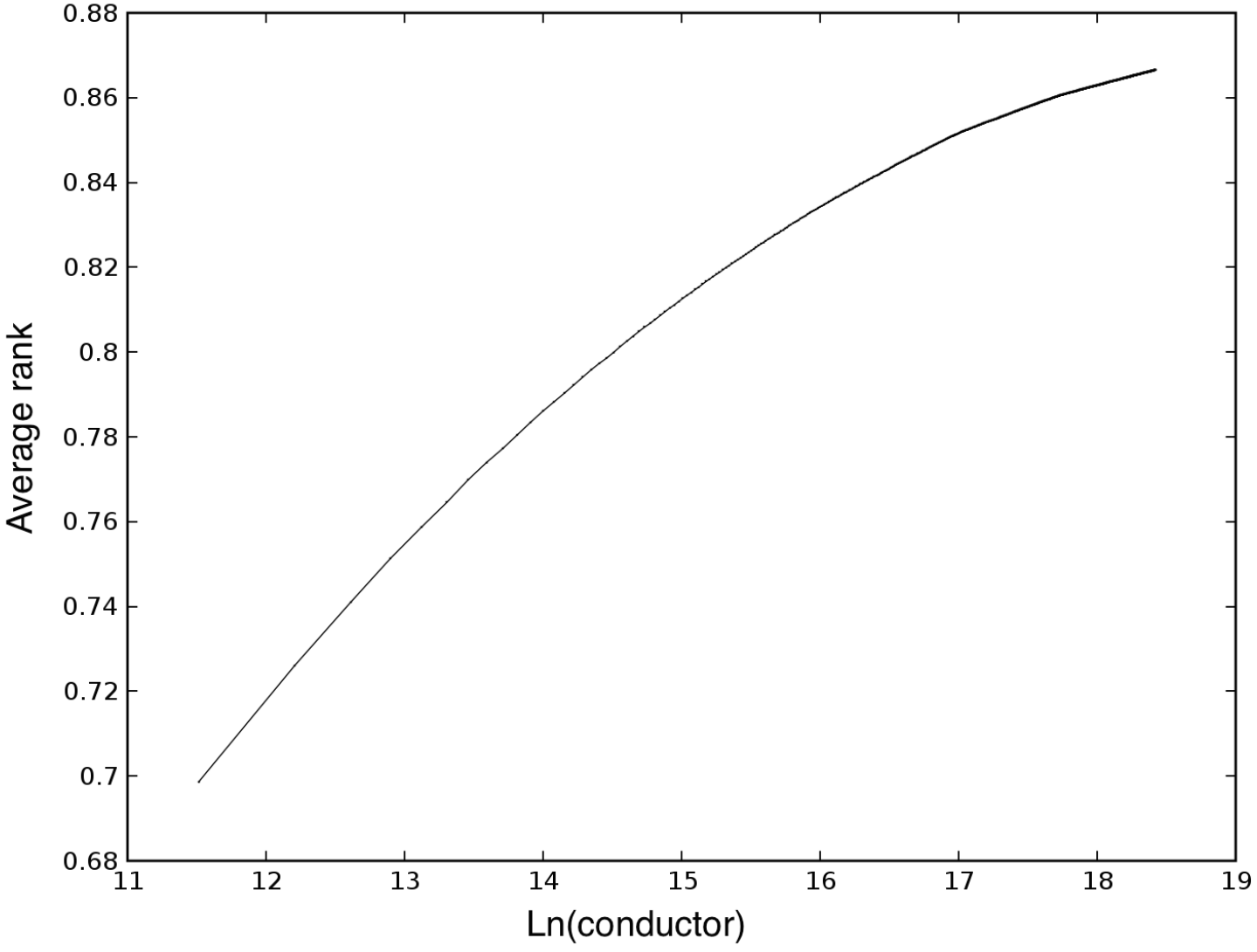
# Data: Elliptic Curves by Height

PIMS Arithmetic Topology Workshop
June 14, 2019

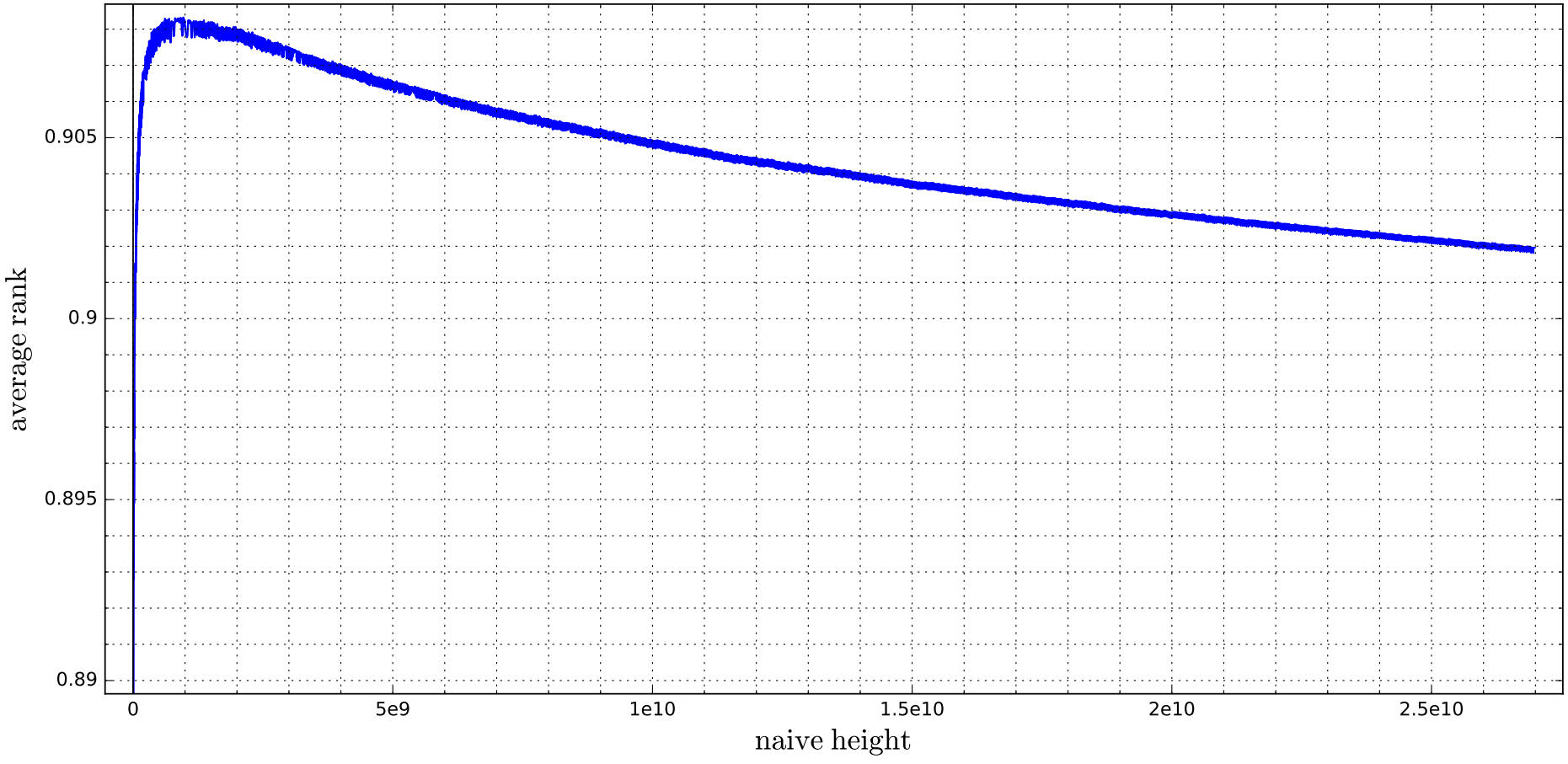
# Average rank versus conductor [BMSW 2007]

from the 136,832,795 curves of conductor $< 10^8$ in the Stein-Watkins db:

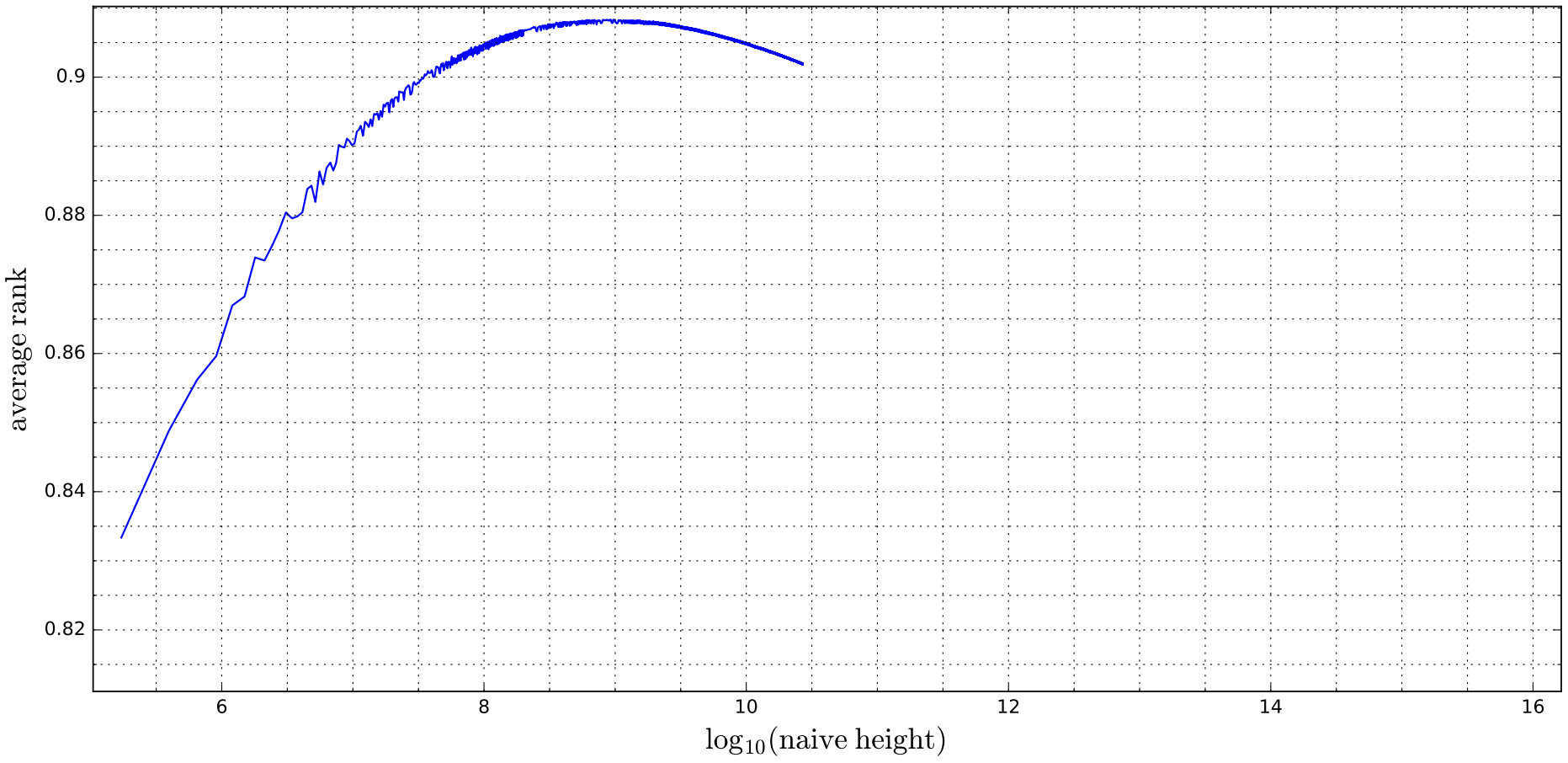


# Average rank versus height [BHKSSW 2016]

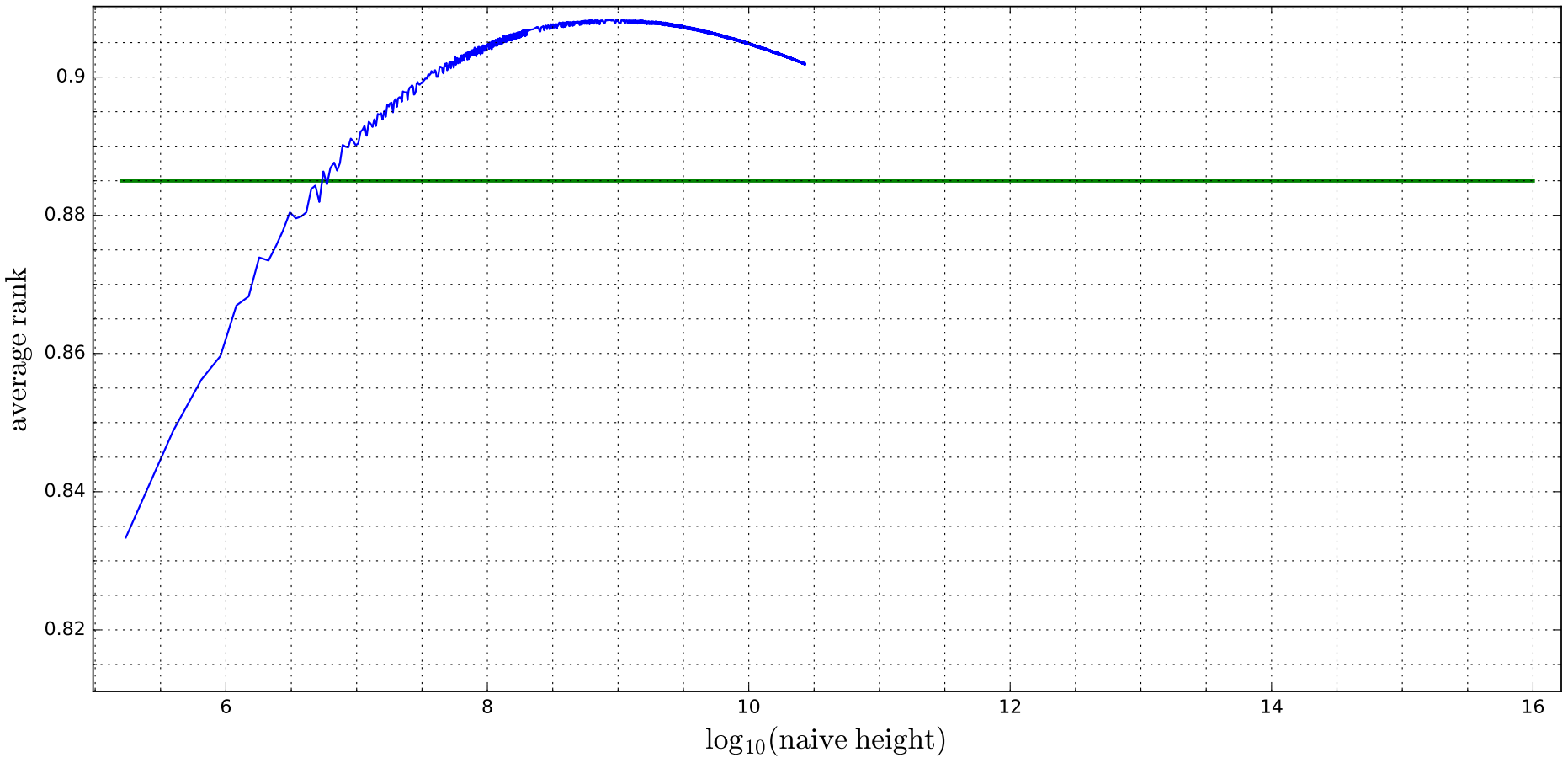
from the 238,764,310 curves of height $< 2.7 \cdot 10^{10}$ in database:



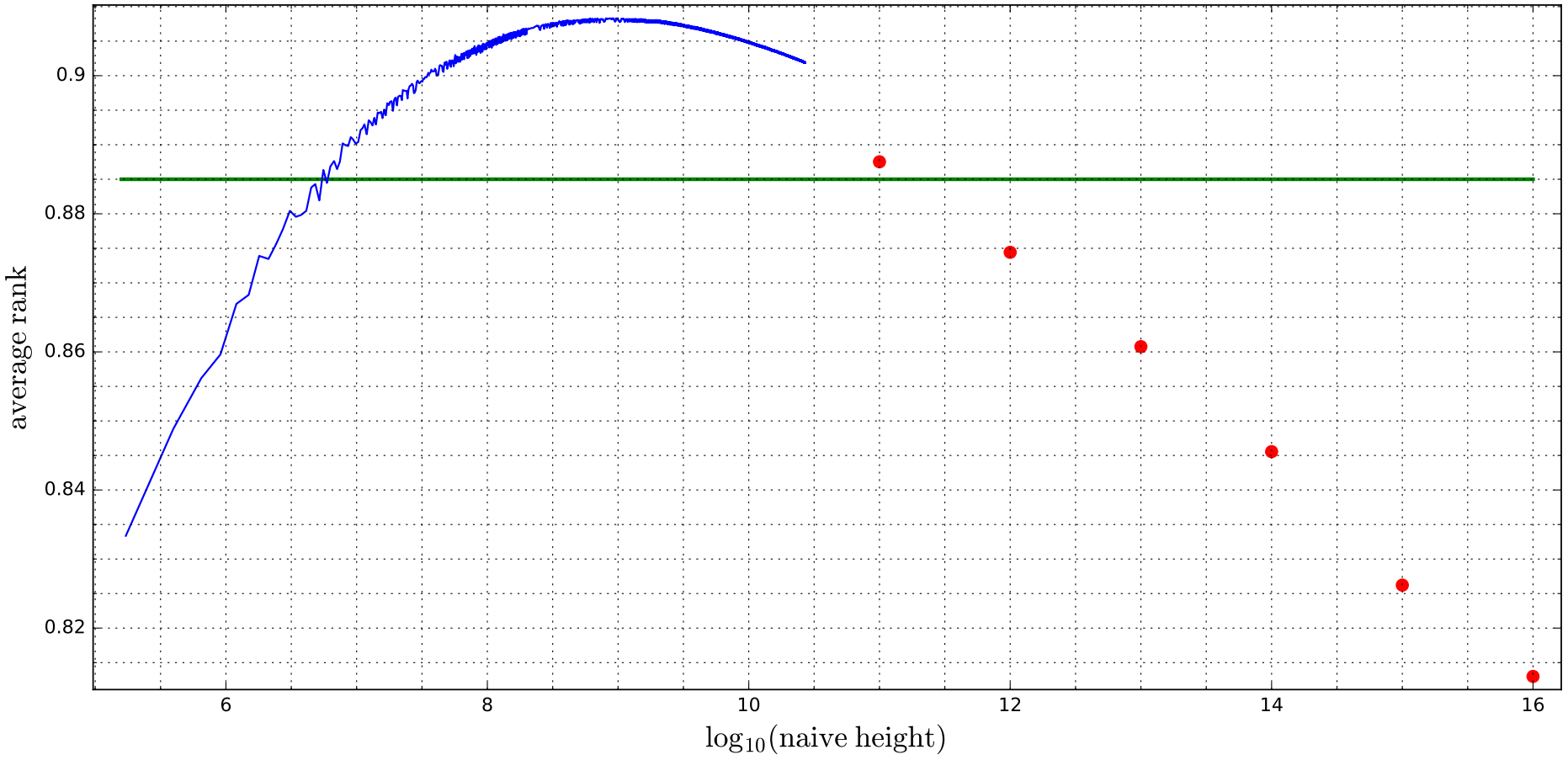
# Average rank ($\log_{10}$ scale)



# Average rank ($\log_{10}$ scale)

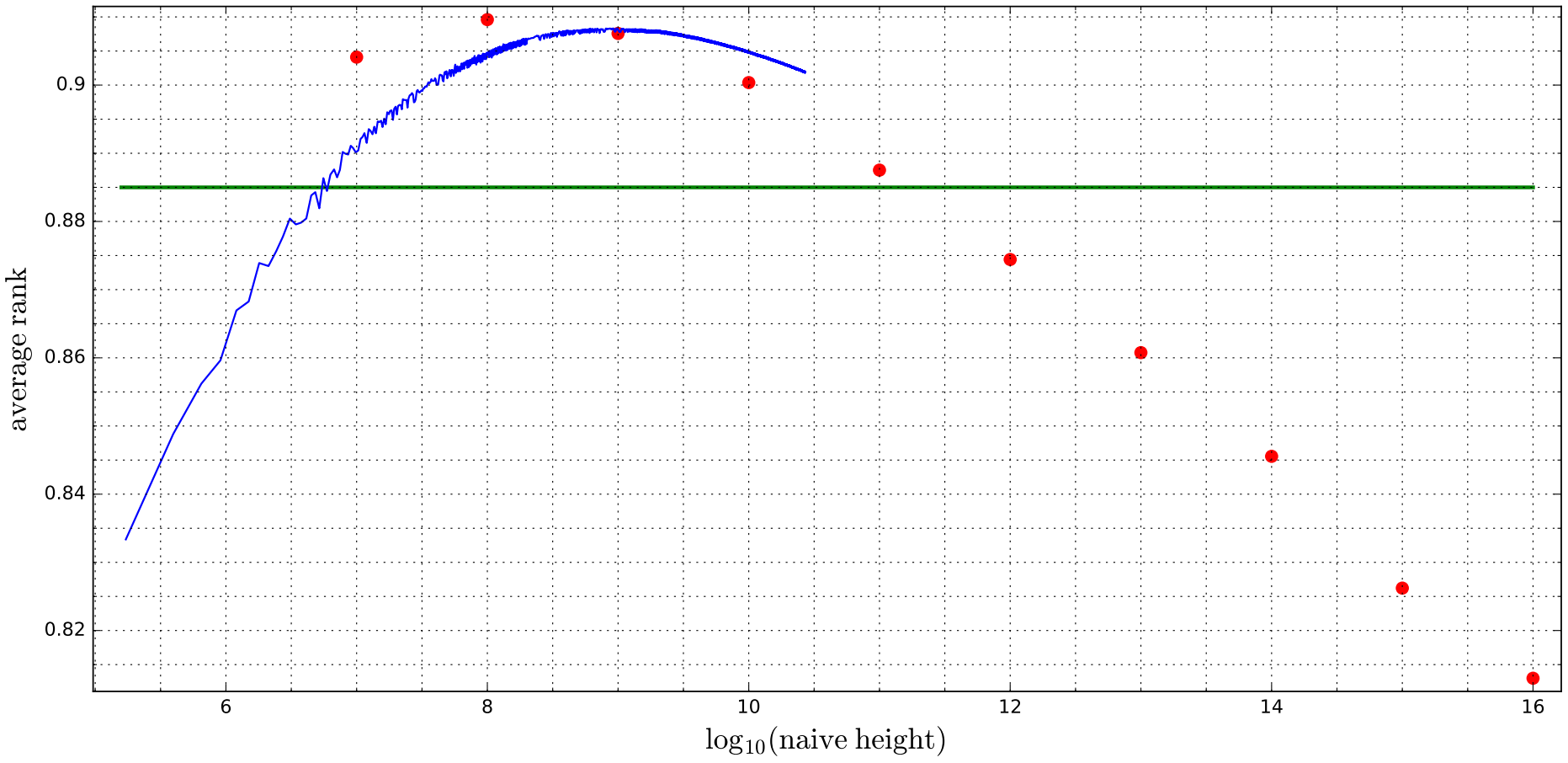


# Average rank ($\log_{10}$ scale)



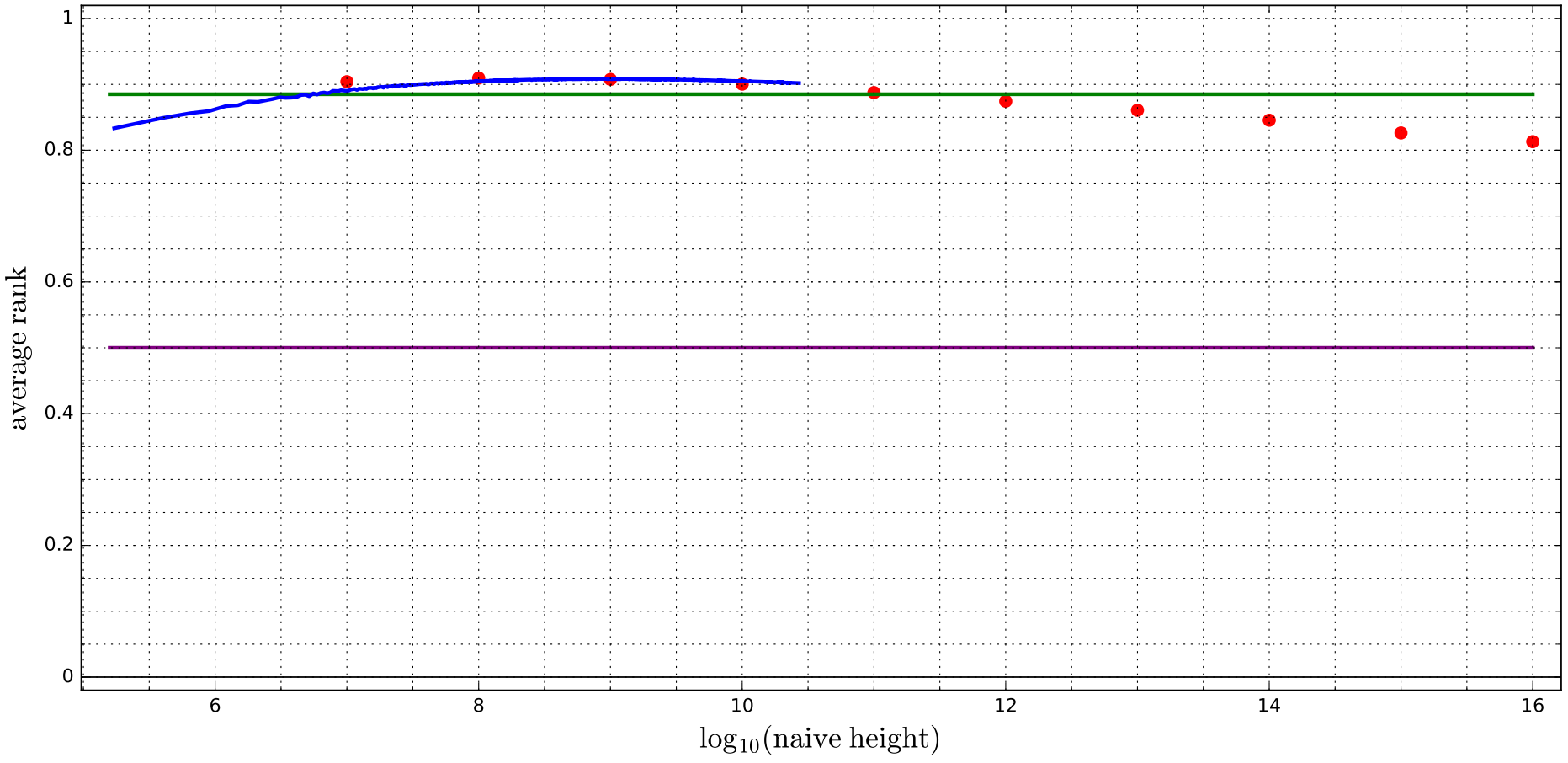

red points: samples of 100,000 curves

# Average rank ($\log_{10}$ scale)




running average versus average in a range/sample

# Average rank ($\log_{10}$ scale) – rescaled



# Distribution of rank

| Rank | # Curves | % Curves |
|---|---|---|
| 0 | 78,039,852 | 32.685% |
| 1 | 113,128,980 | 47.381% |
| 2 | 40,949,307 | 17.151% |
| 3 | 6,259,159 | 2.615% |
| 4 | 380,519 | 0.159% |
| 5 | 6481 | 0.003% |
| 6 | 12 | 0.0005% |
| total | 238,764,310 | |

average rank = 0.901976

# Why does average rank increase at first?

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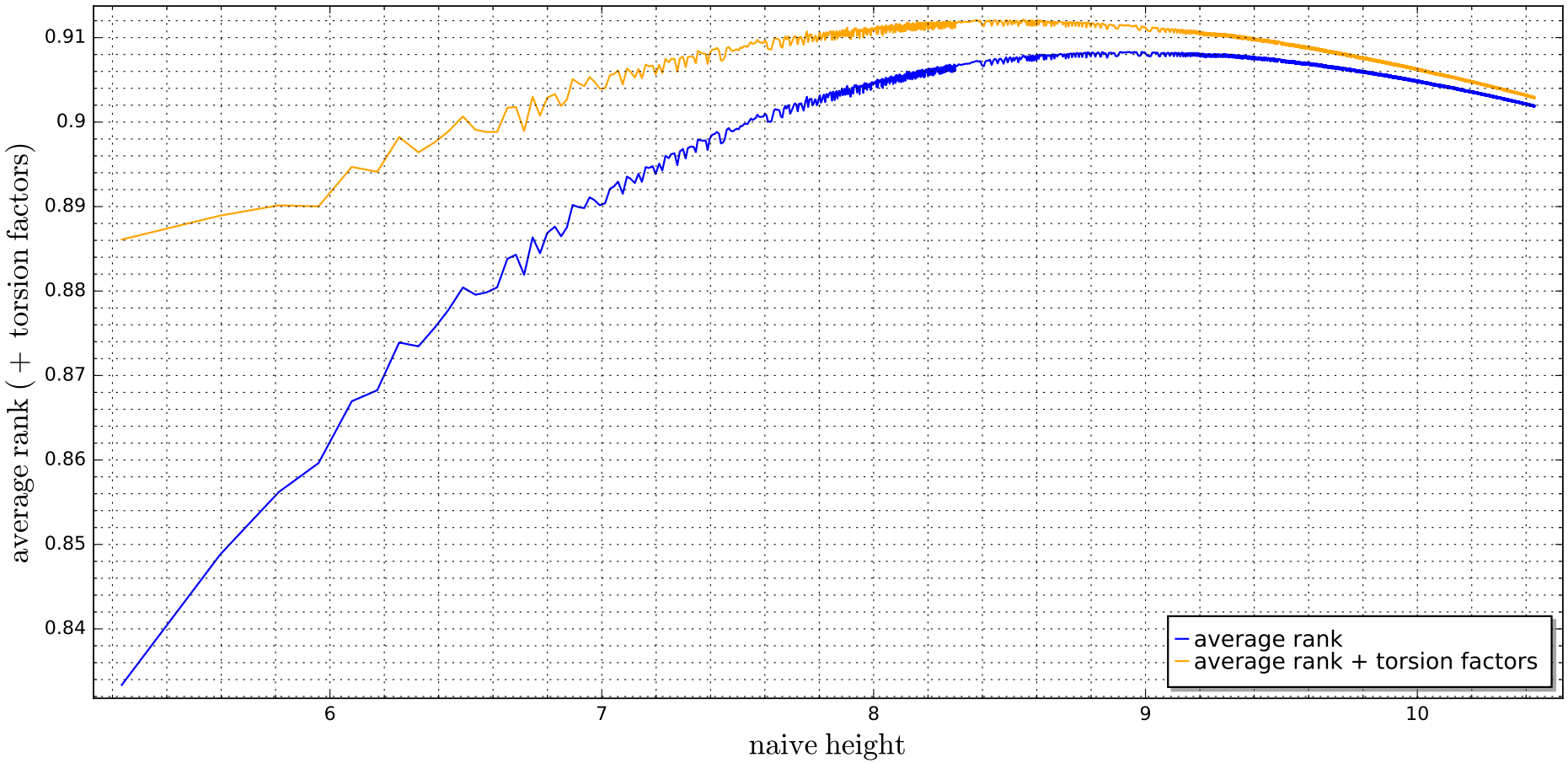
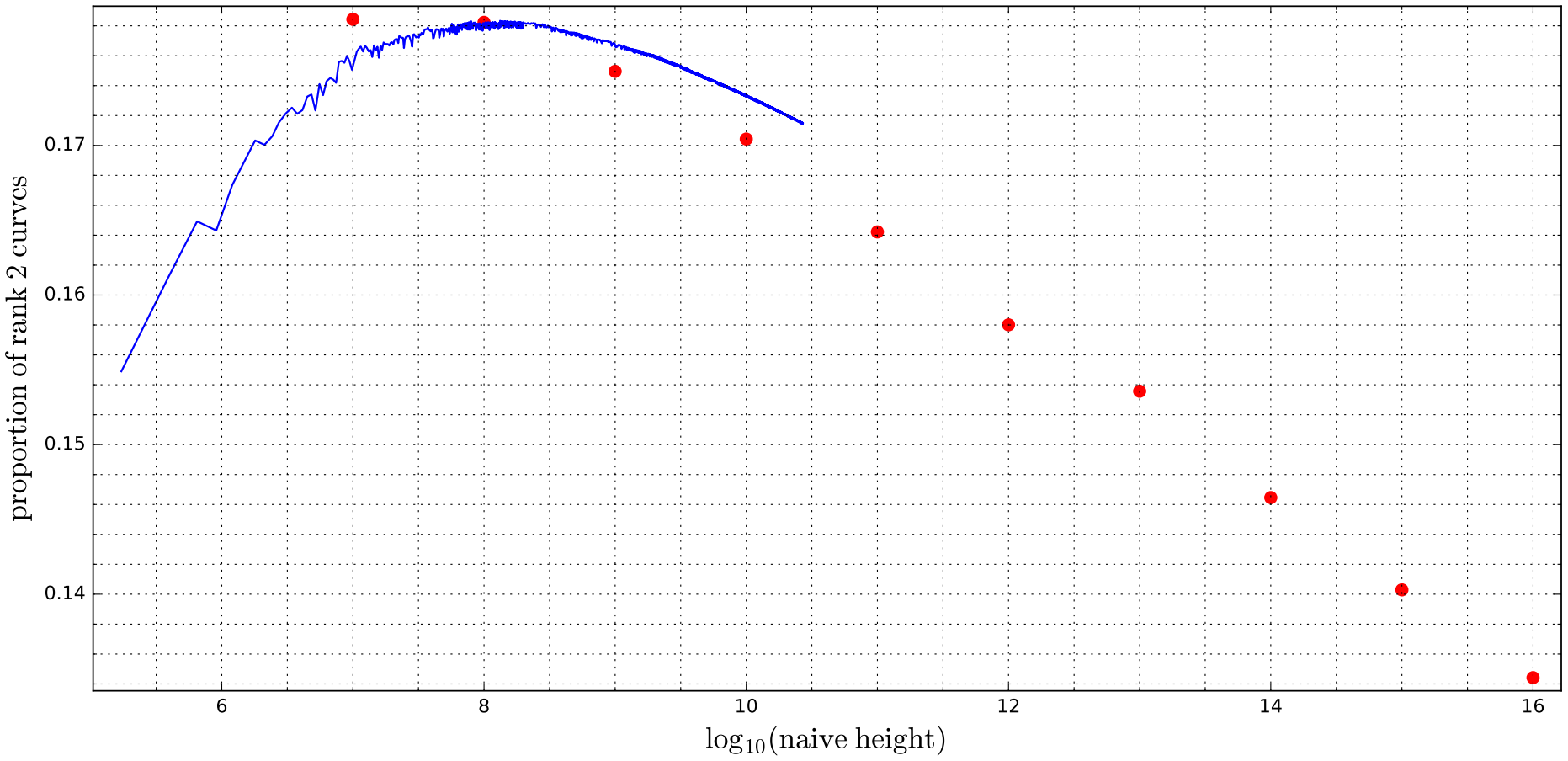
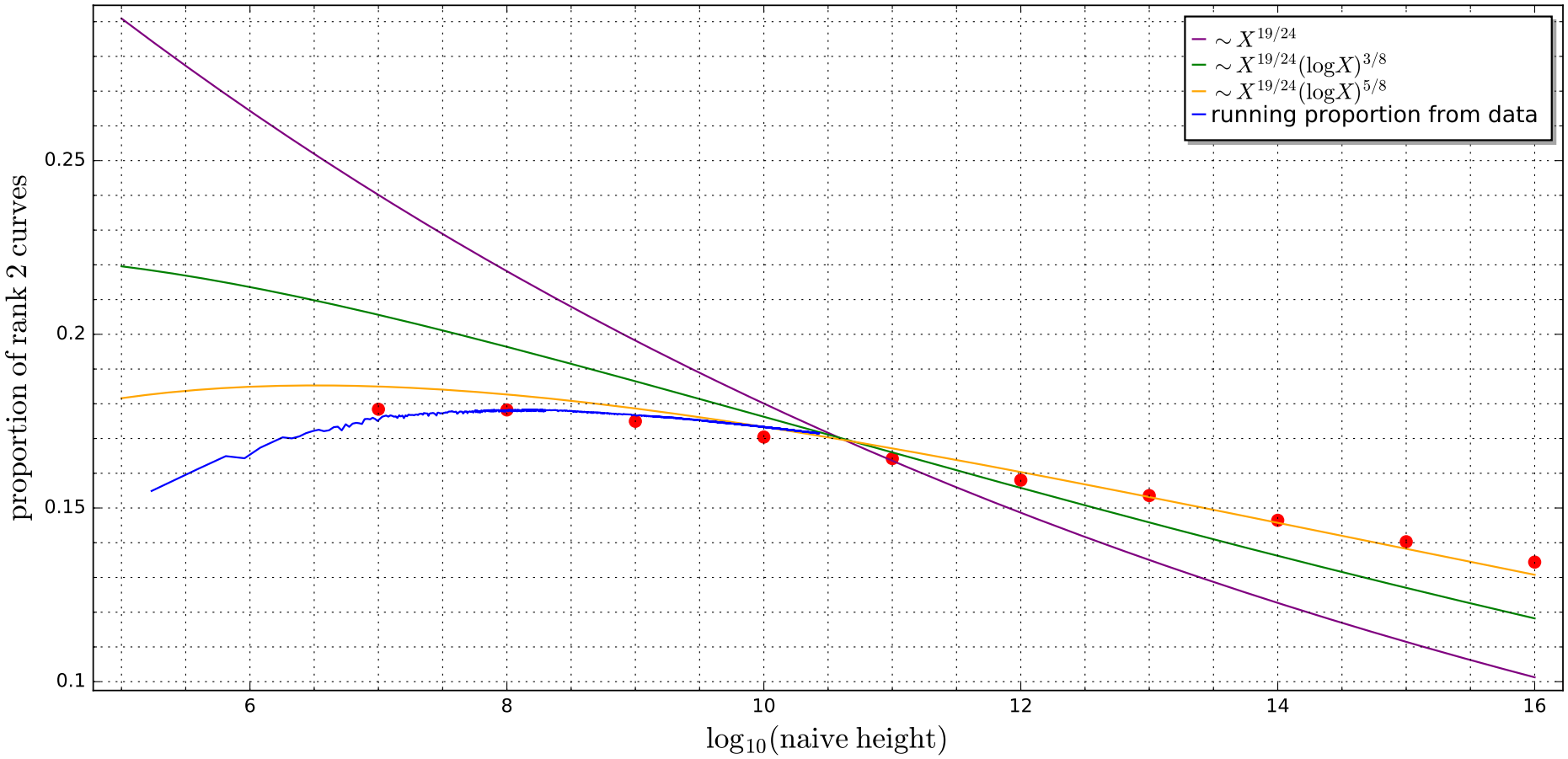


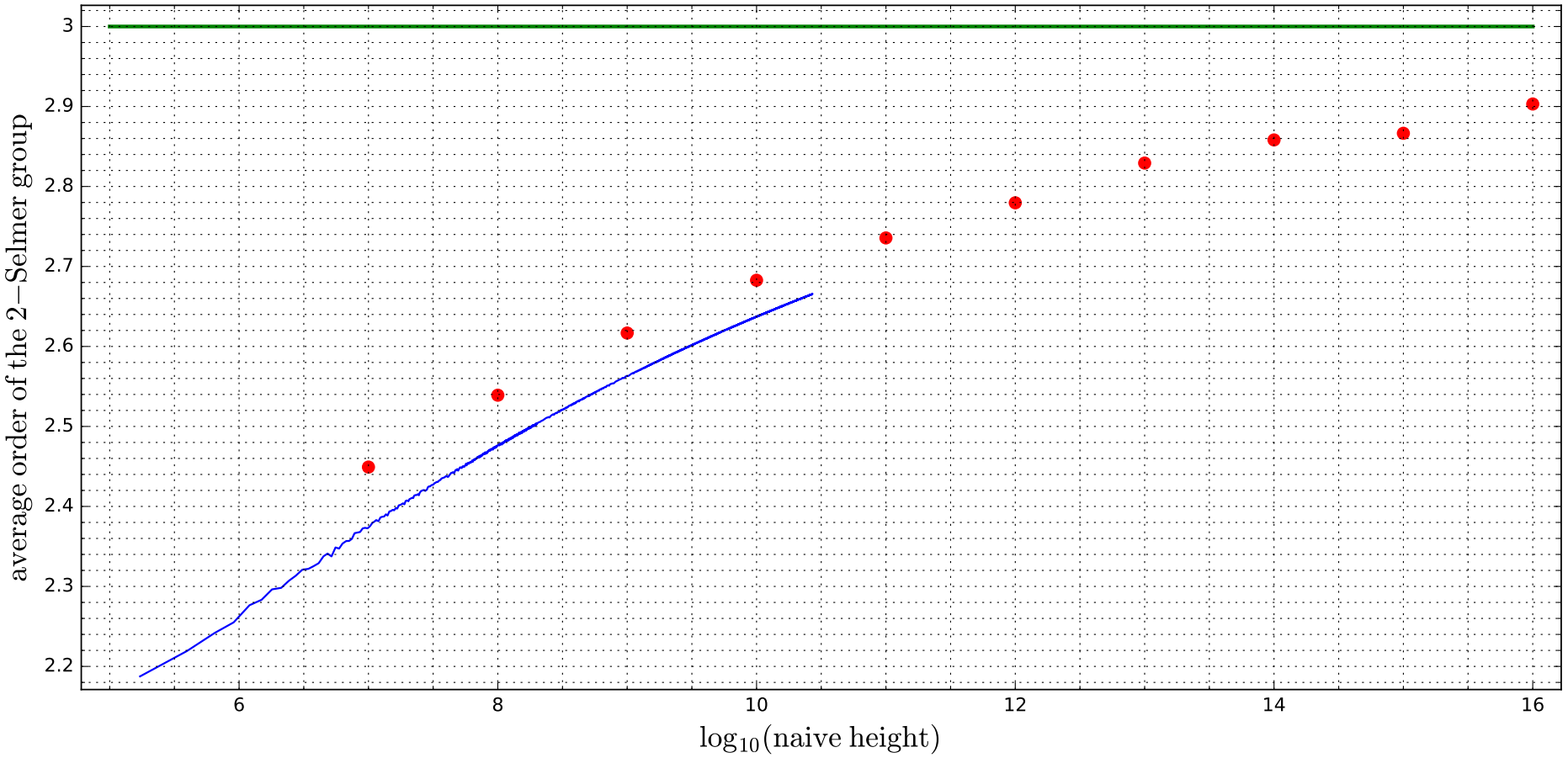
# Proportion of rank 2 curves



# Proportion of rank 2 curves: data versus conjecture



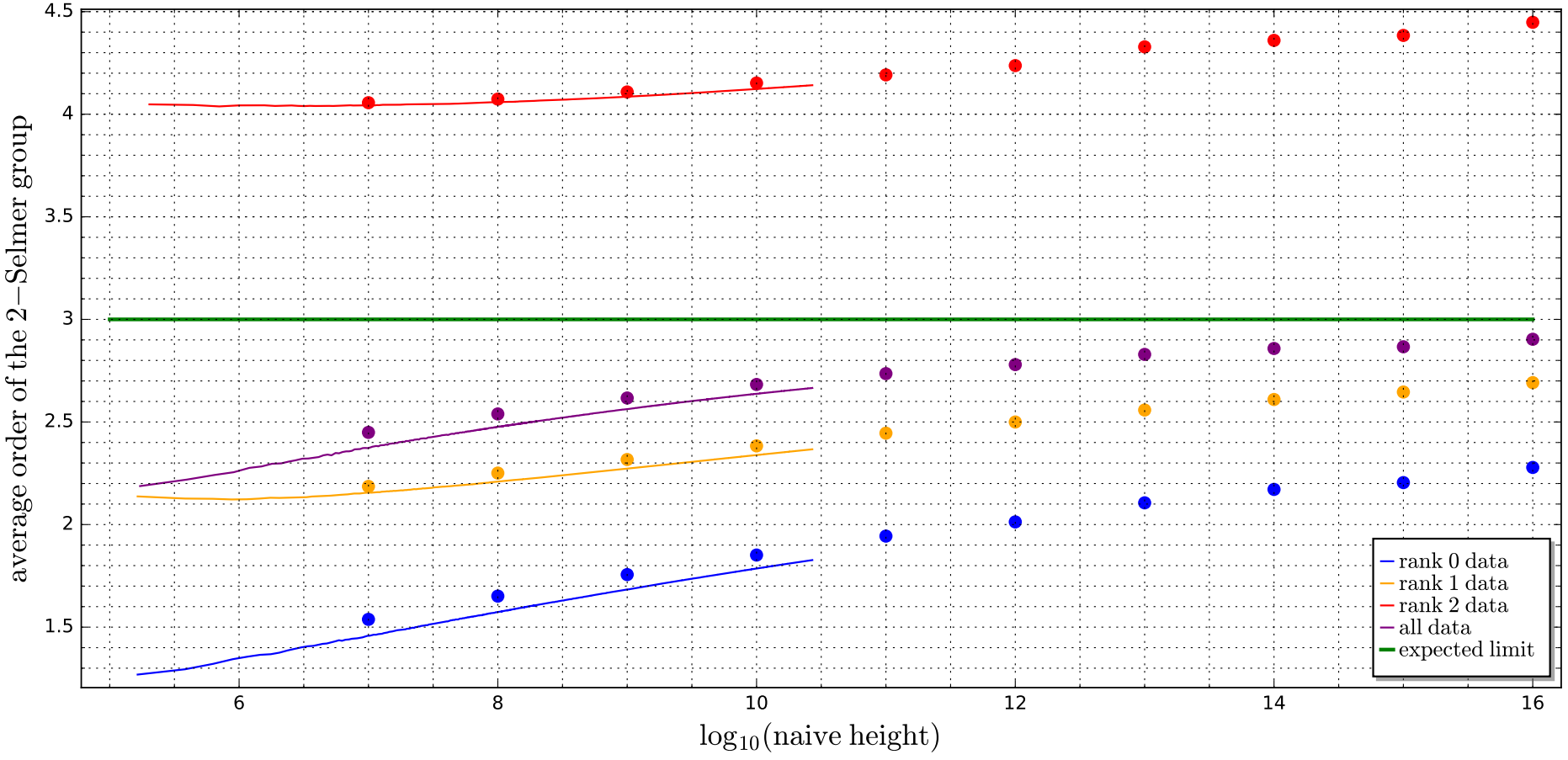
# Average order of the 2-Selmer group



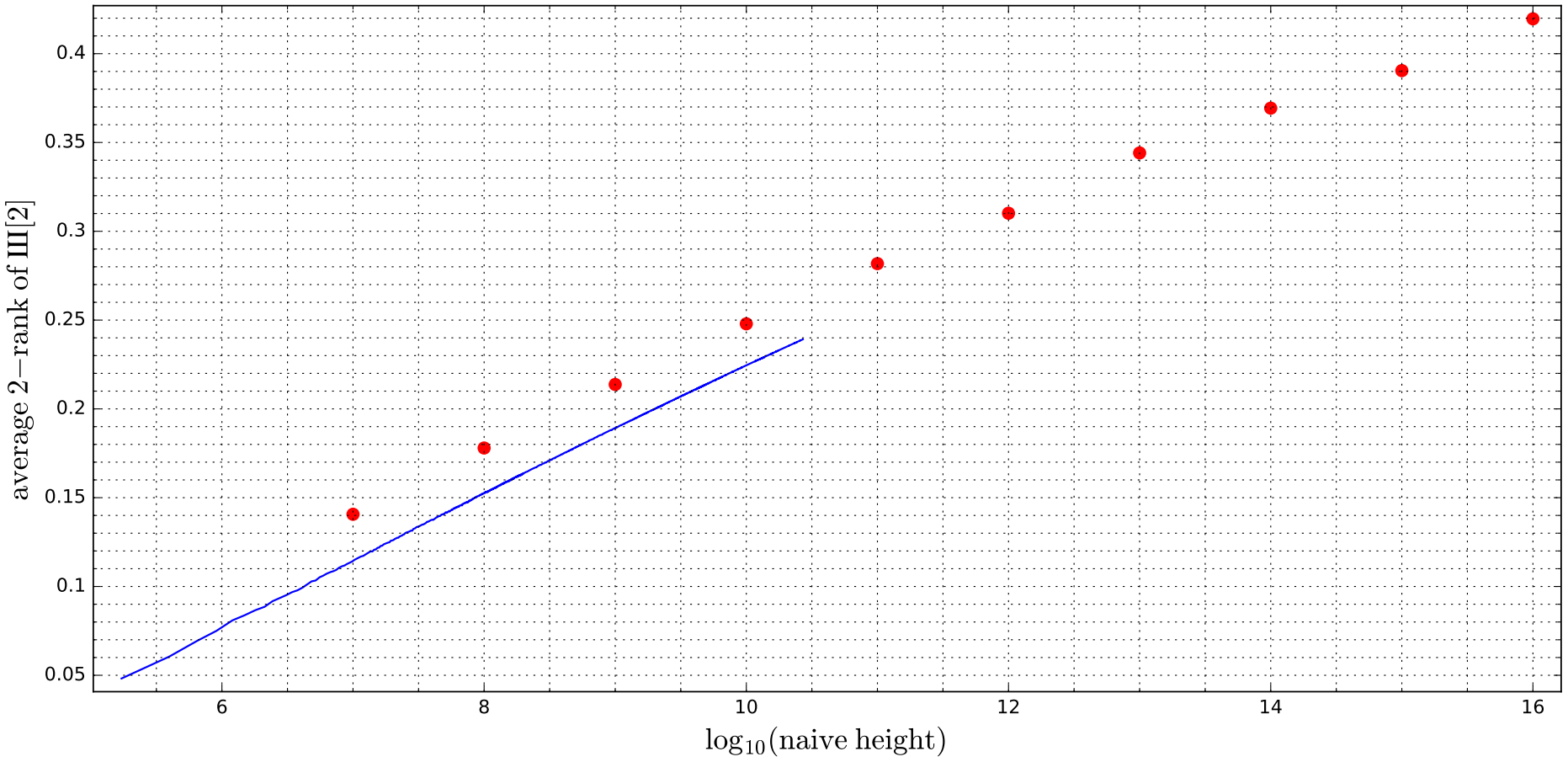
# Distribution of 2-Selmer groups: data versus heuristics

| 2-rank of $\mathrm{Sel}_2$ | predicted % [Poonen–Rains] | % of all curves ht $< 2.7 \cdot 10^{10}$ | % for rank 0 | % for rank 1 |
|---|---|---|---|---|
| 0 | 20.97 | 23.81 | 72.85 | 0 |
| 1 | 41.94 | 44.49 | 0.0865 | 93.83 |
| 2 | 27.96 | 25.78 | 26.93 | 0.098 |
| 3 | 7.989 | 5.511 | 0.0209 | 6.064 |
| 4 | 1.065 | 0.395 | 0.111 | 0.00351 |
| 5 | 0.0687 | 0.00643 | $1.41 \cdot 10^{-5}$ | 0.00173 |

# Average order of the 2-Selmer group, by rank



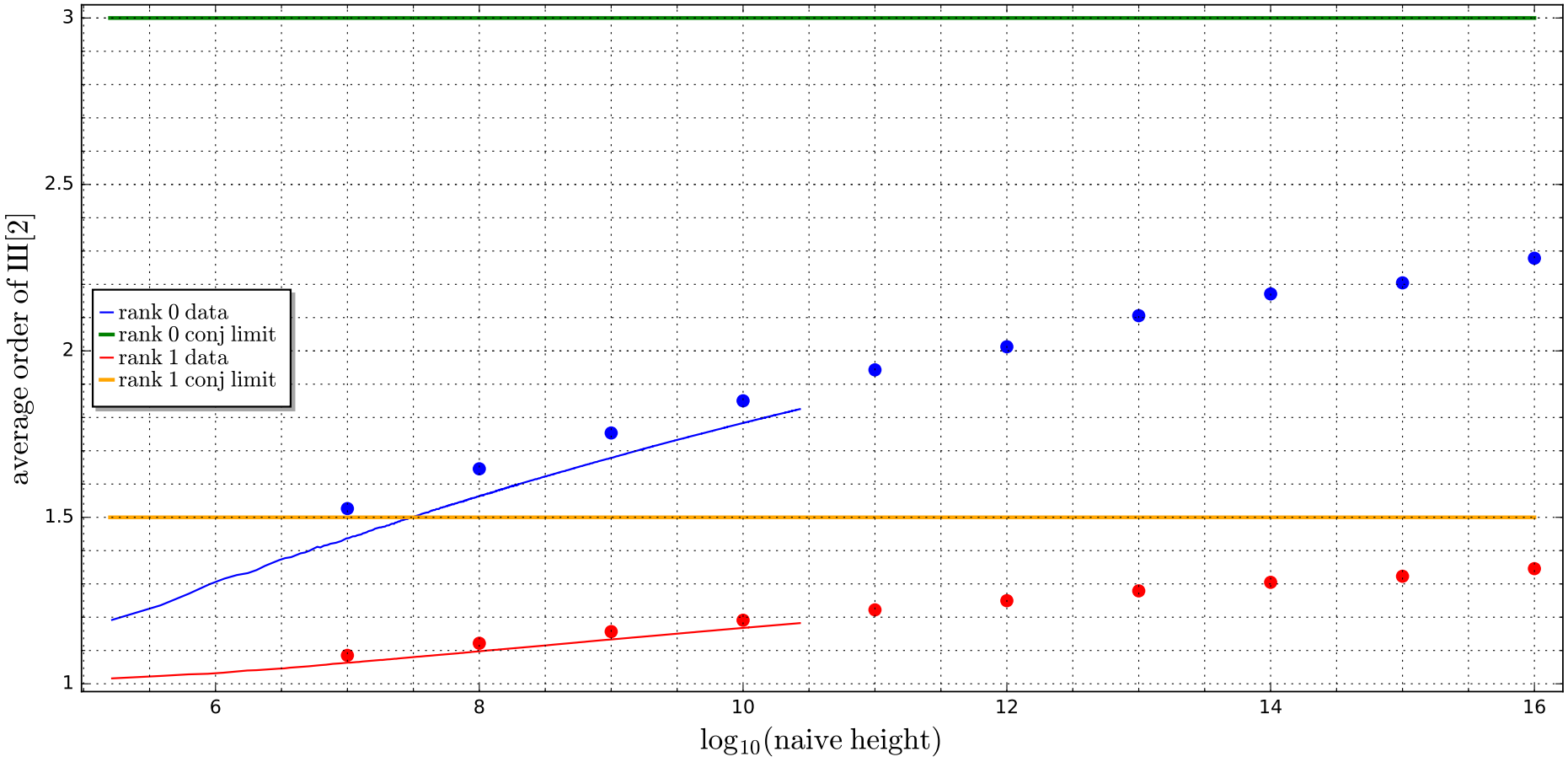
# Average 2-rank of Ш[2]



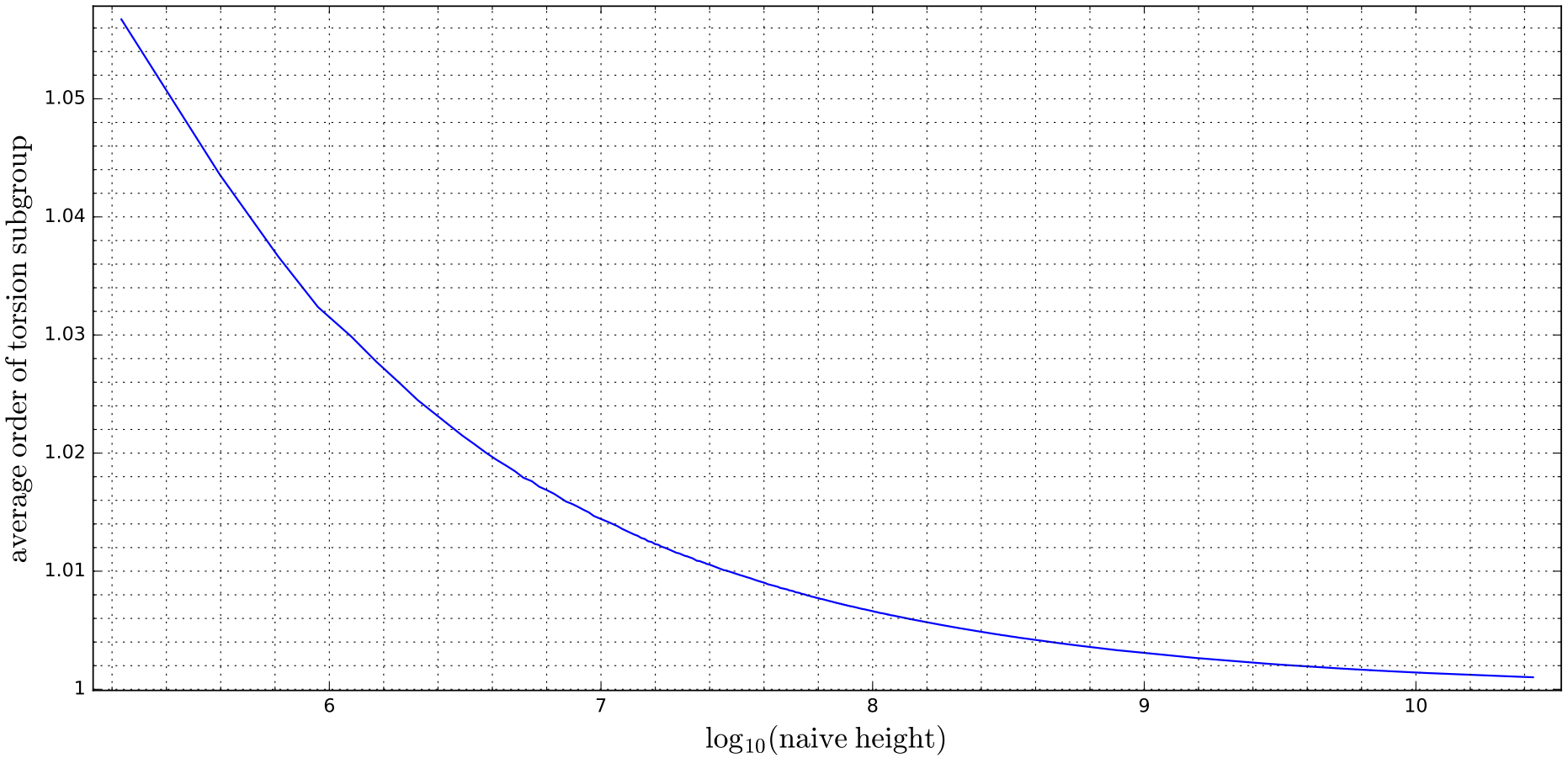
# Distribution of $Ш[2]$: data versus heuristics

| rank | 2-rank of $Ш[2]$ | predicted prop. [Delaunay] | proportion for all curves of height $< 2.7 \cdot 10^{10}$ |
|---|---|---|---|
| 0 | 0 | 0.4194 | 0.7294 |
| 0 | 2 | 0.5592 | 0.2695 |
| 0 | 4 | 0.0213 | 0.0011 |
| 1 | 0 | 0.8388 | 0.9393 |
| 1 | 2 | 0.1598 | 0.0607 |

# Average order of Ш[2]: rank 0 versus rank 1



# Average order of the torsion subgroup



# Distribution of torsion subgroups

| Torsion group | # Curves | Rank 0 | Rank 1 | Rank 2 | Rank $\geqslant 3$ |
|---|---|---|---|---|---|
| trivial | 238528817 (99.901%) | 0.327 | 0.474 | 0.172 | 0.028 |
| $\mathbb{Z}/2\mathbb{Z}$ | 233153 | 0.359 | 0.492 | 0.141 | 0.008 |
| $\mathbb{Z}/3\mathbb{Z}$ | 1020 | 0.463 | 0.466 | 0.072 | 0 |
| $\mathbb{Z}/4\mathbb{Z}$ | 257 | 0.521 | 0.463 | 0.016 | 0 |
| $\mathbb{Z}/6\mathbb{Z}$ | 23 | 0.870 | 0.130 | 0 | 0 |
| $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 1035 | 0.453 | 0.496 | 0.051 | 0 |
| $\mathbb{Z}/5\mathbb{Z}$ | 1 | 1 | 0 | 0 | 0 |
| $\mathbb{Z}/7\mathbb{Z}$ | 1 | 1 | 0 | 0 | 0 |
| $\mathbb{Z}/9\mathbb{Z}$ | 1 | 1 | 0 | 0 | 0 |
| $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ | 2 | 1 | 0 | 0 | 0 |

# Different properties

| Property | # Curves | Rk 0 | Rk 1 | Rk 2 | Rk $\geqslant 3$ |
|---|---|---|---|---|---|
| height $\leqslant 2.7 \cdot 10^{10}$ | 238764310 | 0.327 | 0.474 | 0.172 | 0.028 |
| conductor $\leqslant 10^8$ | 4908673 | 0.305 | 0.474 | 0.193 | 0.027 |
| $E$ has CM | 65732 | 0.328 | 0.474 | 0.170 | 0.028 |

# Different properties

| Property | # Curves | Rk 0 | Rk 1 | Rk 2 | Rk $\geqslant 3$ |
|---|---|---|---|---|---|
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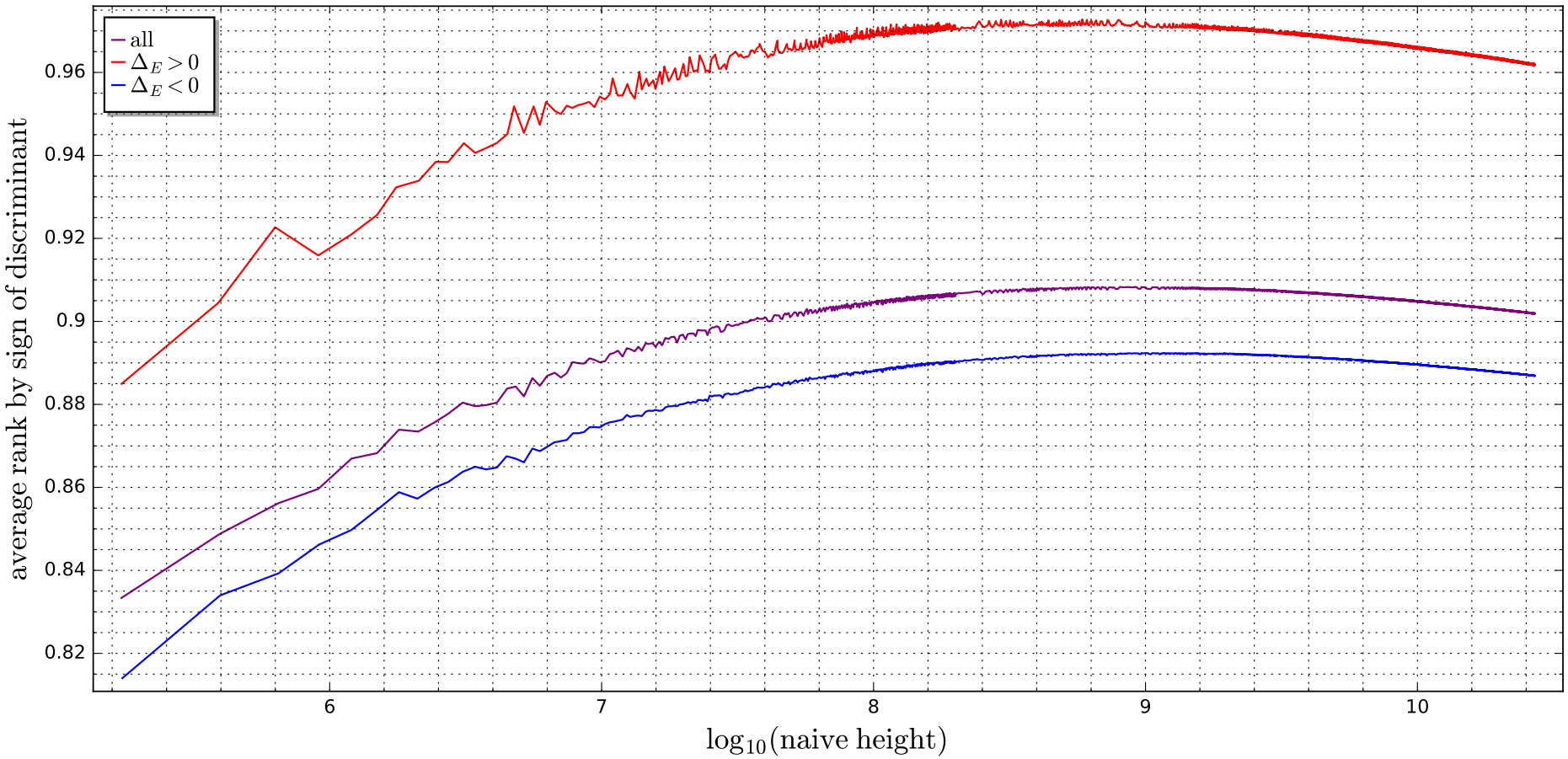
| conductor $\leqslant 10^8$ | 4908673 | 0.305 | 0.474 | 0.193 | 0.027 |
| $E$ has CM | 65732 | 0.328 | 0.474 | 0.170 | 0.028 |
| $\Delta_E > 0$ | 47738800 (19.994%) | 0.305 | 0.466 | 0.192 | 0.036 |
| $\Delta_E < 0$ | 191025510 (80.006%) | 0.332 | 0.476 | 0.166 | 0.026 |

# Average rank: positive versus negative discriminant



# Average order of the torsion subgroup: positive versus negative discriminant