Problems in hyperbolic dynamics - day 2

Current Trends in Dynamical Systems and the Mathematical Legacy of Rufus Bowen
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1 Billiard dynamics (Yuri Lima)

See Bowen's problem 17

Let $T \subset \mathbb{R}^2$ be a connected domain and define the billiard $f : \partial T \times [-\pi/2, \pi/2] \to T \times [-\pi/2, \pi/2]$, where the map is defined by elastic collision with the boundary. If the boundary is not smooth there can be discontinuities. Also, for certain boundaries one can have glancing orbits (orbits that are tangent to the boundary at a point of contact).

For a class of billiards called Sinai billiards a symbolic coding was obtained by Bunimovich, Chernov, and Sinai. This coding is well adapted for the Liouville measure given by $d\mu = \cos\theta dr d\theta$.

In a recent paper Lima and Matheus obtained a countable Markov partition for Bunimovich billiards. Examples of these billiards are pocket billiards, stadium billiards, and flower billiards. These have non-uniform hyperbolicity and the coding obtained works not only for Liouville measure, but also for adapted measures. Adapted measures are measures μ such that if D is the set of discontinuities for the billiard, then $\log d(x, D) \in L^1(\mu)$. Furthermore, if $\chi > 0$ the construction works for all χ-hyperbolic measures simultaneously, where an ergodic measure is χ-hyerbolic if none of the norms of the Lyapunov exponents lie in the interval $(-\chi, \chi)$.

Problem. *For the Bunimovich billiards can we answer the following:*

1. *Do these systems have finite topological entropy?*

2. *Do these systems have a measure of maximal entropy?*
3. *If there is a measure of maximal entropy for these systems, then will the measure be adapted? What boundary conditions will guarantee the measures of maximal entropy are adapted?*

2 Structure of basic sets (Aaron Brown)

See Bowen's problems 7 and 112

In general it is very hard to try to classify the possible structures of basic sets. The most success has been done classifying hyperbolic attractors. If a hyperbolic attractor has codimension-1, then the dynamics on the basin of attraction for the attractor is conjugate to the dynamics of a punctured DA-diffeomorphism. If a hyperbolic attractor is 1-dimensional, then the attractor can be classified as a solenoid over a branched manifold. If the dimension of the unstable splitting is 1-dimensional, but the attractor is higher dimensional, then Brown has obtained a classification. In particular, the above classify all hyperbolic attractors for 3-manifolds.

Problem. *Can we classify larger classes of basic sets.*

1. *What is the classification of hyperbolic attractors when* $\dim E^u = 2$*?*
2. *What are the possible basic sets for 3-manifolds? A starting point could be to examine all basic sets that are not a hyperbolic attractor or repeller, but have topological dimension 1.*
3. *Does there exist a transitive Smale space that is not topologically conjugate to a basic set?*

Problem. (suggested by Sylvain Crovisier). *Can we classify the topology of robustly transitive partially hyperbolic attractors in dimension* 3*?*

3 Extension of Herman-Yoccoz Theory (Konstantin Khanin)

See Bowen's problem 96.

Let T be a C^∞ diffeomorphism of S^1 with rotation number $\rho(T)$ that is Diophantine. Then T is conjugate to the rotation by $\rho(T)$ by a C^∞-diffeomorphism.

We want to see if a result as above can be extended to higher dimension. So let T be a C^∞ diffeomorphism of a torus $\mathbb{T}^n$ and assume that T is minimal and Diophantine (so there exists $C > 0$ and $\delta > 1$ such that for all $x \in M$ and all $n \geq 0$ we have $|T^n x - x| \geq C/n^\delta$).

Problem. *Let S be a C^∞ diffeomorphism such that $T = \varphi^{-1} \circ S \circ \varphi$ where φ is a homeomorphism. Can we conclude that φ is C^∞? What if φ is Hölder continuous?*

4 Develop new machinery for computer assisted estimates (Michael Jakobson)

For the map $x \mapsto x^2 - a$, Luzzatto and Takahashi estimated the measure of stochastic parameters a on the interval $[2-\Delta, 2]$ when $\Delta = 10^{-4990}$) and showed that

$$\frac{Lebesgue(\{a : a \text{ is stochastic}\})}{\Delta} > .97.$$

Yu-Ru Huang has obtained numerical estimates in her PhD thesis *Estimating measure of stochastic parameters nonadjacent to the Chebyshev value*, University of Maryland, 2011, posted on arXiv. For quadratic family $x \mapsto ax(1-x)$ she proved that the measure of stochastic parameters a on the interval of parameter $[3.99512, 3.99513]$ (nonadjacent to 4) is greater than 10^{-15}.

Problem. *Find improved machinery for computer assisted estimates to obtain good estimates for larger intervals of parameter. For instance, lower and upper bounds on the density of stochasticity on the interval* $[3.95, 3.99]$.