PIMS Workshop on Arithmetic Topology

Representation stability and asymptotic stability of factorization statistics

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Arithmetic	**Topology**
factorization statistics of $\mathcal{Y}_n(q) = \mathcal{Y}_n(\overline{\mathbb{F}}_q)^{\mathrm{Fr}_q}$	$\mathcal{W}_n \curvearrowright H^*(\mathcal{X}_n; \mathbb{C})$
factorization statistics of polynomials over $\mathbb{F}_q$	cohomology of hyperplane complements
‘factorization’ statistics of Fr_q-stable maximal tori in algebraic groups	cohomology of flag varieties
asypmtotic stability of polynomial factorization statistics	representation/multiplicity stability

Hyperplane complements of type $\mathcal{W}_n$

Type A_{n-1}	Symmetric group $S_n \curvearrowright \{1, \ldots, n\}$	Permutation matrices $n \times n$
Type B_n/C_n	Hyperocthahedral group $B_n \curvearrowright \{\pm 1, \ldots, \pm n\}$	Signed permutation matrices $n \times n$

$\mathcal{W}_n \curvearrowright \mathbb{R}^n$ by (signed) permutation matrices

$$\mathcal{X}_n = \mathcal{M}_{\mathcal{W}_\mathbf{n}}(\mathbb{C}) := \mathbb{C}^n \setminus \text{complexified reflection hyperplanes}$$

$$\mathcal{W}_n \curvearrowright \mathcal{M}_{\mathcal{W}_n}(\mathbb{C}) \text{ freely}$$

Hyperplane complements and polynomials

$\mathcal{W}_{\mathbf{n}}$	S_n	B_n
$\mathcal{M}_{\mathcal{W}_{\mathbf{n}}}(\mathbb{C})$	$\mathbb{C}^n \backslash \{z_i - z_j = 0\}$ $\mathcal{M}_{S_n}(\mathbb{C}) = \mathrm{PConf}_n(\mathbb{C})$	$\mathbb{C}^n \backslash \{z_i \pm z_j = 0, z_i = 0\}$ $\mathcal{M}_{B_n}(\mathbb{C})$
$(\mathcal{M}_{\mathcal{W}_{\mathbf{n}}}/\mathcal{W}_{\mathbf{n}})(\mathbb{C})$ $\mathcal{Y}_{\mathcal{W}_n}(\mathbb{C})$	$\left\{\{z_1, \ldots, z_n\} : z_i \in \mathbb{C}\right\}$ $\mathcal{Y}_{S_n}(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})$	$\left\{\{\pm z_1, \ldots, \pm z_n\} : z_i \in \mathbb{C}^\times\right\}$ $\mathcal{Y}_{B_n}(\mathbb{C})$
Space of polynomials	$\{(x - z_1)\cdots(x - z_n) : z_i \neq z_j\}$	$\{(x - z_1^2)\cdots(x - z_n^2) : z_i^2 \neq z_j^2,$ $z_i \neq 0\}$

Hyperplane complements and polynomials

$\mathcal{Y}_{S_n}$ and $\mathcal{Y}_{B_n}$ are algebraic varieties defined over $\mathbb{Z}$

$$\mathcal{P}oly_n(\mathbb{K}) := \{f \in \mathbb{K}[x] \colon f \text{ is monic of degree } n\} \quad \text{for a field } \mathbb{K}$$

$\mathcal{Y}_{\mathcal{W}_n}(\mathbb{K}) =$ the $\mathbb{K}$-points of $\mathcal{Y}_{\mathcal{W}_n}$

$$\mathcal{Y}_{S_n}(\mathbb{K}) = \{f \in \mathcal{P}oly_n(\mathbb{K}) \text{ with no repeated roots }\}$$

$$\mathcal{Y}_{B_n}(\mathbb{K}) = \{f \in \mathcal{P}oly_n(\mathbb{K}) \text{ with no repeated roots and } f(0) \neq 0\}$$

$$\mathbb{K} = \mathbb{F}_q, \overline{\mathbb{F}}_q \quad \text{v.s.} \quad \mathbb{K} = \mathbb{C}$$

$$\mathcal{Y}_n(q) := \mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)$$

Factorization type

$f \in \mathcal{P}oly_n(\mathbb{F}_q)$ $\rightsquigarrow$ **factorization type** of f

$\lambda_f \vdash n$ degrees of irreducible factors of $f(x)$

$x^3(x^2+1) \in \mathcal{P}oly_5(\mathbb{F}_3)$ $\rightsquigarrow$ $(1^3\ 2^1) \vdash 5$

$(x+1)(x-1)(x^3-x+1) \in \mathcal{P}oly_5(\mathbb{F}_3)$ $\rightsquigarrow$ $(1^2\ 3^1) \vdash 5$

Remarks:

- $\mathrm{Fr}_q \curvearrowright \mathcal{Y}_{\mathcal{W}_n}(\overline{\mathbb{F}}_q)$ and $\mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q) = \left(\mathcal{Y}_{\mathcal{W}_n}(\overline{\mathbb{F}}_q)\right)^{\mathrm{Fr}_q}$
- factorization types given are given by Fr_q-orbits:
  Type A: $\mathrm{Fr}_q \curvearrowright \{x \in \overline{\mathbb{F}}_q : f(x) = 0\} \rightsquigarrow \lambda_f$
  Type B/C: $\mathrm{Fr}_q \curvearrowright \{x \in \overline{\mathbb{F}}_q : f(x^2) = 0\} \rightsquigarrow (\lambda_f^+, \lambda_f^-)$

$(x^2-1) = (x-1)(x+1) \in \mathcal{Y}_{S_2}(\mathbb{F}_3)$ $\rightsquigarrow$ $(1^2) \vdash 2$

$(x^2-1) = (x-1)(x+1) \in \mathcal{Y}_{B_2}(\mathbb{F}_3)$ $\rightsquigarrow$ $(1^1, 1^1)$ double partition of 2

Factorization statistics for $\mathcal{P}oly_n(\mathbb{F}_q)$ and $\mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)$

A **factorization statistic**

$$P : \mathcal{P}oly_n(\mathbb{F}_q) \to \mathbb{Q} \text{ s.t. } P(f) \text{ only depends on } \lambda_f$$

- $R(f) = \#$ roots of f in $\mathbb{F}_q$
- $X_k(f) = n_k(\lambda_f) = \#$ degree-k irreducible factors of f over $\mathbb{F}_q$

  $X_1(g) = R(g) = 3$ for $g = x(x^2-1)(x^2+1) \in \mathcal{Y}_{S_5}(\mathbb{F}_3) \subset \mathcal{P}oly_5(\mathbb{F}_3)$

- For $f \in \mathcal{Y}_{B_n}(\mathbb{F}_q)$
  $X_k^+(f) = n_k(\lambda_f^+) = \#$ degree-k QR irred factors of f over $\mathbb{F}_q$
  $X_k^-(f) = n_k(\lambda_f^-) = \#$ degree-k NQR irred factors of f over $\mathbb{F}_q$

  $X_1^+(f) = 1$ and $X_1^-(f) = 1$ for $f = (x-1)(x+1) \in \mathcal{Y}_{B2}(\mathbb{F}_3)$

We focus on *polynomial* factorization statistics:

$$P \in \mathbb{Q}[X_1, X_2, \ldots] \quad \text{or} \quad P \in \mathbb{Q}[X_1^+, X_1^-, X_2^+, X_2^- \ldots]$$

$$P : \bigsqcup_{n \geq 1} \mathcal{W}_n \longrightarrow \mathbb{Q} \quad \textbf{character polynomial}$$

Factorization statistics and hyperplane complements

Theorem (Twisted Grothendieck–Lefschetz trace formula)

Let P be a factorization statistic defined over $\mathcal{Y}_n(q)$

$$\frac{1}{q^n} \sum_{f \in \mathcal{Y}_n(q)} P(f) = \sum_{k \geq 0} (-1)^k \frac{\langle P, \psi_n^k \rangle_{\mathcal{W}_n}}{q^k}$$

Arithmetic: $P : \mathcal{Y}_n(q) \to \mathbb{Q}$
factorization statistic for
$\mathcal{Y}_n(q) = \mathcal{Y}_{\mathcal{W}_n}(\overline{\mathbb{F}}_q)^{\mathrm{Fr}_q} \subset \mathcal{P}oly_n(\mathbb{F}_q)$

Topology: $\mathcal{W}_n \curvearrowright H^*(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$
ψ_n^k is the $\mathcal{W}_n$-character of $H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$
$\langle \cdot, \cdot \rangle_{\mathcal{W}_n}$ the standard inner product of $\mathcal{W}_n$-class functions

- Follows from results of Grothendieck, Artin, Lehrer and Kim

  Type A: Church–Ellenberg–Farb (2014)

  Type B/C: J.R.–Wilson, Casto, Matei (2017)

Factorization statistics and hyperplane complements

Example: counting polynomials

$P(\sigma) = 1$ for all $\sigma \in \mathcal{W}_n$

$$\frac{1}{q^n} \sum_{f \in \mathcal{W}_n(\mathbb{F}_q)} 1 = \sum_{k=0}^{n} (-1)^k q^{-k} \dim_{\mathbb{C}} H^k(\mathcal{Y}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$$

$$|\mathcal{Y}_{\mathcal{W}_n}(\mathbb{F}_q)| = q^n P_{\mathcal{Y}_{\mathcal{W}_n(\mathbb{C})}}(-q^{-1})$$

Arithmetic:

$$|\mathrm{Conf}_n(\mathbb{F}_q)| = q^n - q^{n-1}$$

$$|\mathcal{Y}_{B_n}(\mathbb{F}_q)| = q^n - 2q^{n-1} + 2q^{n-2} - \dots$$

Topology:

$$H^k(\mathrm{Conf}_n(\mathbb{C})) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

(Arnol'd, F.Cohen)

$$H^k(\mathcal{Y}_{B_n}(\mathbb{C})) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q}^2 & 0 < k < n \\ \mathbb{Q} & k = n \\ 0 & k > 2 \end{cases}$$

(Brieskorn, Lehrer)

Unrestricted factorization statistics

Theorem (Hyde 2017)

Let P be a factorization statistic defined over $\mathcal{P}oly_n(\mathbb{F}_q)$

$$\frac{1}{q^n} \sum_{f \in \mathcal{P}oly_n(\mathbb{F}_q)} P(f) = \sum_{k \geq 0} \frac{\langle P, \phi_n^k \rangle_{\mathcal{S}_n}}{q^k}$$

Arithmetic: P is a factorization statistics for $\mathcal{P}oly_n(\mathbb{F}_q)$

Topology: ϕ_n^k is the $\mathcal{S}_n$-character of $H^{2k}(\mathrm{PConf}_n(\mathbb{R}^3); \mathbb{C})$

- His proof uses a splitting measure interpretation.
- Same approach recovers type A formula for $\mathcal{Y}_{S_n}(\mathbb{F}_q)$. *Bonus:* gives an efficient, direct way to compute the characters ψ_n^k and ϕ_n^k.

Questions:

- Can this approach be used to recover the type B/C formula for $\mathcal{Y}_{B_n}(\mathbb{F}_q)$?

- **(Hyde)** Is there a geometric interpretation of the connection between factorization statistics on $\mathcal{P}oly_n(\mathbb{F}_q)$ and the cohomology of $\mathrm{PConf}_n(\mathbb{R}^3)$?

Statistics for Fr_q-stable maximal tori and flag varieties

Theorem (Lehrer)

Let P be a factorization statistic defined over $\mathcal{Y}_n(q)$

$$\frac{1}{|\mathcal{Y}_n(q)|}\sum_{T\in\mathcal{Y}_n(q)} P(T) = \sum_{k\geq 0}(-1)^k \frac{\langle P, \psi_n^k\rangle_{\mathcal{W}_n}}{q^k}$$

Arithmetic: P is a 'factorization' statistic for $\mathcal{Y}_n(q) = \{\mathrm{Fr}_q\text{-stable maximal tori in } \mathbf{G}_n(\overline{\mathbb{F}}_q)\}$

Topology: $\mathcal{W}_n \curvearrowright H^*(\mathcal{F}\mathbf{G}_n;\mathbb{C})$
ψ_n^k is the $\mathcal{W}_n$-character of $H^{2k}(\mathcal{F}\mathbf{G}_n;\mathbb{C})$
$\langle\cdot,\cdot\rangle_{\mathcal{W}_n}$ the standard inner product of $\mathcal{W}_n$-class functions

- (Steinberg): $|\mathcal{Y}_n(q)| = q^{n^2-n}$ if $\mathbf{G}_n = \mathrm{GL}_n$
  $|\mathcal{Y}_n(q)| = q^{2n^2}$ if $\mathbf{G}_n = \mathrm{Sp}_{2n}$ (or $\mathbf{G}_n = \mathrm{SO}_{2n+1}$)
- Can be obtain from a Twisted Grothendieck–Lefschetz trace formula (CEF 2014)
  Type A: Church–Ellenberg–Farb (2014)
  Type B/C: J.R.–Wilson, Casto, Matei (2017)

Families of algebraic groups

Algebraic groups	$\mathbf{G}_n$	**Weyl group** $\mathcal{W}_\mathbf{n}$	**Type**
general linear groups	GL_n	S_n (symmetric group)	A
special orthogonal groups	SO_{2n+1}	B_n (hyperoctahedral group)	B
symplectic groups	Sp_{2n}	B_n (hyperoctahedral group)	C

$\mathbf{G}_n$ is defined over a field $\mathbb{K} = \mathbb{F}_q, \overline{\mathbb{F}}_q, \mathbb{C}$

$\mathbf{G}_n(\mathbb{K}) = \mathbb{K}$-points of the algebraic group

Fr_q-stable maximal tori in $\mathbf{G}_n(\overline{\mathbb{F}}_q)$

T torus in $\mathbf{G}_n(\mathbb{K})$: subgroup $\mathbb{K}$-isom. to a product of $\mathrm{GL}_1(\mathbb{K})$

T maximal torus: if T' is a torus and $T \subseteq T'$, then $T = T'$

When $\mathbb{K} = \overline{\mathbb{F}}_q$

T ***Fr_q*****-stable:** T is stable under the *Frobenius action*:

$$Fr_q : \mathbf{G}_n(\overline{\mathbb{F}}_q) \to \mathbf{G}_n(\overline{\mathbb{F}}_q) \quad \text{given by} \quad (x_{i,j}) \mapsto (x_{i,j}^q)$$

Examples:

$T_0 = \{diag\{\lambda_1, \ldots, \lambda_n\} : \lambda_i \in \overline{\mathbb{F}}_q^\times\} \subseteq \mathrm{GL}_n(\overline{\mathbb{F}}_q)$

$T_0 = \{diag\{1, \lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}\} : \lambda_i \in \overline{\mathbb{F}}_q^\times\} \subseteq \mathrm{SO}_{2n+1}(\overline{\mathbb{F}}_q)$

$T_0 = \{diag\{\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}\} : \lambda_i \in \overline{\mathbb{F}}_q^\times\} \subseteq \mathrm{Sp}_{2n}(\overline{\mathbb{F}}_q)$

$$\mathcal{Y}_n(q) := \{T \leq \mathbf{G}_n(\overline{\mathbb{F}}_q) : \; T \text{ is a } Fr_q\text{-stable maximal torus}\}$$

‘Factorization’ statistics for Fr_q-stable maximal tori

$T \in \mathcal{Y}_n(q) \rightsquigarrow w_T \in \mathcal{W}_n \rightsquigarrow$ ’factorization type’ of T:

$$\lambda_T \vdash n \quad \text{or} \quad (\lambda_T^+, \lambda_T^-)$$

‘Factorization’ statistic: $P : \mathcal{Y}_n(q) \longrightarrow \mathbb{Q}$

Type A	$X_r(T) = n_r(\lambda_T) =$ # of r-dimensional Fr_q-stable subtori of T irreducible over $\mathbb{F}_q$
Type B/C	$X_r(T) = n_r(\lambda_T^+) =$ # of r-dimensional Fr_q-stable subtori of T irreducible over $\mathbb{F}_q$ that split over $\mathbb{F}_{q^r}$ $Y_r(T) = n_r(\lambda_T^-) =$ # of r-dimensional Fr_q-stable subtori of T irreducible over $\mathbb{F}_q$ that **do not** split over $\mathbb{F}_{q^r}$

A **split** Fr_q-stable maximal torus:

$$T_0 = \left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{bmatrix} \middle| \lambda_1 \in \overline{\mathbb{F}}_p^\times \right\}$$

$T_0 \rightsquigarrow (1)(\bar{1}) \in B_1$

$X_1(T_0) = 1;\ Y_1(T_0) = 0$

For $\epsilon \in \mathbb{F}_q^\times$ not a square in $\mathbb{F}_q$ a **non-split** torus

$$T_\epsilon = \left\{ \begin{bmatrix} x & y \\ \epsilon y & x \end{bmatrix} \middle| x, y \in \overline{\mathbb{F}}_p,\ x^2 - \epsilon y^2 = 1 \right\}$$

$T_0 \rightsquigarrow (1\ \bar{1}) \in B_1$

$X_1(T_\epsilon) = 0;\ Y_1(T_\epsilon) = 1.$

Flag varieties

$\mathbf{G}_{(}\mathbb{C})$	**Generalized flag variety** $\mathcal{X}_n = \mathcal{F}\mathbf{G}_n$	$\mathbf{H}^{2*}(\mathcal{F}\mathbf{G}_\mathbf{n};\mathbb{C}) \cong \mathbf{R}_\mathbf{n}^*$ **(Borel)**
$\mathrm{GL}_n(\mathbb{C})$	Complete flags in $\mathbb{C}^n$ $\{0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n : \dim V_m = m\}$	$S_n \curvearrowright \mathbb{C}[x_1, \ldots, x_n]/I_{S_n}$ $I_{S_n} = \langle S_n\text{-invariant polynomials}\rangle$
$\mathrm{SO}_{2n+1}(\mathbb{C})$	Complete flags in $\mathbb{C}^{2n+1}$ equal to their orthogonal complements	$B_n \curvearrowright \mathbb{C}[x_1, \ldots, x_n]/I_{B_n}$ $I_{B_n} = \langle B_n\text{-invariant polynomials}\rangle$
$\mathrm{Sp}_{2n}(\mathbb{C})$	Complete flags in $\mathbb{C}^{2n}$ equal to their symplectic complements	$B_n \curvearrowright \mathbb{C}[x_1, \ldots, x_n]/I_{B_n}$ $I_{B_n} = \langle B_n\text{-invariant polynomials}\rangle$

Multiplicity stability: $n \to \infty$

Sequence of $\mathcal{W}_n$-representations satisfies *multiplicity stability*:

the decomposition into irreducibles "stabilizes" for n large

$$H^1(\mathcal{M}_{S_4}(\mathbb{C});\mathbb{C}) = V((4)) \oplus V((3,1)) \oplus V((2,2))$$

$$H^1(\mathcal{M}_{S_5}(\mathbb{C});\mathbb{C}) = V((5)) \oplus V((4,1)) \oplus V((3,2))$$

$$H^1(\mathcal{M}_{S_6}(\mathbb{C});\mathbb{C}) = V((6)) \oplus V((5,1)) \oplus V((4,2))$$

$$\vdots$$

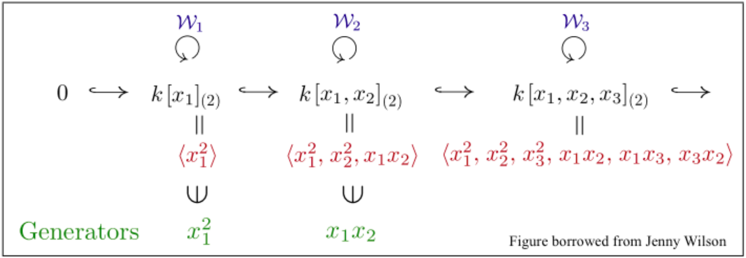
$$H^1(\mathcal{M}_{S_n}(\mathbb{C});\mathbb{C}) = V((n)) \oplus V((n-1,1)) \oplus V((n-2,2)) \text{ for } n \geq 4$$

$$H^1(\mathcal{M}_{B_n}(\mathbb{C});\mathbb{C}) = V((n), \emptyset)^{\oplus 2} \oplus V((n-2,1,1), \emptyset)^{\oplus 2} \oplus V((n-2,2), \emptyset)^{\oplus 2}$$
$$\oplus V((n-2), (2)) \qquad \text{for } n \geq 4$$

$\mathrm{FI}_{\mathcal{W}}$-modules

Category $FI_{\mathcal{W}}$: Obj: $\mathbf{n} = \{1, \bar{1}, \ldots, n, \bar{n}\}$; Morph: $\mathrm{End}_{FI_{\mathcal{W}}}(\mathbf{n}) = \mathcal{W}_n$ & $I : \mathbf{n} \hookrightarrow \mathbf{n+1}$

$\mathrm{FI}_{\mathcal{W}}$-module (over R): functor from $\mathrm{FI}_{\mathcal{W}}$ to R-modules



$\mathbf{n} \mapsto H^1(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C}) \supset \{\alpha_{ij} : \alpha_{ij} = \alpha_{ji}, i \neq j, i, j \in [n]\} \curvearrowleft \mathcal{W}_n$

generated by $\alpha_{12} \in H^1(\mathcal{M}_{\mathcal{W}_2}(\mathbb{C}); \mathbb{C})$ ($\mathrm{FI}_{\mathcal{W}}$-deg 2)

$\mathrm{FI}_{\mathcal{W}}$-algebras:

$\mathbf{n} \mapsto H^*(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$

generated (as $\mathrm{FI}_{\mathcal{W}}$-algebra) by $H^1(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C}); \mathbb{C})$ ($\mathrm{FI}_{\mathcal{W}}$-deg 2)

$\mathbf{n} \mapsto \mathbb{K}[x_1, x_2, \ldots, x_n] \curvearrowleft \mathcal{W}_n$

generated (as $\mathrm{FI}_{\mathcal{W}}$-algebra) by $\mathbb{K}[x_1]_{(1)}$ ($\mathrm{FI}_{\mathcal{W}}$-deg 1)

Representation stability and $\mathrm{FI}_{\mathcal{W}}$-modules

[CEF, Wilson] **Repstability** = f.g. $\mathrm{FI}_{\mathcal{W}}$-module $\Longrightarrow$ multiplicity stability

Theorem (Church–Farb–Ellenberg, Wilson)

The sequences of $\mathcal{W}_n$-representations

- $\{H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C});\mathbb{C})\}_n$
- $\{H^k(\mathcal{F}\mathbf{G}_n(\mathbb{C});\mathbb{C})\}_n$

are finitely generated $\mathrm{FI}_{\mathcal{W}}$-modules.

Key consequences: Let $\psi_n^k = \mathcal{W}_n$-character of $H^k(\mathcal{M}_{\mathcal{W}_n}(\mathbb{C});\mathbb{C})$ or $H^{2k}(\mathcal{F}\mathbf{G}_n(\mathbb{C});\mathbb{C})$

- Then ψ_n^k are given by a unique character polynomial for $n >> k$.
- For every character polynomial P

$$\langle P, \psi_n^k \rangle_{\mathcal{W}_n} \quad \text{is constant for } n >> k$$

Representation stability ⇒ asymptotic stability

Theorem

Let P a polynomial factorization statistic, then

$$C(n,q)\sum_{f\in\mathcal{Y}_n(q)} P(f) = \sum_{k=0}^{n}(-1)^k q^{n-k}\langle P,\psi_n^k\rangle_{\mathcal{W}_n}$$

Theorem (Church–Ellenberg–Farb, J. R.–Wilson, Casto ⋆)

Let P a polynomial factorization statistic, then

$$\lim_{n\to\infty}\frac{\sum_{f\in\mathcal{Y}_n(q)} P(f)}{|\mathcal{Y}_n(q)|} = \sum_{k=0}^{\infty}(-1)^k q^{-k}\lim_{n\to\infty}\langle P,\psi_n^k\rangle_{\mathcal{W}_n}$$

and the series converges.

Convergence: $\mathrm{FI}_{\mathcal{W}}$-degree of generators

Question (J. R.– Wilson) What's the underlying structure of $\{A_n^k \circlearrowleft \mathcal{W}_n\}$ such that for every character polynomial P the formula $\sum_{k=0}^{\infty}(-1)^k q^{-k} \lim_{n\to\infty}\langle P, \chi_{A_n^k}\rangle_{\mathcal{W}_n}$ converges?

- If $A_n^k \subset H_n^k$, where H^* is a $\mathrm{FI}_{\mathcal{W}}$-algebra f.g. in $\mathrm{FI}_{\mathcal{W}}$-deg $\leq 1 \Rightarrow$ ✓

  Ex. Co-invariant algebras $\Rightarrow$ asymptotic for Fr_q-stable maximal tori

- Generators in $\mathrm{FI}_{\mathcal{W}}$-degree two or more $\not\Rightarrow$ convergence:

  Theorem (J.R.–Wilson) Let A^* be a graded $\mathrm{FI}_{\mathcal{W}}$-algebra containing the free symmetric algebra on $\{\alpha_{i,j} : i \neq j\}$. Then the series $\sum_{k=0}^{\infty}(-1)^k q^{-k} \lim_{n\to\infty}\langle 1, \chi_{A_n^k}\rangle_{\mathcal{W}_n}$ does NOT converge.

- Hyperplane arrangements have generators in $\mathrm{FI}_{\mathcal{W}}$-degree two:

  to get convergence we consider relations

Computing asymptotics using topology

Table 1
Some statistics for Frobenius-stable maximal tori of $\mathrm{Sp}_{2n}(\overline{\mathbb{F}_q})$ and $\mathrm{SO}_{2n+1}(\overline{\mathbb{F}_q})$.

Fr_q-stable maximal tori statistic for $\mathrm{Sp}_{2n}(\overline{\mathbb{F}_q})$ and $\mathrm{SO}_{2n+1}(\overline{\mathbb{F}_q})$	Hyperoctahedral character	Formula in terms of n	Limit as $n \to \infty$
Total number of Fr_q-stable maximal tori	1	q^{2n^2} (Steinberg)	
Expected number of 1-dimensional Fr_q-stable subtori	$X_1 + Y_1$	$1 + \frac{1}{q^2} + \frac{1}{q^4} + \cdots \frac{1}{q^{2n-2}}$	$\frac{q^2}{(q^2-1)}$
Expected number of split 1-dimensional Fr_q-stable subtori	X_1	$\frac{1}{2}\left(1 + \frac{1}{q} + \frac{1}{q^2} + \cdots \frac{1}{q^{2n-1}}\right)$	$\frac{q}{2(q-1)}$
Expected value of reducible minus irreducible Fr_q-stable 2-dimensional subtori	$\binom{X_1+Y_1}{2} - (X_2 + Y_2)$	$\frac{\left(q^4 - \frac{1}{q^{2n}}\right)\left(1 - \frac{1}{q^{2(n-1)}}\right)}{(q^2-1)(q^4-1)}$	$\frac{q^4}{(q^2-1)(q^4-1)}$
Expected value of split minus non-split Fr_q-stable 2-dimensional irreducible subtori	$X_2 - Y_2$	$\frac{q^2\left(1 - \frac{1}{q^{2n}}\right)\left(1 - \frac{1}{q^{2(n-1)}}\right)}{2(q^4-1)}$	$\frac{q^2}{2(q^4-1)}$

Computing asymptotics using other techniques

Theorem (Fulman – J. R.– Wilson)

(i) **(Type A).** *Let λ be a fixed partition. Let $\mathcal{Y}_n(q)$ denote the set of Fr_q-stable maximal tori of $GL_n(\overline{\mathbb{F}}_q)$. Then*

$$\lim_{n\to\infty} \frac{1}{q^{n^2-n}} \sum_{T\in\mathcal{Y}_n(q)} \binom{X}{\lambda}(T) = \frac{1}{z_\lambda} \prod_{r=1}^{|\lambda|} \left(\frac{q^r}{q^r-1}\right)^{n_r(\lambda)}.$$

(ii) **(Type B/C).** *Fix partitions μ and λ. Let $\mathcal{Y}_n(q)$ denote the set of Fr_q-stable maximal tori of $SO_{2n+1}(\overline{\mathbb{F}}_q)$ or $Sp_{2n}(\overline{\mathbb{F}}_q)$. Then*

$$\lim_{n\to\infty} \frac{1}{q^{2n^2}} \sum_{T\in\mathcal{Y}_n(q)} \binom{X}{\mu}\binom{Y}{\lambda}(T) = \frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{q^r}{q^r-1}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{q^r}{q^r+1}\right)^{n_r(\lambda)}.$$

$$\binom{X}{\lambda}(\sigma) := \prod_{r=1}^{|\lambda|} \binom{X_r(\sigma)}{n_r(\lambda)} \text{ for all } \sigma \in S_n \qquad z_\lambda = \prod_{r=1}^{|\lambda|} n_r(\lambda)!\, r^{n_r(\lambda)}$$

Asymptotic statistics ⇒ compute stable multiplicities

Corollary (Chen, Fulman - J. R.- Wilson)

Given P a polynomial factorization statistics

$$\beta_i := \lim_{n\to\infty} \langle P, \psi_n^i \rangle_{\mathcal{W}_n} = \lim_{n\to\infty} \dim_{\mathbb{C}} H^i(\mathcal{Y}_n(\mathbb{C}); V_n^P)$$

The generating function $\sum_{i=0}^{\infty} \beta_i z^i$ is a rational function.

Example 3.9. (Example: $\mathbb{C}^n$).

Character polynomial:

$$\begin{aligned} P^{\mathbb{C}^n} &= X_1 - Y_1 \\ &= \binom{X}{\square} - \binom{Y}{\square}. \end{aligned}$$

Betti numbers:

$$\begin{aligned} \sum_{i=0}^{\infty} \beta_i z^i &= \frac{z}{(1-z)(1+z)} \\ &= z + z^3 + z^5 + z^7 + z^9 + \cdots + z^{2d+1} + \cdots . \end{aligned}$$

Recurrence: $\beta_d = \beta_{d-2}$ for $d \geq 3$.

Asymptotic statistics $\Rightarrow$ multiplicity stability

Corollary (Chen, Fulman - J. R.- Wilson)

- *Obtain a generating function for the characters* ψ_n^k
- ψ_n^k *are eventually given by character polynomials* Q_k
- *Recover multiplicity stability:*
  $\langle P, \psi_n^k \rangle_{\mathcal{W}_n}$ *is constant for* $n >> k$ *if* P *is any character polynomial*
- *Obtain a generating function for those character polynomials* Q_k
- *Have another proof of Theorem* $\star$

Question: Identify other families where asymptotic stability of statistics could be used to compute stable multiplicities or stable characters