme $2^2 < 7$ et $3^2 > 7$, on peut prendre 2 comme longueur du côté du carré inclus dans celui de départ. On obtient ainsi un carré de côté 2 (et donc d'aire 4) contenu dans le carré d'aire 7. Mais 2 constitue une approximation plutôt grossière de $\sqrt{7}$. Comment faire pour améliorer la situation?

Regardons la région en forme de « L » inversé (vers la gauche) entourant le carré de côté 2. Appelant c la largeur d'une patte de ce « L » (c'est-à-dire en posant $c = \sqrt{7} - 2$), on remarque que cette région peut être partagée en trois morceaux : deux rectangles de côtés 2 et c, plus un petit carré de côté c. On a donc : $2(2c) + c^2 - 7 - 4 - 3$.

Afin de simplifier la discussion, on peut « oublier » le carré de côté c – après tout ce carré semble « petit » lorsqu'on le compare aux autres morceaux formant le carré d'aire 7. On obtient ainsi l'approximation : $2(2c) \approx 3$, c'est-à-dire $c \approx 3/4$. Une meilleure valeur approchée de $\sqrt{7}$ est donc donnée par : $2 + 3/4 - 11/4$.

Peut-être l'approximation $\sqrt{7} \approx 11/4$ suffit-elle quant à la précision désirée. Mais si tel n'est pas le cas, on peut poursuivre le calcul, à partir cette fois de la valeur que nous venons tout juste d'obtenir – qui est certes plus près de $\sqrt{7}$ que la valeur initiale 2. Cependant,

$$\left(\frac{11}{4}\right)^2 = 7\frac{9}{16},$$

de sorte que le carré de côté 11/4 ne peut pas être inclus dans le carré d'aire 7. Il faut donc adapter le raisonnement précédent.

Pour trouver une meilleure estimation, il faut soustraire de 11/4 le côté *d* de la nouvelle région en « L » inversé. Or, comme on le voit sur la figure ci-contre,

$$\left(\frac{11}{4}\right)^2 = 7 + 2\left(\frac{11}{4}\right)d - d^2$$

(en prenant les deux rectangles de côtés 11/4 et *d*, on se trouve à compter deux fois le carré de côté *d*). Négligeant d^2, on obtient :

$$\frac{11}{2}d = \frac{121}{16} - 7.$$

On trouve alors $d \approx 9/88$ et la nouvelle estimation de $\sqrt{7}$ est :

$$\frac{11}{4} - \frac{9}{88} = \frac{233}{88}.$$

Dossier Racines

Accromath vol. 1 (Été-Aut. 2006)

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Dans son étude sur le cercle, Archimède met l'accent sur un lien explicite entre l'aire et la circonférence. Si ce résultat peut nous sembler presque banal aujourd'hui, il n'en demeure pas moins fondamental en vue de la compréhension de cette figure, qui est loin d'être simple quand vient le temps d'exprimer ses grandeurs (aire, périmètre). On ne dispose cependant d'aucun indice quant à la façon dont l'éminent Syracusain aurait mis le doigt sur ce résultat.

Regard archimédien sur le cercle : quand la circonférence prend une bouffée d'aire

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Dans son traité *De la mesure du cercle*, Archimède (~287–~212) établit trois propositions à propos de cette figure fondamentale. Dans l'une d'elles, qui a déjà été présentée dans *Accromath*,[1] Archimède introduit des bornes numériques fines permettant d'exprimer la circonférence C d'un cercle en fonction de son diamètre d :

$$3\frac{10}{71}d < C < 3\frac{1}{7}d.$$

Nous nous intéressons ici à la toute première proposition du traité,[2] où l'auteur relie d'une manière fort originale l'aire du cercle à son périmètre. L'énoncé de ce résultat se lit comme suit :

> Tout cercle équivaut au triangle rectangle dans lequel l'un des côtés de l'angle droit est égal au rayon du cercle, et la base (c'est-à-dire l'autre côté de l'angle droit), égale au périmètre.

Il est question ici d'équivalence selon l'aire. À noter qu'Archimède ne se préoccupe pas de la manière dont le périmètre du cercle aurait été « rectifié », c'est-à-dire transformé en un segment de droite qui vient jouer le rôle d'un des côtés du triangle en cause. Il tient simplement pour acquis qu'un tel segment existe.[3]

Si on se permet une notation moderne – n'ayant évidemment pas cours à l'époque hellénistique –, la proposition 1 peut se lire comme suit. Étant donné un cercle de rayon r et de circonférence C, son aire A est donnée par la formule

$$A = \frac{1}{2}rC.$$

Réécrivant cette dernière sous la forme

$$A = r \times \frac{C}{2} = \frac{d}{2} \times \frac{C}{2},$$

on peut aussi paraphraser en présentant l'aire du cercle comme étant celle d'un rectangle dont les côtés sont respectivement le demi-diamètre et le demi-périmètre du cercle.

1. *Voir notre texte* Regard archimédien sur le cercle : la quête du fameux 22/7, Accromath, *vol. 7, été-automne 2012, p. 24-29.*
2. *La deuxième proposition de ce même traité est présentée dans la* Section problèmes.
3. *Comme la base du triangle vaut πd, il ne saurait être question, par exemple, de la tracer à la règle et au compas. Voir à ce sujet l'encadré* Une pseudo-quadrature du cercle.

Bien sûr, pour nous qui sommes habitués à penser de nos jours au cercle en termes de formules telles $C = 2\pi r$ ou $A = \pi r^2$, le résultat d'Archimède n'est pas trop étonnant : une simple manipulation algébrique suffit en effet à l'établir à partir de ces deux égalités. Mais il faut bien comprendre que tel n'était pas du tout le contexte dans lequel évoluait le grand géomètre grec. Il est donc intéressant à cet égard de se pencher sur la preuve qu'il donne de sa proposition 1.

La preuve d'Archimède

Les mathématiciens, qui aiment à l'occasion forger des mots nouveaux, appellent *loi de trichotomie*[4] le fait suivant : étant donné deux quantités (disons, X et Y), alors de trois choses l'une : soit qu'elles soient égales, soit que l'une soit plus grande que l'autre, ou l'inverse. En symboles,

$$X = Y \text{ ou } X > Y \text{ ou } X < Y.$$

Pour montrer qu'on a bel et bien X et Y égaux, il suffit donc d'éliminer les deux autres possibilités. Cette observation est au cœur de la démarche proposée par Archimède. L'argument se fait ainsi en deux étapes : il s'agit d'un *double raisonnement par l'absurde*.

Afin de simplifier la notation, nous remplaçons l'expression $rC/2$ par A_T, l'aire du triangle mentionné dans l'énoncé de la proposition 1. Il nous faut donc prouver que $A = A_T$.

Dans un premier temps, nous supposons au contraire que $A > A_T$ et montrons que ce scénario ne tient pas la route. Comme la différence $A - A_T$ est positive, il est possible d'inscrire consécutivement dans le cercle d'abord un carré, puis un octogone régulier, ensuite un 16-gone,… , jusqu'à ce qu'on arrive à un certain polygone régulier inscrit dont l'aire diffère de celle du cercle d'une quantité moindre que l'excédent de A sur A_T.

4. *Une généralisation, bien sûr, de la notion de dichotomie.*

Calculer l'aire d'un cercle à partir de son rayon

Observons au passage que la formule usuelle $A = \pi r^2$, qui établit un lien direct entre l'aire du cercle et la longueur de son rayon, est davantage applicable en pratique que la relation d'Archimède : on n'a qu'à *mesurer* le rayon du cercle et vlan! un simple *calcul* nous donne son aire. Le résultat d'Archimède relève plutôt d'un cadre théorique, mais il propose néanmoins une vision à la fois riche et pertinente du cercle (voir notamment à cet égard l'encadré *Quelques retombées du résultat d'Archimède*).

Illustration tirée de Les vrais pourtraits et vies des hommes illustres grecz, latins et payens d'André Thevet (1584)

Si on désigne par A_I l'aire de ce dernier polygone inscrit, on a donc

$$A - A_I < A - A_T,$$

c'est-à-dire $A_I > A_T$.

Mais l'aire de tout polygone régulier peut s'exprimer comme étant le demi-produit de son apothème a par son périmètre p. Or clairement, dans le cas du polygone inscrit en cause, l'apothème est inférieur au rayon du cercle, tandis que le périmètre, comme celui de tout polygone inscrit dans le cercle, est certainement plus petit que la circonférence du cercle. Bref, on a

$$A_I = \frac{1}{2}ap < \frac{1}{2}rC = A_T,$$

ce qui vient s'opposer à l'inégalité $A_I > A_T$ établie précédemment. Cette contradiction nous amène à rejeter l'hypothèse voulant que $A > A_T$.

L'aire de l'octogone régulier

Dans la figure ci-contre, on voit bien que l'octogone régulier se décompose en huit triangles isocèles de base q, le côté de l'octogone, et de hauteur a, l'apothème. Son aire est donc

$$8\left(\frac{1}{2}aq\right) = \frac{1}{2}a(8q) = \frac{1}{2}ap,$$

où p désigne le périmètre de l'octogone.

3,141 592 653 589 793 238 462 643 383 279 502 884 197 169 399 375 105 820 974

Regard archimédien sur le cercle :

la quête du fameux 22/7

La constante 22/7 a longtemps été utilisée afin d'exprimer la circonférence du cercle en fonction de son diamètre. Connue de tous les écoliers jusqu'à des temps très récents, c'est dans un court traité dû à l'un des mathématiciens les plus importants de l'Antiquité que cette remarquable approximation de π a été introduite, sa justification faisant intervenir tant des arguments géométriques élémentaires que de fascinantes prouesses numériques.

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Université Laval

On doit au Grec Archimède[1] (~287–~212) et à son appétit insatiable des mathématiques une douzaine de traités qui constituent autant de monuments de la littérature mathématique de l'Antiquité. Plusieurs de ces ouvrages se présentent sous des titres résolument évocateurs, tels que *De la sphère et du cylindre*, *Des spirales*, *La quadrature de la parabole* ou encore *La méthode relative aux théorèmes mécaniques*, pour n'en nommer que quelques-uns.

Dans son opuscule intitulé *De la mesure du cercle*, Archimède expose trois propositions dépeignant le cercle de manière aussi astucieuse que jolie.[2] Nous abordons ici la dernière de ces propositions, dans laquelle le Syracusain coince la circonférence du cercle entre deux bornes remarquablement fines. L'argumentation que déploie Archimède afin de justifier ce résultat, bien que passablement dense, permet de goûter la saveur des mathématiques de l'époque. Nous suivons de près dans ce texte la démarche archimédienne.

1. *On trouvera deux textes consacrés à Archimède dans le volume 2 (hiver-printemps 2007) de la revue* ***Accromath***.
2. *Aux dires des experts, il semble que seul un fragment du traité original d'Archimède ne soit parvenu jusqu'à nous.*

Une circonférence bien encadrée

La troisième proposition du traité *De la mesure du cercle* renferme l'une des plus célèbres approximations du nombre π : le fameux 22/7, toujours d'intérêt aujourd'hui en raison de sa grande simplicité et susceptible même d'être utilisé en pratique, quand le niveau de précision visé n'est pas trop élevé. Cette valeur a été abondamment utilisée au fil des âges, notamment par Héron d'Alexandrie, au premier siècle de notre ère, ou encore par al-Khwarizmi, plus de mille ans après Archimède.

« Pour tout cercle, la multiplication du diamètre par trois et un septième est la circonférence qui l'entoure; ceci est une convention entre les gens, sans nécessité. »

Al-Khwarizmi, *Livre d'algèbre et d'al-muqabala*, c. 820.[3]

3. *Cité dans R. Rashed, al-Khwarizmi : Le commencement de l'algèbre, p. 204, (Éditions Albert Blanchard, 2007)*

De fait, Archimède introduit dans cette troisième proposition deux bornes pour la circonférence d'un cercle en termes de son diamètre, bornes qui, il convient de le souligner, font non seulement montre d'une précision inégalée dans un tel contexte historique, mais sont également les plus anciennes dont la justification nous soit parvenue. En dépit de son caractère un brin vieillot, il convient sans doute de s'arrêter à la façon même dont l'éminent géomètre grec énonce son résultat :

Le périmètre de tout cercle vaut le triple du diamètre augmenté de moins de la septième partie, mais de plus des dix soixante et onzièmes parties du diamètre.

On aura reconnu dans la première partie de cet énoncé le 22/7 au cœur du présent texte. Étant donné un cercle de circonférence C et de diamètre d, le résultat d'Archimède peut donc s'écrire, en notation moderne :

$$3\frac{10}{71}\times d < C < 3\frac{10}{70}\times d.$$

(On a utilisé ici deux fractions de même numérateur afin de mieux faire ressortir la finesse de l'encadrement obtenu par Archimède.) Habitués comme nous le sommes aujourd'hui de penser à π via une représentation décimale telle 3,1416 (voire 3,1415926536, valeur que nous révèle immédiatement la plus élémentaire des calculatrices), les deux approximations obtenues par Archimède peuvent, à première vue, ne nous sembler que modérément percutantes. Il suffit pourtant de se rappeler le cadre dans lequel évoluait Archimède, et en particulier le système de numération peu performant utilisé en Grèce antique, pour apprécier toute la force de son résultat, et comprendre les prouesses numériques qu'il doit déployer pour l'obtenir.

L'argument d'Archimède est l'archétype des méthodes géométriques visant à obtenir π avec une précision de plus en plus fine. Il repose, dans un premier temps, sur l'idée d'approximer le cercle donné par des polygones réguliers bien choisis (circonscrits ou inscrits, selon la partie de la double inégalité précédente sous discussion). À cette fin, Archimède démarre avec un hexagone régulier, qu'il remplace successivement, en effectuant les bissections appropriées, par des polygones réguliers de 12, 24, 48, et enfin 96 côtés. À ce stade, il a atteint le niveau de précision souhaité, tout en demeurant dans un cadre numérique qui ne rend pas les calculs inabordables.

Un Québécois au π-nacle

En 1975, le Québécois Simon Plouffe (prix Reconnaissance 2004 de l'UQAM) s'est rendu célèbre en figurant dans le *Livre des records Guinness* pour avoir récité par cœur les 4096 premières décimales de π – une jolie puissance de 2, selon ses dires. Il a alors détenu le record mondial jusqu'en 1977. (Le record à ce jour, datant de 2005, dépasse les 67 000 décimales.) Simon Plouffe est reconnu aujourd'hui notamment pour la mise au point de méthodes numériques permettant de connaître la n-ième décimale de π sans avoir à calculer les précédentes, de même que pour l'« inverseur de Plouffe », un programme informatique permettant, à partir d'une suite de décimales, d'identifier un « réel remarquable » qui lui correspond en révélant une ou des formules dont il peut être issu (voir http://pi.lacim.uqam.ca/fra/).

La base géométrique de l'encadrement archimédien

Voyons comment Archimède obtient

$$3\frac{1}{7}\times d$$

à titre de borne supérieure pour la circonférence. Considérons un cercle de centre O et de diamètre $\overline{AH}$, ainsi que la tangente au cercle en A (coupant donc perpendiculairement le diamètre). Plaçons un point B sur cette tangente de telle sorte que l'angle BOA corresponde au tiers de l'angle droit, c'est-à-dire 30°.

Soulignons dès maintenant que les côtés du triangle rectangle BAO sont dans des rapports fort singuliers. En effet, comme ce triangle peut être vu comme un « demi- triangle équilatéral », on a

$$\frac{m(\overline{BO})}{m(\overline{AB})} = 2 \text{ et } \frac{m(\overline{AO})}{m(\overline{AB})} = \sqrt{3}.$$

Cette dernière valeur numérique jouera un rôle-clé un peu plus loin. En raison de la

Dossier Histoire

Accromath vol. 7 (Été-Aut. 2012)

www.accromath.ca

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PLAN OF THE TALK

I- Introductory remarks *(done)*

II- History of mathematics and the school mathematics curriculum

III- Resources to support the teaching of history of mathematics to prospective teachers

IV- Examples of topics in the history of mathematics suitable for prospective teachers

V- Concluding remarks

V- Concluding remarks

Mathematics as a domain with a long and rich history, still on-going, and with a significant impact on today's society.

Such a large vision is essential for "***Changing the Culture***" that is being carried in the classroom.

History can help reaching these goals!

Teachers need to be introduced to the potentialities (and ***limits***!) of the use of history of math in education

Part of the mission incumbent upon the university mathematics departments

Discussion / Questions (Act 2)

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