Choosing points on plane cubic curves

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June 10, 2019

Partially joint with Ishan Banerjee (University of Chicago)

The main question

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A smooth plane cubic curve

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Question (Benson Farb): Are the algebraic constructions the only ways to continuously choose n distinct points on each smooth plane cubic curve?

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  Are the algebraic ones the only sections?

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- Does ξ_{18} has a section? (Open)

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Thank you.

An Enriched Degree of the Wronski Map

Thomas Brazelton
University of Pennsylvania
PIMS Workshop on Arithmetic Topology

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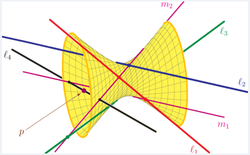

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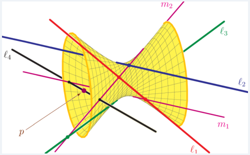

The answer is always two lines. Classical enumerative geometry produces integral answers to questions of the form "*how many ..?*"

$$\{\text{Geometric questions over } \mathbb{C} \text{ or } \mathbb{R}\} \to \mathbb{Z}.$$

$\mathbb{A}^1$-enumerative geometry

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$$\begin{aligned} \langle a \rangle : k \times k &\to k \\ (x, y) &\mapsto axy, \end{aligned}$$

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for $a \in (k^*)/(k^*)^2$, subject to certain relations.

Then $\mathbb{A}^1$-enumerative geometry produces answers to enumerative questions which are valued in $\mathrm{GW}(k)$:

$$\{\text{Geometric questions over } k\} \to \mathrm{GW}(k).$$

Four lines in three space, revisited

We can reformulate the "four lines in three-space problem" as follows:

Theorem (Srinivasan-Wickelgren)

The number of lines meeting four lines in $\mathbb{P}^3_k$ is the hyperbolic element

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As we might expect, $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ by taking the rank, so we recover the classical computation in the complex case.

The Wronski

The *Wronski map*, for any functions $f_1, \ldots, f_m$ is defined to be the determinant

$$\mathrm{Wr}(f_1, \ldots, f_m) := \det \begin{vmatrix} f_1 & \cdots & f_m \\ f_1' & \cdots & f_m' \\ \vdots & \ddots & \vdots \\ f_1^{(m-1)} & \cdots & f_m^{(m-1)} \end{vmatrix}.$$

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If we consider $f_1, \dots, f_m \in k_{m+p-1}[t]$ as polynomials of degree at most $m+p-1$, then we get a well-defined morphism of mp-dimensional k-schemes

$$\begin{aligned} \mathrm{Wr} : \mathrm{Gr}_k(m, m+p) &\to \mathbb{P}_k^{mp} \\ \mathrm{span}\,\{f_1, \dots, f_m\} &\mapsto \mathrm{Wr}(f_1, \dots, f_n). \end{aligned}$$

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2. the number of p-planes intersecting mp general m-planes in $(m+p)$-space
3. the number

$$d(m,p) = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!},$$

which counts *standard Young tableau* of size $m \times p$.

The $\mathbb{A}^1$-Degree of the Wronski

Theorem (B.)

In the case where m and p are even, we have that the $\mathbb{A}^1$-degree of the Wronski $Wr : Gr_k(m, m+p) \to \mathbb{P}_k^{mp}$ is

$$\deg^{\mathbb{A}^1} Wr = \frac{d(m,p)}{2}\boldsymbol{H}.$$

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$$\deg^{\mathbb{A}^1} Wr = \frac{d(m,p)}{2} \boldsymbol{H}.$$

This provides an enriched count of p-planes intersecting mp general m-planes in $(m+p)$-space, which encodes geometric information about the intersection.

Proving this theorem

Note that the global degree of the Wronski is a sum of local degrees:

$$\deg^{\mathbb{A}^1} \mathrm{Wr} = \sum_{x \in \mathrm{Wr}^{-1}(y)} \deg_x^{\mathbb{A}^1} \mathrm{Wr}.$$

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In the case where the line bundle is relatively orientable over the Grassmannian, *and* the relative orientation is compatible with the Wronski, then the Euler class of the line bundle $e(\mathcal{V})$ will be exactly the $\mathbb{A}^1$-degree. This occurs if and only if m and p are both even.

Proving this theorem (continued)

We now apply a result of Levine that indicates that the Euler class $e(\mathcal{V})$ is an integer multiple of the hyperbolic element $\mathbf{H}$.

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Finally, by referencing the classical computation of Schubert of the degree of the complex Wronski, we conclude that

$$\deg^{\mathbb{A}^1} \mathrm{Wr} = \frac{d(m,p)}{2}\mathbf{H}.$$

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Thank You!

A brief survey of Arithmetic Equivalence

Santiago Arango Piñeros

Universidad de Los Andes, Colombia.

PIMS Workshop on Arithmetic Topology, June 2019

The Dedekind zeta function

Let K be a number field, and let $\mathcal{O}_K$ be its ring of integers. The *Dedekind zeta function* of K is defined by the Dirichlet series

$$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} N(I)^{-s} = \prod_{\mathfrak{p}} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

where the sum ranges over nonzero ideals in $\mathcal{O}_K$, the product ranges over nonzero prime ideals in $\mathcal{O}_K$ and $N(I) := \#(\mathcal{O}_K/I)$ is the absolute norm.

$\zeta_K(s)$ admits an analytic continuation to $\mathbb{C} - \{1\}$ and satisfies a functional equation relating the argument s to $1 - s$.

Example

$\zeta_{\mathbb{Q}}(s) = \sum_{I \subseteq \mathbb{Z}} N(I)^{-s} = \sum_{n \geq 1} n^{-s} = \zeta(s)$ is the Riemann zeta function.

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Arithmetic Equivalence

Definition

Two number fields K_1 and K_2 are said to be arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. We denote this by $K_1 \approx K_2$.

$\zeta_K(s)$ governs the arithmetic of K to the extent that each rational prime has the same decomposition type in two arithmetically equivalent fields.

Theorem (Perlis, 1977)

Arithmetically equivalent number fields have the same degree, discriminant, signature, roots of unity, normal closure, normal core and product of the class number with the regulator.

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Almost Conjugate Subgroups

Definition

Let H_1, H_2 be subgroups of a finite group G. We say that H_1 and H_2 are almost conjugate if for every G-conjugacy class $\mathcal{C}$,

$$\#(H_1 \cap \mathcal{C}) = \#(H_2 \cap \mathcal{C}).$$

We denote this by $H_1 \approx H_2$.

Example (Gassman)

Let $G = S_6$ and consider the subgroups

$$H_1 = \{e, (12)(34), (13)(24), (14)(23)\},$$

$$H_2 = \{e, (12)(56), (34)(56), (14)(23)\}.$$

Both H_1 and H_2 are isomorphic to the Klein four group, but they are not conjugate within G. Nevertheless, $H_1 \approx H_2$.

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Theorem (Gassman, 1926)

$\zeta_{K_1}(s) = \zeta_{K_2}(s)$ if and only if $H_1 \approx H_2$.

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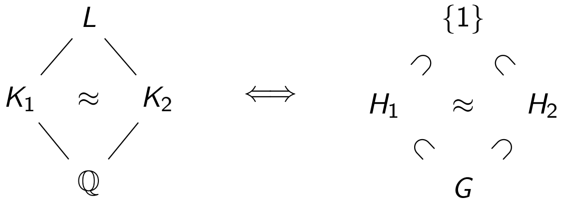

Isospectral Riemmanian Manifolds

Let (M, g) be a compact and connected Riemmanian manifold.

Let Δ_M be the Laplace-Beltrami operator,

$$\Delta_M : C^\infty(M) \to C^\infty(M), \quad f \mapsto -\mathrm{div}\,\mathrm{grad}\, f.$$

Theorem

For (M, g) as above, the eigenspaces of Δ_M are finite dimensional, and the corresponding eigenvalues form a countable discrete sequence of non-negative real numbers $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

The ordered sequence of nonzero eigenvalues of Δ_M (listed with multiplicity) is the eigenvalue spectrum of M, denoted by $\lambda(M)$.

Riemannian manifolds with the same spectrum are called isospectral.

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The zeta function in Riemannian Geometry

(M, g) compact connected Riemmanian manifold. The zeta function of M is defined by

$$\zeta_M(s) := \sum_{i \geq 1} \lambda_i^{-s}, \quad \text{for } \mathrm{Re}(s) \gg 0,$$

where $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots\} = \lambda(M)$.

$\zeta_M(s)$ has a meromorphic continuation to the whole plane with at worst finitely many simple poles.

Also, $\zeta_{M_1}(s) = \zeta_{M_2}(s)$ if and only if $\lambda(M_1) = \lambda(M_2)$.

Example

Consider $\mathbb{S}^1$ with the usual metric. The Laplacian is $\Delta = -\mathrm{d}^2/\mathrm{d}\theta^2$, and $\lambda(\mathbb{S}^1) = \{0, 1, 1, 4, 4, 9, 9, 16, 16, \ldots\}$. Therefore,

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Galois Theory - Riemannian Coverings

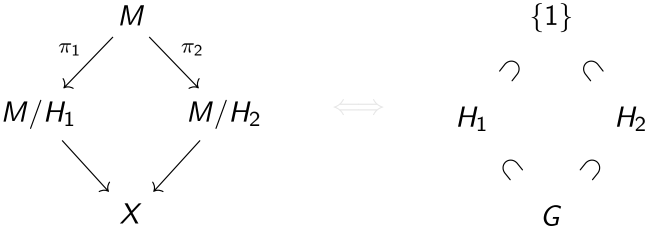
Field Extensions	Riemannian Covers
L/K	$\pi : M \to X$
Galois	Normal
$\mathrm{Gal}(L/K)$	$\mathrm{Deck}(\pi)$
$[L : K]$	$\deg \pi$
L^H/K	$\pi_H : M/H \to K$

Sunada's Theorem

Let $\pi : M \to X$ be a finite normal Riemannian covering of a compact connected Riemannian manifold X, and let $G := \mathrm{Deck}(\pi)$. H_1 and H_2 subgroups of G.

Theorem (Sunada, 1985)

$\zeta_{M/H_1}(s) = \zeta_{M/H_2}(s)$ *if and only if* $H_1 \approx H_2$.



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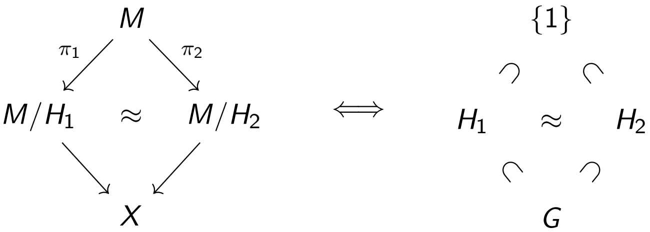

Other Contexts

- If Γ is a finite connected graph, and H_1, H_2 are subgroups of $G = \mathrm{Aut}(\Gamma)$ whose non-trivial elements have no fixed points, then $\zeta_{\Gamma/H_1}(s) = \zeta_{\Gamma/H_1}(s)$ iff $H_1 \approx H_2$. (Halbeisen & Hungerbühler, 1999)
- Let X/k be a projective algebraic curve curve with an action of a finite group G, and let H_1, H_2 be almost conjugate subgroups of G. Then, the Jacobians of the curves X/H_1 and X/H_2 are isogenous over k. (Prasad & Rajan, 2002)
- Let $p : X \to Y$ be a Galois étale cover of smooth projective varieties over k, with Galois group G. If H_1, H_2 are almost conjugate subgroups of G, then the effective Chow motives $M(X/H_1)$ and $M(X/H_2)$ are isomorphic. (Arapura et al., 2017)

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Points and lines on cubic surfaces

Ronno Das

University of Chicago

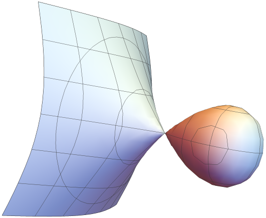
June 10, 2019

Workshop on Arithmetic Topology

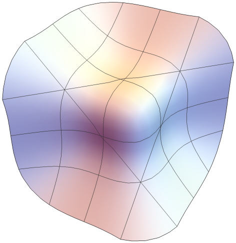
What is a (smooth) cubic surface?

Definition

zero set $S \subset \mathbb{P}^3$ of a homogeneous cubic polynomial $F(x,y,z,w)$



singular
$$x^2(x+w) = w(y^2+z^2)$$



smooth
$$x^3+y^3+z^3+w^3 = 0$$

The Cayley–Salmon Theorem

- $M = \left\{ S \;\middle|\; S \subset \mathbb{C}P^3 \text{ smooth cubic surface} \right\}$
  $= \{F \mid F \text{ homogeneous smooth degree } 3 \text{ in } \mathbb{C}[x,y,z,w]\} \big/ \mathbb{C}^\times$

The Cayley–Salmon Theorem

- $M_{\text{line}} = \{(S,L) \mid S \in M,\ L \subset S,\ L \text{ line}\}$

$$\downarrow$$

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Theorem (Cayley–Salmon)

The projection $M_{\text{line}} \to M$ is a $27:1$ covering map.

The lines on the Clebsch surface

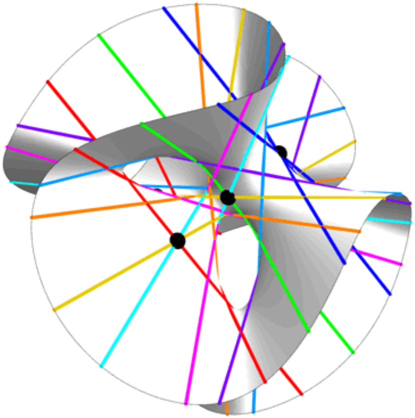

Figure: 27 lines on the Clebsch surface: $x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$

Image credit: Greg Egan, via the AMS Visual Insight blog by John Baez

What about points?

- $M_{\text{point}} = \{(S, p) \mid S \in M,\ p \in S\}$

$$\Big\downarrow \text{universal bundle}$$

- $M = \big\{S \;\big|\; S \subset \mathbb{C}P^3 \text{ smooth cubic surface}\big\}$

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From topology to point counts

- $M(\mathbb{F}_q)$: smooth cubic surfaces over $\mathbb{F}_q$

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- Average number of lines: $\dfrac{\#M_{\text{line}}(\mathbb{F}_q)}{\#M(\mathbb{F}_q)} = 1 + O\left(\frac{1}{\sqrt{q}}\right)$

- M_{line} connected $\implies H^0(M) \cong H^0(M_{\text{line}})$

From topology to *point* counts

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- Need more knowledge about: $H^*(M)$, $H^*(M_{\text{line}})$, $H^*(M_{\text{point}})$

Cohomology of M

Theorem (Vassiliev 1999)

$H^*(M;\mathbb{Q}) \cong H^*(\mathrm{PGL}(4,\mathbb{C});\mathbb{Q})$.

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- Why $\mathrm{PGL}(4,\mathbb{C})$? Automorphism group of $\mathbb{C}P^3$

Cohomology of M

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$H^*(M;\mathbb{Q}) \cong H^*(\mathrm{PGL}(4,\mathbb{C});\mathbb{Q})$.

- Why $\mathrm{PGL}(4,\mathbb{C})$? Automorphism group of $\mathbb{C}P^3$
- Fix $S_0 \in M \rightsquigarrow$ orbit map $g \mapsto g(S_0)$, $\mathrm{PGL}(4,\mathbb{C}) \to M$

Cohomology of M

Theorem (Vassiliev 1999)

$H^*(M;\mathbb{Q}) \cong H^*(\mathrm{PGL}(4,\mathbb{C});\mathbb{Q})$.

- Why $\mathrm{PGL}(4,\mathbb{C})$? Automorphism group of $\mathbb{C}P^3$
- Fix $S_0 \in M \rightsquigarrow$ orbit map $g \mapsto g(S_0)$, $\mathrm{PGL}(4,\mathbb{C}) \to M$

Theorem (Peters–Steenbrink 2003)

The orbit map induces $H^*(M;\mathbb{Q}) \cong H^*(\mathrm{PGL}(4,\mathbb{C});\mathbb{Q})$.

Cohomology of M_{line}

Theorem (D.) arXiv:1803.04146

$H^*(M_{\text{line}};\mathbb{Q}) \cong H^*(M;\mathbb{Q})$; induced by the covering map.

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Corollary

Average number of lines $= 1$.

- In fact, $\#M_{\text{line}}(\mathbb{F}_q) = \#M(\mathbb{F}_q) = q^4(\#\,\mathrm{PGL}(4,\mathbb{F}_q))$.

Cohomology of M_{point}

Theorem (D.) arXiv:1902.00737

$H^*(M_{\text{point}};\mathbb{Q}) \cong H^*(M \times \mathbb{C}P^2;\mathbb{Q})$.

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Corollary

Average number of points $= q^2 + q + 1$.

Improvement in progress: distribution of points*

t	$51840q^4 \times$ (proportion of surfaces with q^2+tq+1 points)
-2	$80q^4 + 240q^3 - 400q - 240$
-1	$3465q^4 - 1935q^3 + 2025q^2 - 8145q - 1890$
0	$11664q^4 + 4320q^3 - 5184q^2 + 6480q + 4320$
1	$20820q^4 - 3060q^3 + 1620q^2 + 9660q - 720$
2	$13104q^4 - 720q^3 + 5184q^2 - 5040q - 2160$
3	$2430q^4 + 1350q^3 - 4050q^2 - 2430q + 540$
4	$240q^4 + 240q$
5	$36q^4 - 180q^3 + 324q^2 - 180q$
7	$q^4 - 15q^3 + 81q^2 - 185q + 150$

*using results from Bergvall–Gounelas '19

Improvement in progress: other markings[†]

marking	average count
1 line	1
pair of skew lines	$1 - \frac{1}{q} + \frac{1}{q^4}$
pair of intersecting lines	1
2 (or 3) intersecting lines	$1 - \frac{1}{q^3} + \frac{1}{q^4}$
"tritangent"	1
sextet of skew lines	$1 - \frac{1}{q} + \frac{1}{q^4}$
"double six"	$1 - \frac{1}{q}$
27 lines	$1 - \frac{15}{q} + \frac{81}{q^2} - \frac{185}{q^3} + \frac{150}{q^4}$
$\vdots$	$\vdots$

[†] using results from Bergvall–Gounelas '19

Embrace the singularity

- $M = (\mathbb{C}^{20} \setminus \Sigma)/\mathbb{C}^\times$, where
  $\Sigma = \{$singular cubic polynomials$\}$, the discriminant locus

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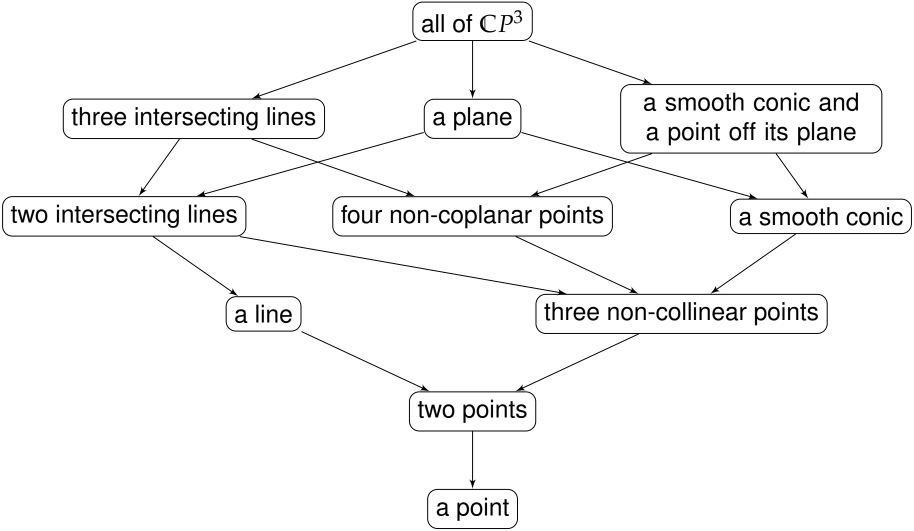
- Alexander duality $\implies H^*(M) \leftrightsquigarrow H_*(\Sigma)$

- Break up (stratify) Σ based on where $F \in \Sigma$ is singular

Singular sets of singular cubics



Resolving Σ

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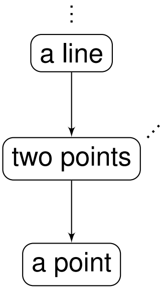

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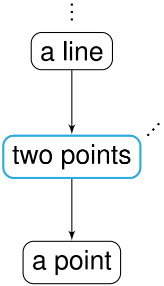

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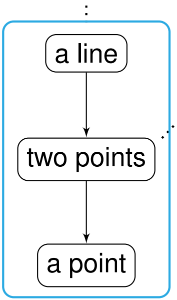

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    point $\subset$ two points $\subset$ line

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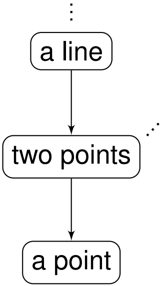

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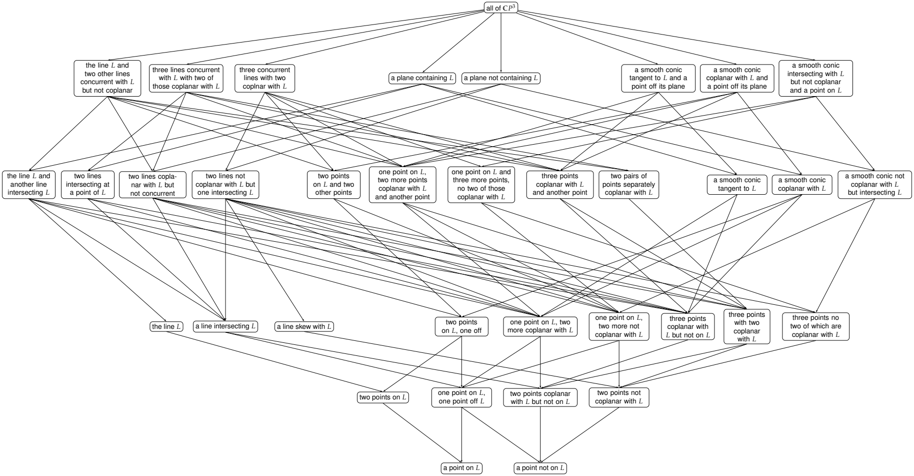
- Combine (co)homology of all the pieces in a spectral sequence ...

Keeping track of the lines and points

Just the surface	With line L	With point p
two points	two points on L	p and another point
	one on L, one off L	two points collinear with p
	both off L, coplanar with L	two points not collinear with p
	two points not coplanar with L	
11 pieces	36 pieces	32 pieces

Singular sets of singular cubics containing the line L



Representation Stability and $\overline{M}_{g,n}$

Phil Tosteson

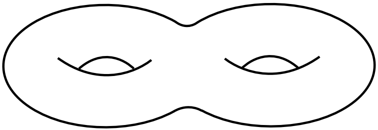
PIMS, June 10th, 2019

Moduli Space of Curves

M_g is the moduli space of complex curves of genus g.

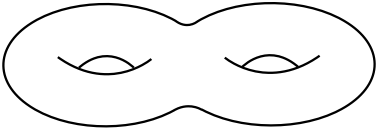
$$M_g = \frac{\{C \text{ smooth complex genus } g \text{ curve}\}}{C \sim C' \text{ if } C \text{ and } C' \text{ are isomorphic}}$$

A point in M_2 is a genus 2 curve with a complex structure.

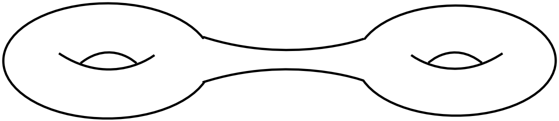


Moduli Space of Curves

We can try to visualize M_g by associating a **hyperbolic metric** to each complex structure (our drawings are not accurate).

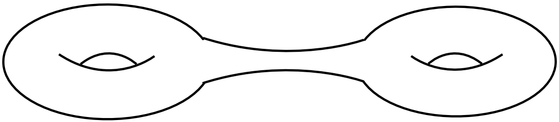


As the complex structure on C changes, the metric deforms, and we trace out a path in M_g.



Compactification by Nodal Curves

As the neck of the surface stretches longer and longer, we obtain a sequence of curves with **no limiting smooth curve.**



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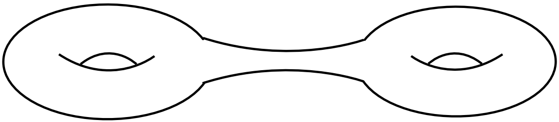
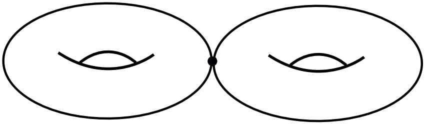


To compactify M_g, we consider a larger space $\overline{M}_g$ that has nodal curves.



Moduli Spaces of Marked Curves

$M_{g,n}$ is the moduli space of complex curves of genus g and n marked points

$$M_{g,n} := \frac{\{C \text{ complex genus } g \text{ curve}, p_1, \dots, p_n \in C\}}{\{\text{ isomorphisms } C \simeq C' \text{ preserving } p_1, \dots, p_n\}}$$

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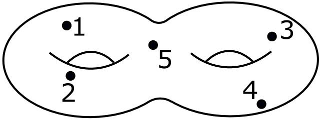
This is a point in $M_{2,5}$.



Deligne–Mumford Compactification

A **stable marked curve** is a compact, connected, one dimensional algebraic variety C, together with a collection of marked points $p_i \in C$, $i = 1, \ldots, n$, that satisfy

- Every marked point p_i is smooth.
- Every singular point $c \in C$ is a double point.
- Each genus 0 irreducible component contains ≥ 3 marked or double points.
- Each genus 1 irreducible component contains ≥ 1 marked or double points.

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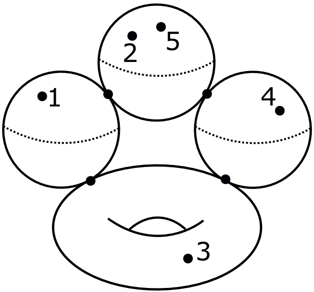
Definition

$\overline{M}_{g,n}$ is the **moduli space of stable marked complex curves**.

$$\overline{M}_{g,n} := \frac{\{C, p_1, \ldots, p_n |\ C \text{ is a stable marked curve of genus } g\}}{(C, p_i) \sim (D, q_i) \text{ if } C, D \text{ are isomorphic as marked curves}}.$$

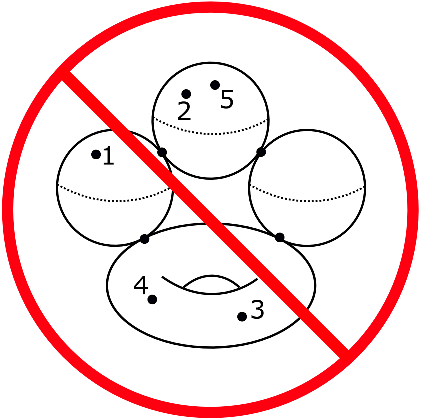
Deligne–Mumford Compactifications

This nodal curve **is stable**. It defines an element of $\overline{M}_{2,5}$



Deligne–Mumford Compactifications

This nodal curve is not stable, because one of the genus 0 components has only 2 special points



The symmetric group $\mathbf{S}_n$ acts on $M_{g,n}$ and $\overline{M}_{g,n}$, by **relabelling points**, so we can ask:

Question

What are the $\mathbf{S}_n$ representations $H_i(M_{g,n}, \mathbb{Q})$ and $H_i(\overline{M}_{g,n}.\mathbb{Q})$?

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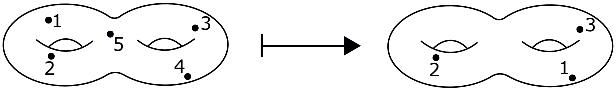
- In general, this is hard. Can we say something qualitative about these $\mathbf{S}_n$ representations for $n \gg 0$?
- Church-Ellenberg-Farb introduced a strategy for answering this type of question.

FI action on $M_{g,n}$

An injection $f : [n] \hookrightarrow [m]$ gives a map $M_{g,n} \leftarrow M_{g,m}$, by **forgetting and relabelling points.**

For example, the injection $[3] \to [5]$, given by $1 \mapsto 4, 2 \mapsto 2, 3 \mapsto 3$ acts by



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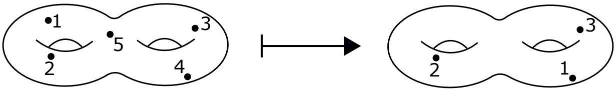

This means that $H^i(M_{g,n})$ **is an FI module.**

Representation Stability for $M_{g,n}$

Theorem (Jiménez Rolland)

Fix i and g. The **FI** *module $H^i(M_{g,n}, \mathbb{Q})$ is* **finitely generated**.

This means there is a finite list of classes $x_k \in H^i(M_{g,n_k})$ from which the rest can be obtained by **acting by injections**, and **taking linear combinations**.

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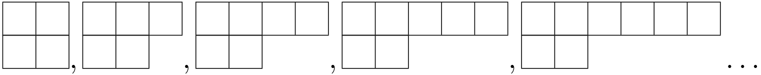
Here are some consequences:

- The function $n \mapsto \dim M_n$ agrees with a polynomial for $n \gg 0$
- The groups M_n/S_n stabilize
- The $\mathbf{S}_n$ representation $H_i(M_{g,n})$ **eventually agrees** with a finite direct sum of representations with "growing top rows"



Case of $\overline{M}_{0,n}$

Keel computed the cohomology of $\overline{M}_{0,n}$. In particular we have:

Theorem (Keel)

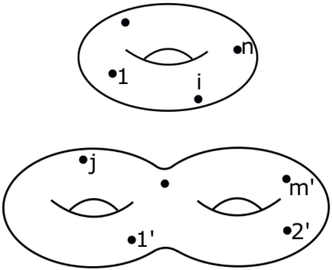
The vector space $H^2(\overline{M}_{0,n})$ has dimension $2^{n-1} - \frac{n^2-n+2}{2}$.

- Thus $H^2(\overline{M}_{0,n})$ cannot be a finitely generated **FI** module.
- A **different** algebraic structure is required to study $H_i(\overline{M}_{g,n})$.

A source of algebraic structure: Gluing maps

Let $i \in [n] = \{1, \ldots, n\}$ and $j \in [m] = \{1', \ldots, m'\}$. There is a gluing map

$$\mathrm{glue}_{i,j} : \overline{M}_{g,n} \times \overline{M}_{g,m} \to \overline{M}_{g,n+m-2}$$



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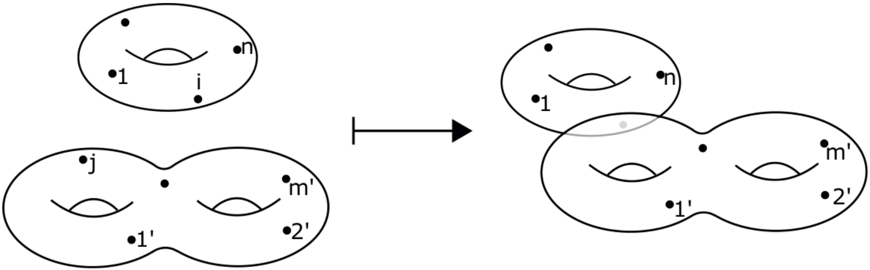

$\mathbf{FS}^{\mathrm{op}}$ modules

FS is the category of finite sets and surjections.

$$\emptyset \quad \{1\} \leftarrow \{1,2\} \overset{6}{\Lleftarrow} \{1,2,3\} \overset{36}{\Lleftarrow} \{1,2,3,4\} \overset{240}{\Lleftarrow} \{1,2,3,4,5\} \Lleftarrow \cdots$$

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An $\mathbf{FS}^{\mathrm{op}}$ module is **contravariant functor** from **FS** to the category of abelian groups.

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As for **FI**, an $\mathbf{FS}^{\mathrm{op}}$ module is a sequence of $\mathbf{S}_n$ representations, related by transition maps.

$\mathbf{FS}^{\mathrm{op}}$ modules have **more transition maps** than **FI** modules: notice that $\mathbf{FS}(n,k)$ grows exponentially $O(k^n)$, whereas $\mathbf{FI}(k,n)$ only grows polynomially $O(n^k)$.

An $\mathbf{FS}^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

The category $\mathbf{FS}^{\mathrm{op}}$ is generated by permutations $\sigma \in \mathbf{S}_n$, and surjections $[n+1] \twoheadrightarrow [n]$.

- We already know how permutations should act

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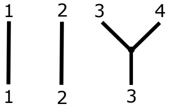
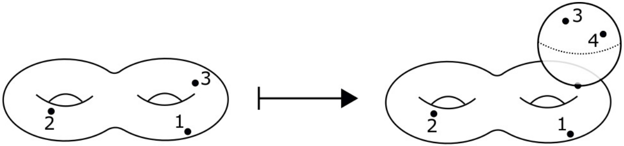


we associate a map $f : \overline{M}_{g,3} \to \overline{M}_{g,4}$:



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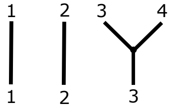

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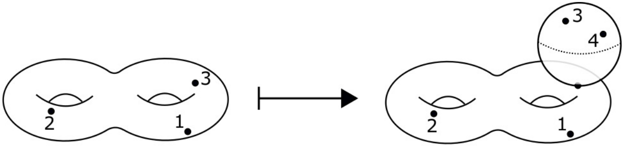

On homology, this defines an action of $\mathbf{FS}^{\mathrm{op}}$.

An $\mathbf{FS}^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

Why? At space level this construction generates an action by the category, **BT**, of binary forests.

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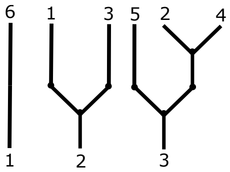

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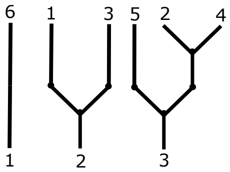

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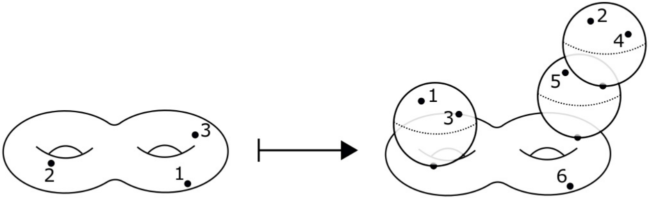
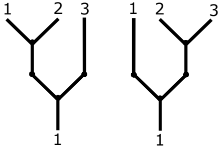


An $\mathbf{FS}^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

On homology, different trees induce the same map.

For example, the trees



give two maps $\overline{M}_{g,1} \underset{T_2}{\overset{T_1}{\rightrightarrows}} \overline{M}_{g,3}$.

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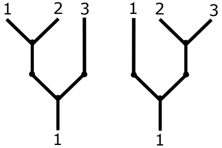

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There is a single gluing map

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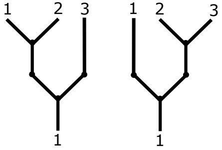
T_1 and T_2 are given by evaluating this gluing map at two different points of $\overline{M}_{0,4}$.

An $\mathbf{FS}^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

On homology, different trees induce the same map.

For example, the trees



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T_1 and T_2 are given by evaluating this gluing map at two different points of $\overline{M}_{0,4}$. Since $\overline{M}_{0,4}$ is connected, there is a path between them, and they induce the **same** map on homology.

Finite generation

Arguing like this, we see that the action of **BT** induces an $\mathbf{FS}^{\mathrm{op}}$ action on homology.

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Theorem

Let $g, i \in \mathbb{N}$. Then the $\mathbf{FS}^{\mathrm{op}}$ module

$$n \mapsto H_i(\overline{M}_{g,n}, \mathbb{Q})$$

is a subquotient of an $\mathbf{FS}^{\mathrm{op}}$ module that is finitely generated in degree $\leq p(g,i)$ where $p(g,i)$ is a polynomial in g and i of order $O(g^2 i^2)$.

Consequences of finite generation

Applying results of Sam–Snowden on finitely generated $\mathbf{FS}^{\mathrm{op}}$ modules, we obtain the following

Corollary

Let $C = p(g, i)$. Then

- *The generating function for the dimension of $H_i(\overline{M}_{g,n})$ is rational and takes the form*

$$\sum_n \dim H_i(\overline{M}_{g,n}) t^n = \frac{f(t)}{\prod_{j=1}^{C}(1 - jt)^{d_j}}$$

for some polynomial $f(t)$ and $d_j \in \mathbb{N}$.

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  for some polynomial $f(t)$ and $d_j \in \mathbb{N}$.

- *In particular, there exist polynomials $f_1(n), \ldots, f_C(n)$ such that for $n \gg 0$ we have $\dim H_i(\overline{M}_{g,n}) = \sum_{j=1}^{C} f_j(n) j^n$*

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- *Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_C$ be an integer partition of k, and $\lambda + n$ be the partition $\lambda_1 + n \geq \lambda_2 \geq \cdots \geq \lambda_C$. The multiplicity of $\lambda + n$ in $H_i(\overline{M}_{g,n+k})$,*

$$n \mapsto \dim \mathrm{Hom}_{\mathbf{S}_{n+k}}(M_{\lambda+n}, H_i(\overline{M}_{g,n+k})),$$

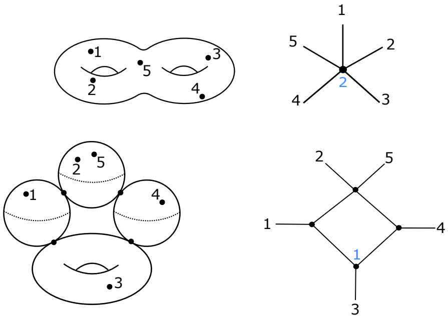
is bounded by a polynomial of degree $C - 1$.

Thanks!

Thanks for listening!

Idea of proof

To each curve, we can associate a stable graph. This stratifies $\overline{M}_{g,n}$



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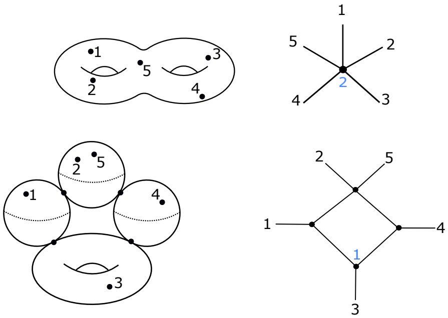
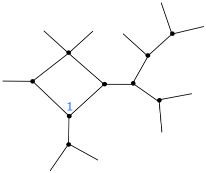


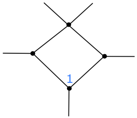
To bound $H_i(\overline{M}_{g,n})$, we bound the (Borel–Moore) homology of these strata.

Idea of proof

BT acts on the strata by tacking on trees. Any homology class coming from this stratum



is pushed forward from a smaller stratum



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Want to show that only finitely many graphs can contribute $\mathbf{FS}^{\mathrm{op}}$ module generators to $H_i(\overline{M}_{g,n})$. The above argument shows that graphs with **external** Y**'s** do not give generators.

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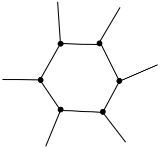

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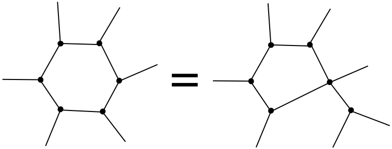

This is a possible exception. But any class from first stratum should be a homologous to one from the second.

Technical problems

The category **BT** is not known to be Noetherian, and $\mathbf{FS}^{\mathrm{op}}$ does not act on the spectral sequence associated to the filtration.

Need to define a **coarsening** of the stratification to make the argument work– requires more combinatorics and some algebraic geometry.

Cohomology of the space of polynomial maps on $\mathbb{A}^1$ with prescribed ramification

Oishee Banerjee

University of Chicago

June 10, 2019

Motivation

Varieties $/\mathbb{C}$.

- G finite group. Homology of the (components of the) moduli space of branched G-covers of $\mathbb{A}^1$

$$\left\{ X \xrightarrow{f} \mathbb{A}^1 : f \text{ is a branched covering map}, \; Gal(X/\mathbb{A}^1) \cong G \right\}$$

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  - Constraint: $Gal(X/\mathbb{A}^1)$ is fixed.
  - Variable: $\#$ branch points $\{X \xrightarrow{f} \mathbb{A}^1\}$ varies $\implies$ genus of X varies.

An orthogonal problem

- (A component of) the moduli space of genus 0, degree n branched covers of $\mathbb{A}^1$:

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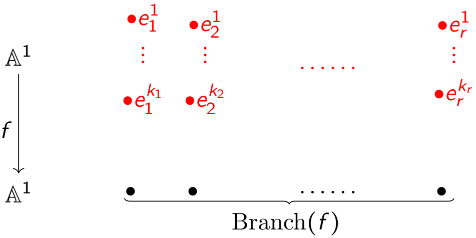
- Constraints on ramification data $\longrightarrow$ subspaces of M_{n-1}.
- Orthogonal problem: Can we prove a homological stability result à la EVW as $n \to \infty$?

Ramification data of $\mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1$

$f \in \mathrm{M}_n$.



Ramification length of f, $\mathit{length}(f) := \sum_i \Big(\sum_{1 \le j \le k_i} (e_i^j - 1) \Big) - 1.$

Some examples

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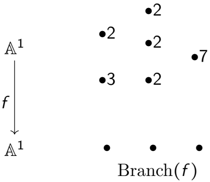

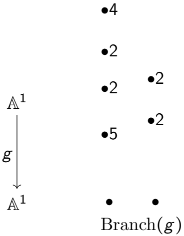

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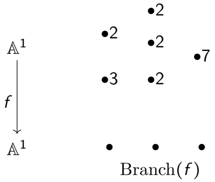

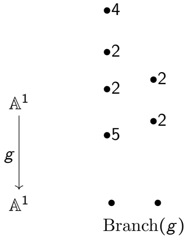

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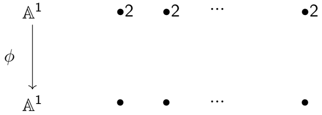

$$length(\phi) = 0.$$

Define $Simp_n^m := \{f \in \mathrm{M}_n : length(f) < m\} \subset \mathrm{M}_n$.

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Theorem

Let $m, n \geq 1$. Then for all $n \geq 3m$:

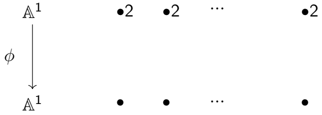
$$H^i(Simp_n^m(\mathbb{C}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i = 0, \\ \mathbb{Q}^{\oplus c(m)} & \text{for } i = 2m-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $p(N) := \#$ partitions of N and

$$c(m) := \sum_{k \geq 1} \left(\sum_{\substack{n_1 + \ldots + n_k = m \\ n_1 \leq \ldots \leq n_k}} p(n_1+1) \ldots p(n_k+1) \right).$$

The space of simply-branched polynomials $Simp_n^1$

Schematic of $\phi \in Simp_n^1$:



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$$\begin{array}{lcccc} \mathbb{A}^1 & \bullet 2 & \bullet 2 & \cdots & \bullet 2 \\ \phi \big\downarrow & & & & \\ \mathbb{A}^1 & \bullet & \bullet & \cdots & \bullet \end{array}$$

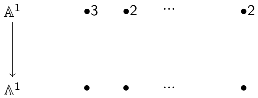
Theorem

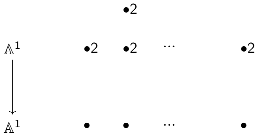
Let $n \geq 4$. Then

$$H^i(Simp_n^1(\mathbb{C}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } i = 0, \\ \mathbb{Q}^{\oplus 2} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A few words about the proof

- $H^1(Simp_n^1; \mathbb{Q}) \cong \mathbb{Q}^2$; $\mathrm{M}_n - Simp_n^1$ has two components:





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Plan of action: Formulate a ‘generalised nerve covering theorem’ in the language of sheaves. Key players-

1. Combinatorics of the poset that encodes the ramification data,
2. Geometry of the strata in the resulting stratification of M_n.

Arithmetic application

Corollary

Let $m, n \geq 1$ and let $q = p^d$, where p is a prime and $d \geq 1$. Then

$$\#Simp_n^m(\mathbb{F}_q) = q^n - c(m)q^{n-m}$$

for all $n < p - 1$ and $m \leq \frac{n}{3}$.

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Case $m = 1$: $\# Simp_n^1(\mathbb{F}_q) = q^n - 2q^{n-1}$.

FAQs

- Are there natural maps $Simp_n^m \to Simp_{n+1}^m$ that induce stability on cohomology?
- How about when genus of X is positive?
- How (dis)similar is $Simp_n^1$ to the configuration space of points on $\mathbb{C}$?

Representation stability in the level 4 braid group

Kevin Kordek
joint with Dan Margalit
Georgia Tech

The level m braid group

Integral Burau representation:

$$\rho_n : \mathsf{B}_n \to \mathsf{GL}_n(\mathbb{Z}[t, t^{-1}]) \to \mathsf{GL}_n(\mathbb{Z})$$

The level m braid group

Definition

$$\mathsf{B}_n[m] = \ker\left(\mathsf{B}_n \xrightarrow{\rho_n} \mathsf{GL}_n(\mathbb{Z}) \to \mathsf{GL}_n(\mathbb{Z}/m\mathbb{Z})\right)$$

The level m braid group

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$\mathrm{B}_n[4] \cong \pi_1(X_n(\mathbb{C}))$

$$X_n = \operatorname{Spec} \mathcal{O}(\mathrm{PConf}_n(\mathbb{C})) \left[\sqrt{x_i - x_j} \, \right]_{i<j}$$

The universal mod 2 abelian cover of $\mathrm{PConf}_n(\mathbb{C})$.

Homology of $\mathrm{B}_n[4]$

Basic question: How does $H_k(\mathrm{B}_n[4])$ vary with n?

Theorem (K-Margalit, 2019)

$\{H_1(\mathrm{B}_n[4];\mathbb{C})\}$ *is uniformly representation stable.*

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Not split!

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Theorem (K-Margalit)

$H^*(\mathrm{B}_n[4];\mathbb{Q})$ *is not generated in degree 1 for* $n \geq 15$.

Application 2: Combinatorial structure

Fact: $B_n[m]$ contains all mth powers of half-twists.

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No.

Theorem (K-Margalit)

$B_n[4]$ *is not generated by 4th powers of half-twists for* $n \geq 3$.

Application 3: Characteristic varieties

$$V = \begin{array}{c}\text{Space of}\\ \text{complex 1-dim. local systems}\\ \text{over } \mathrm{PConf}_n(\mathbb{C})\end{array} \subset (\mathbb{C}^\times)^{\binom{n}{2}}$$

All components are (torsion-translated) subtori. (Arapura)

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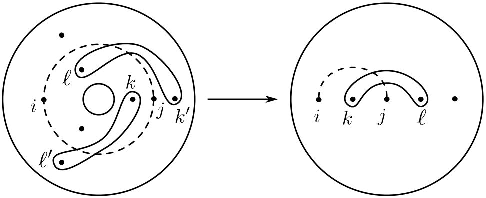
Theorem (K-Margalit)

There are no 2-torsion translated components of V.



Thank you!

$$H_1(\mathrm{B}_n[4];\mathbb{C}) \cong \begin{cases} V_2(1,(0)) & n=2 \\ V_3(1,(0)) \oplus V_3(1,(1)) \oplus V_3(\rho_3,(0)) & n=3 \\ V_n(1,(0)) \oplus V_n(1,(1)) \oplus V_n(1,(2)) \oplus V_n(\rho_3,(0)) \oplus V_n(\rho_4,(0)) & n \geq 4. \end{cases}$$

Adding points to configurations
In 2-sphere

Lei Chen
Joint work with Nick Salter

General question

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$$\mathrm{Conf}_{n,m}(M) \xrightarrow{\text{forget}} \mathrm{Conf}_n(M)$$

General question

$$\mathrm{Conf}_{n,m}(M) \underset{\text{section?}}{\overset{\text{forget}}{\rightleftarrows}} \mathrm{Conf}_n(M)$$

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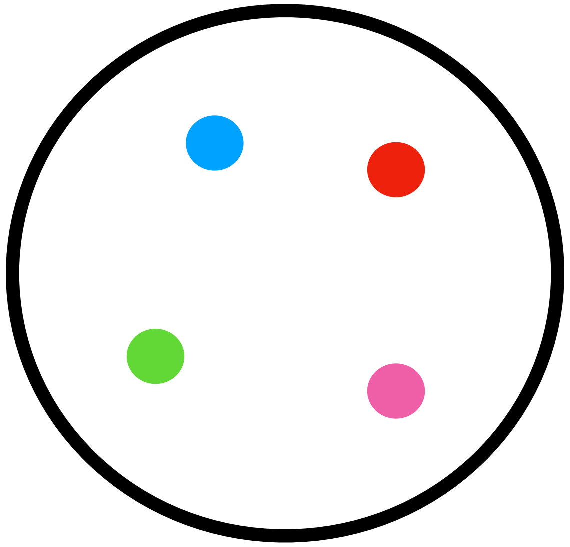

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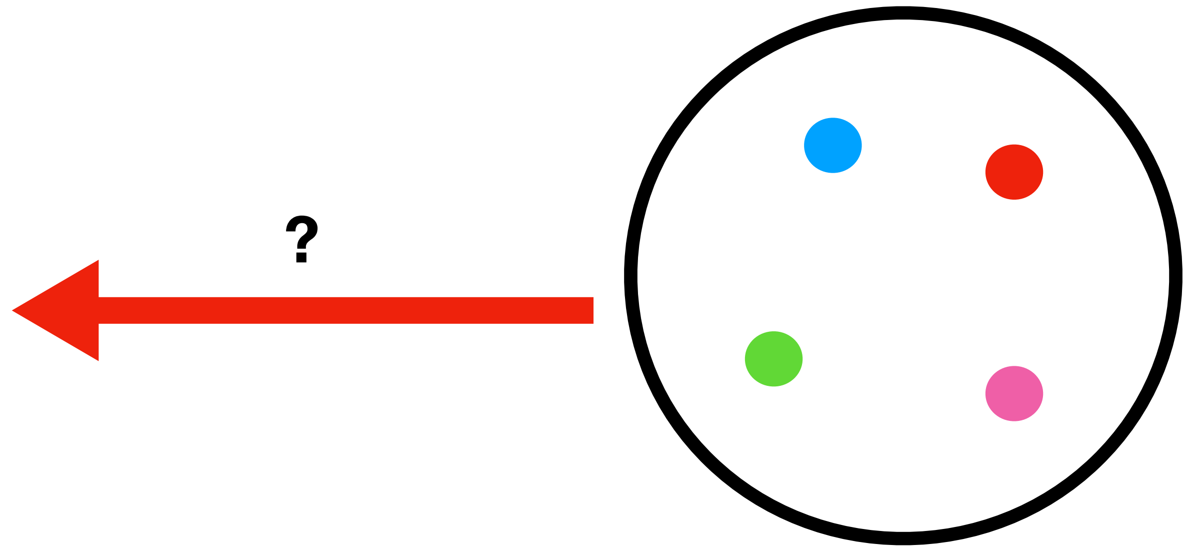

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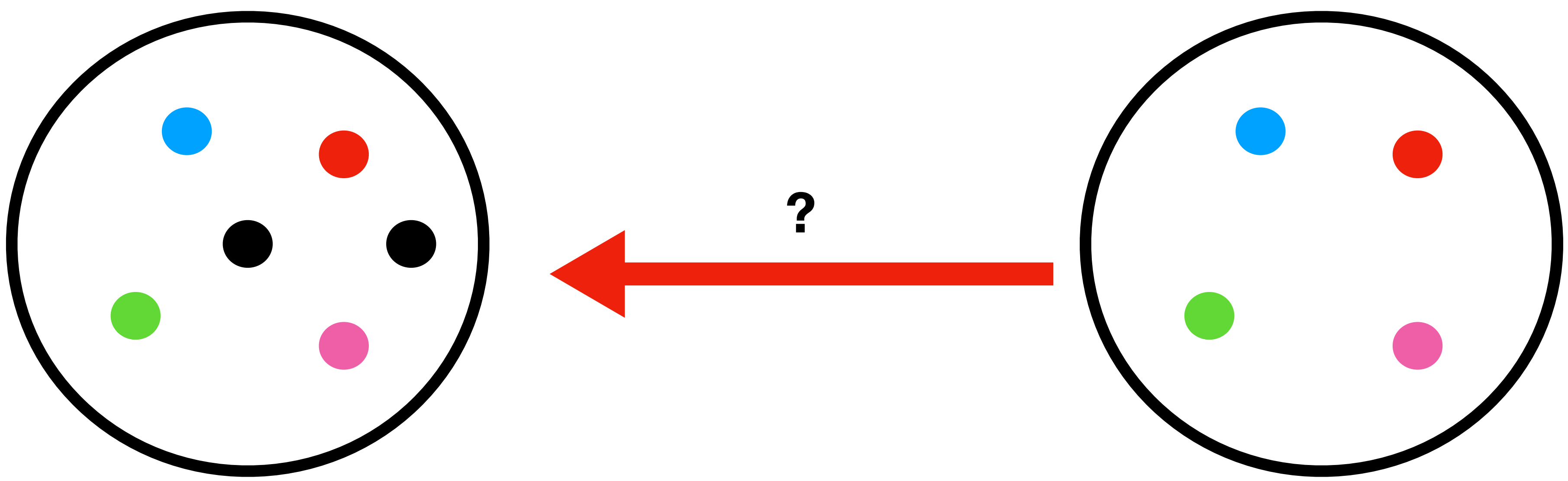

Example: Euclidean space

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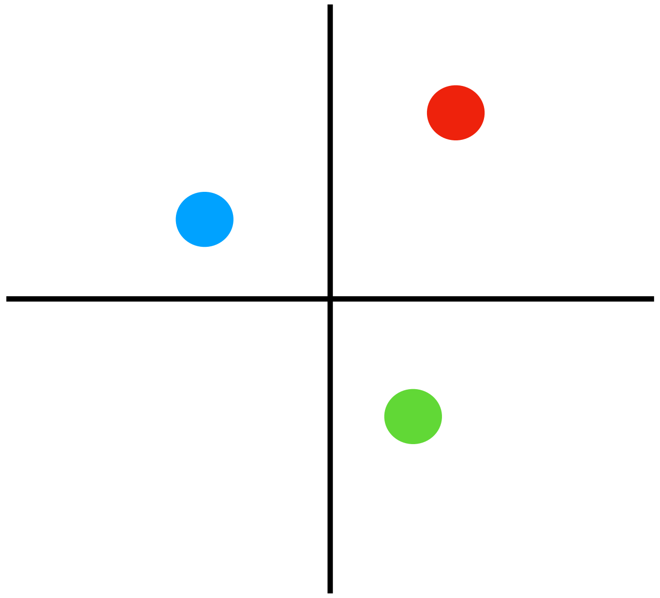

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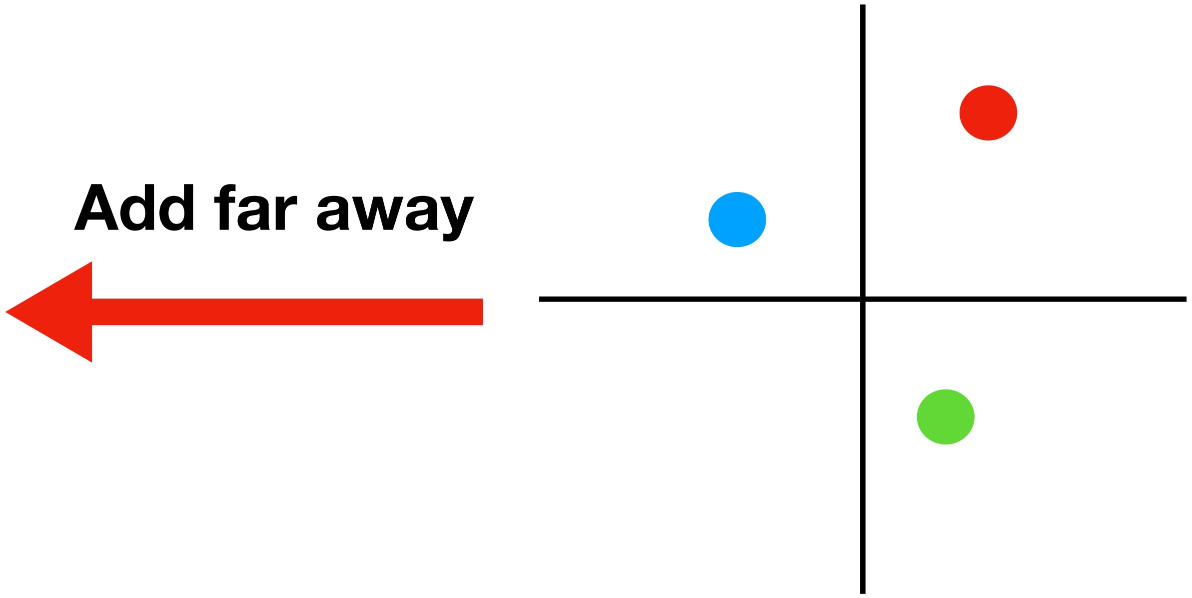

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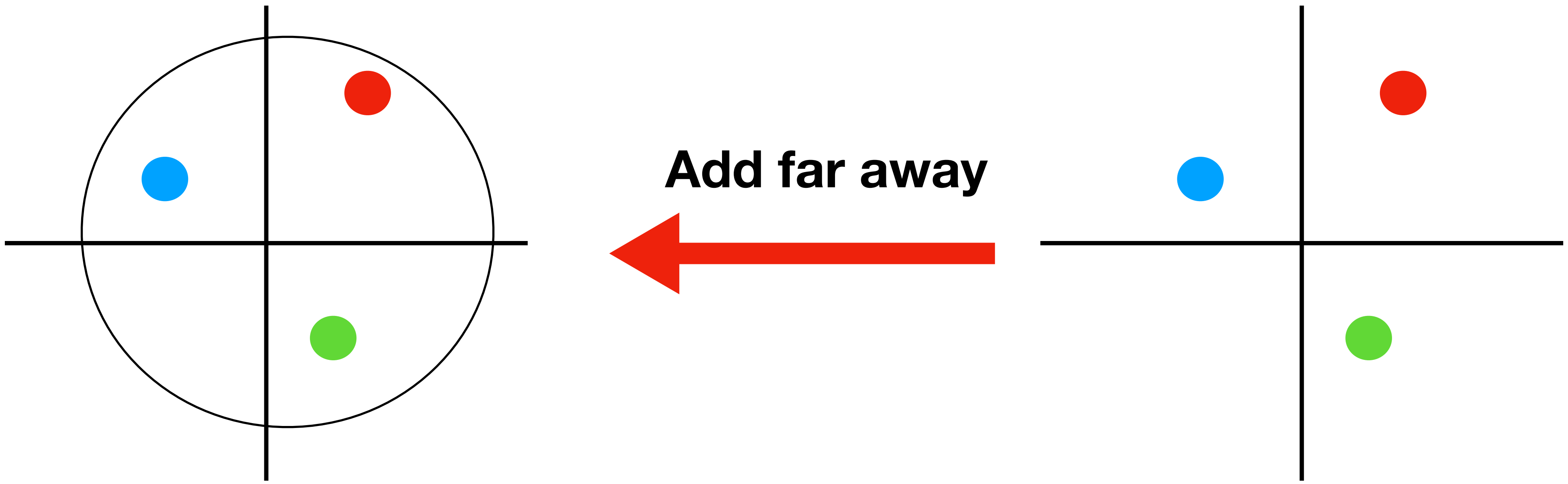

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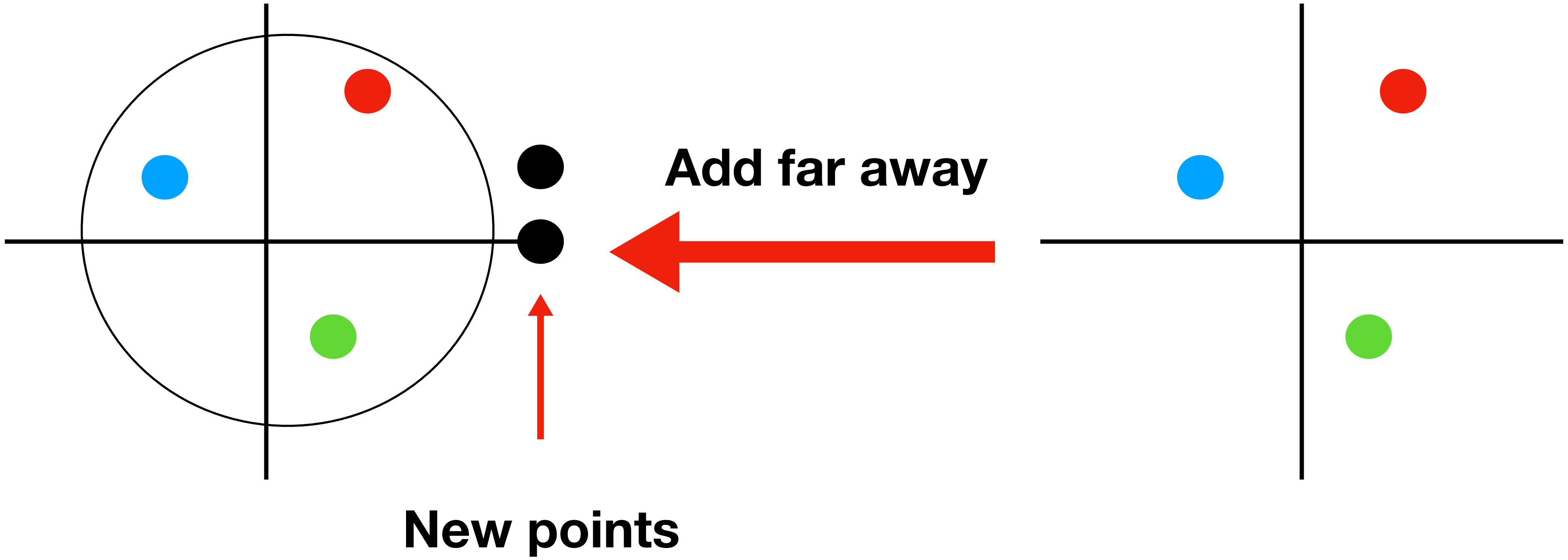

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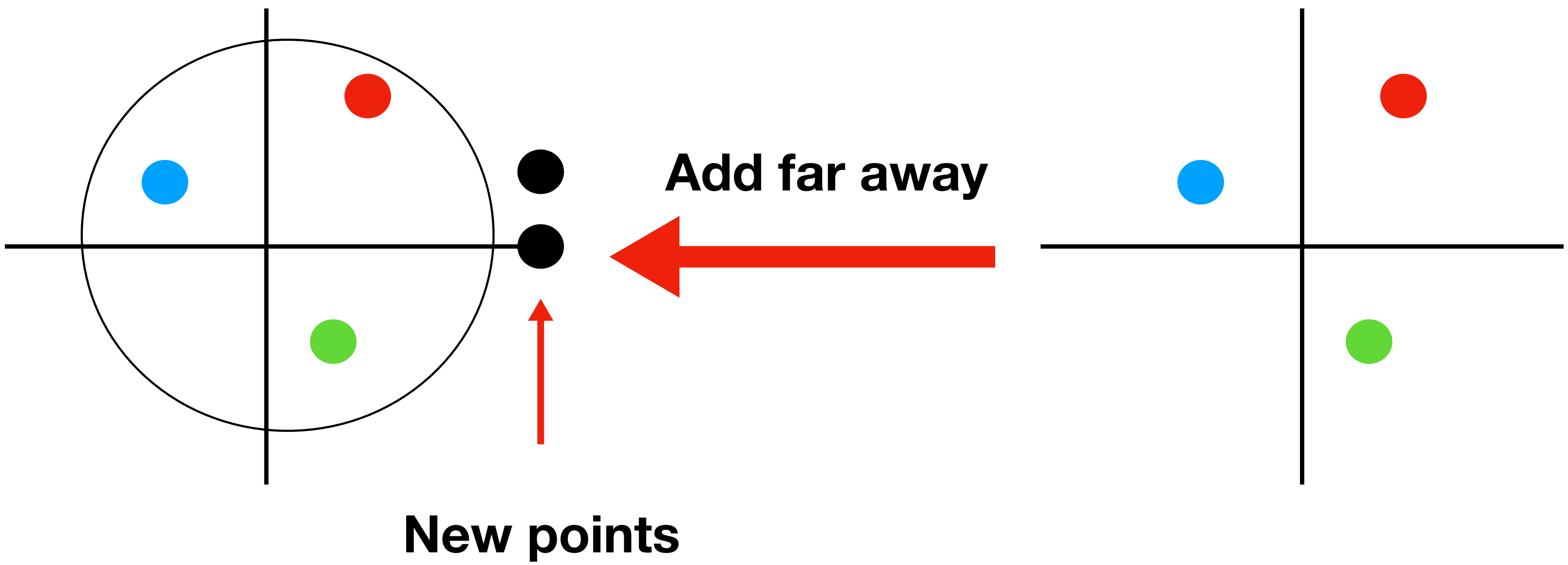

Add one point: This is the only construction in R^2

2-Sphere: previous results

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Goncalves-Guaschi:

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$$\mathrm{Conf}_{3,m}(S^2) \to \mathrm{Conf}_3(S^2)$$

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No Construction!!!

No holomorphic construction

For n>4

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Earle-Kra, Hubbard: Teichmuller theory

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Even for the universal cover

A continuous construction:
“Adding close by”

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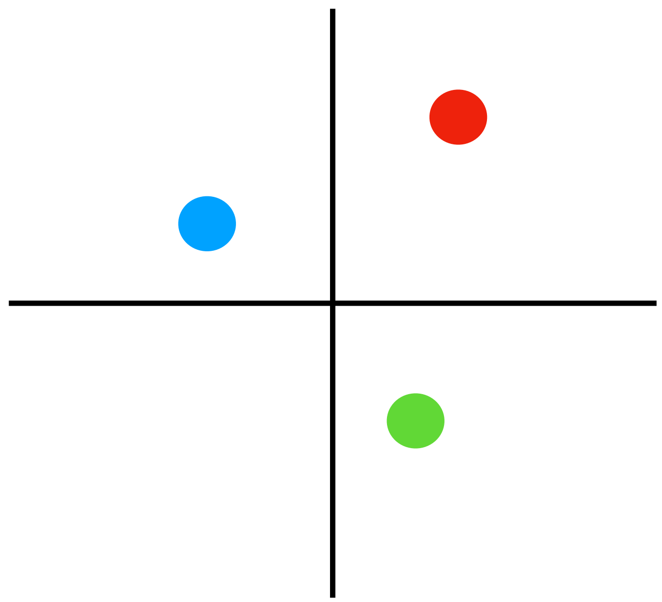

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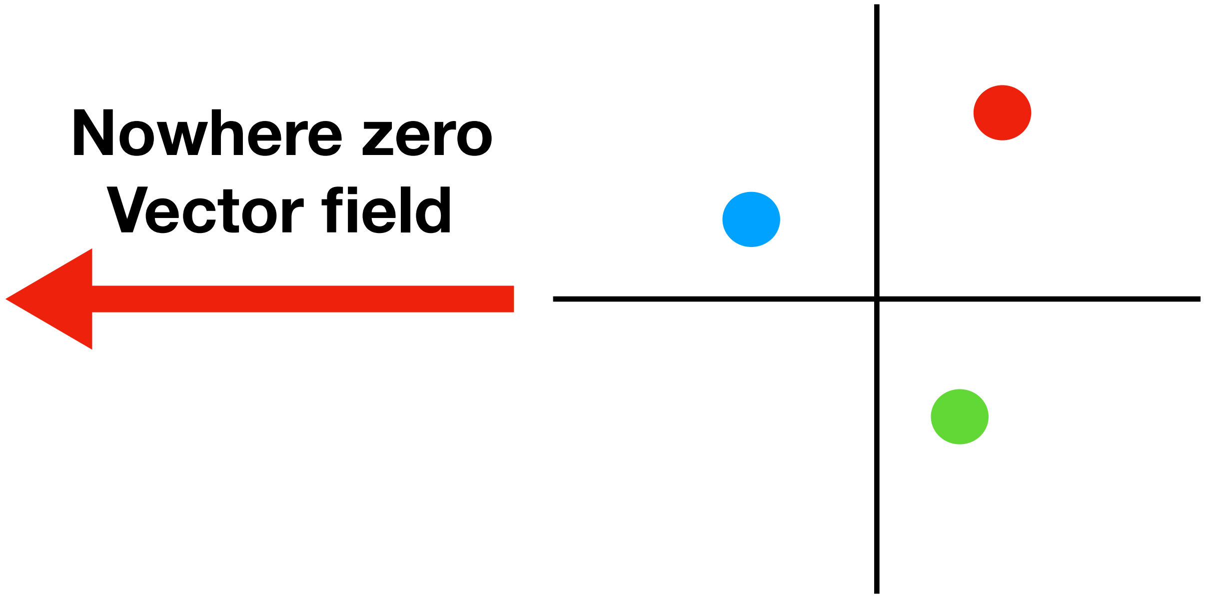

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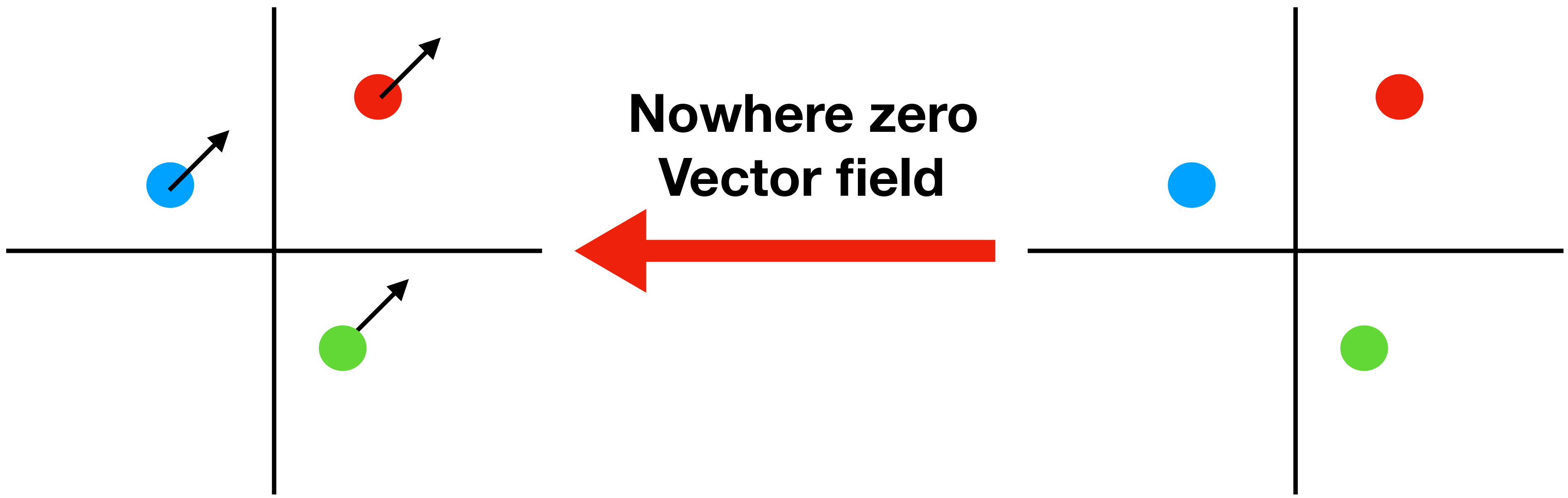

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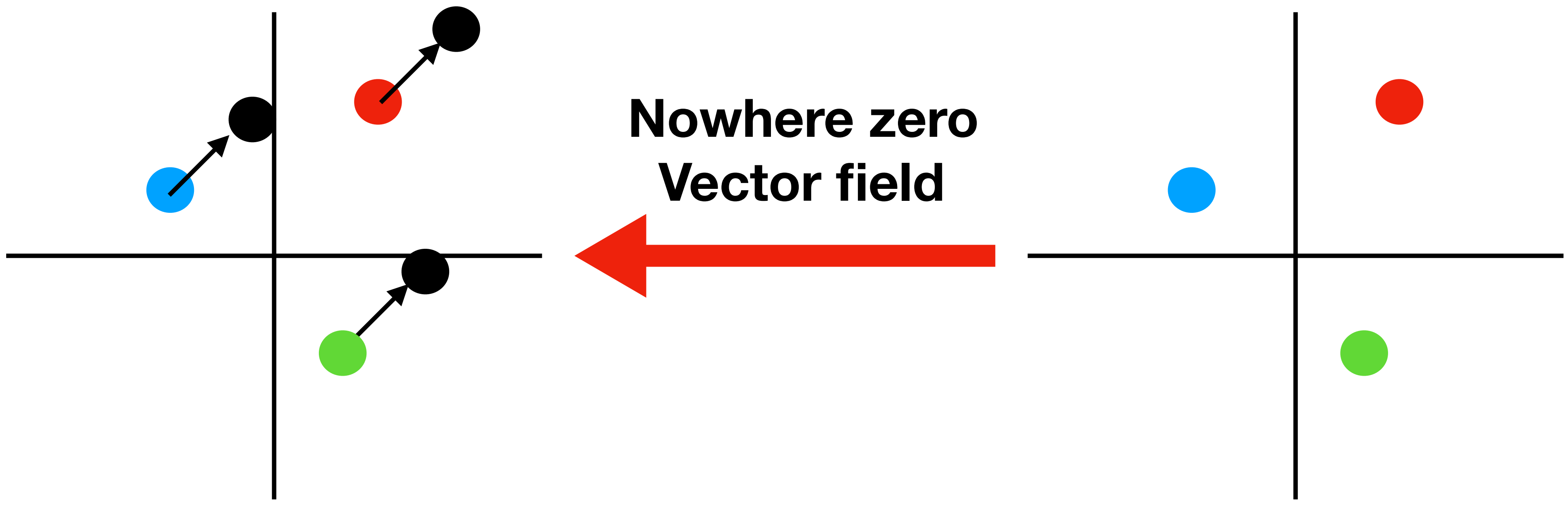

Result (Chen — Salter)

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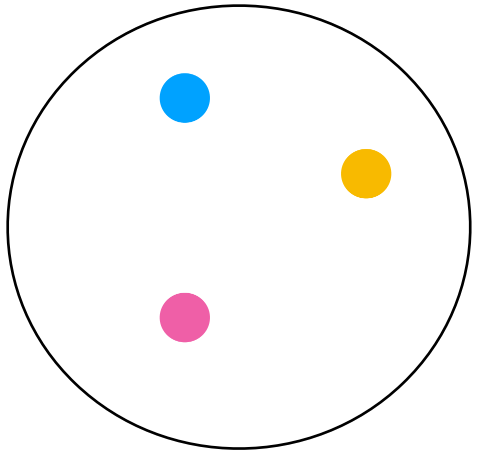

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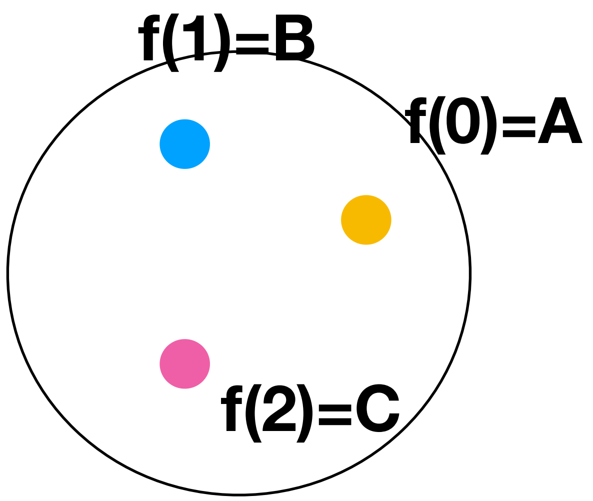

Unique Mobius map f: f(0)=A, f(1)=B, f(2)=C

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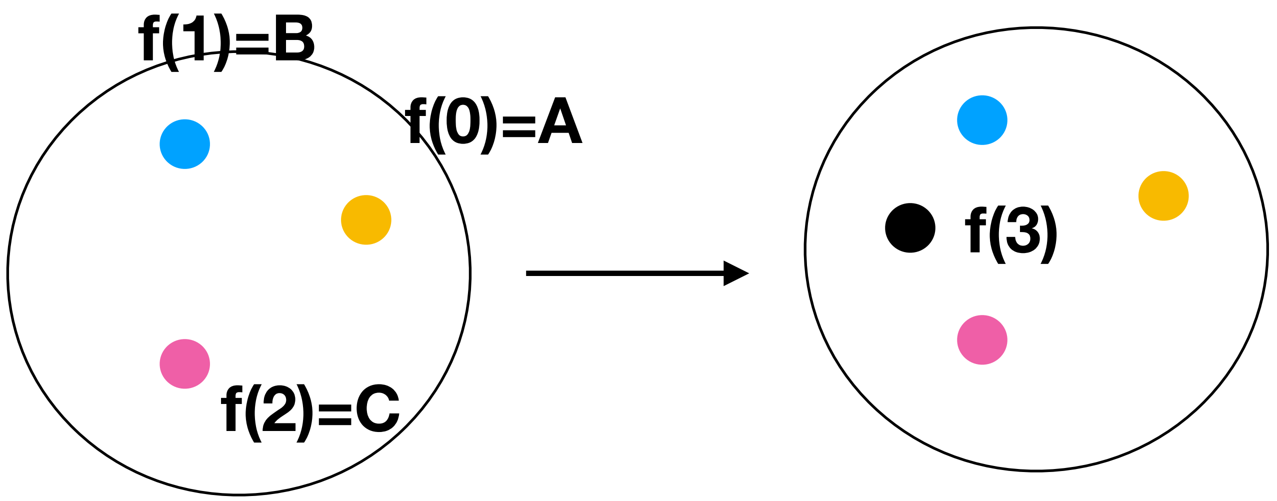

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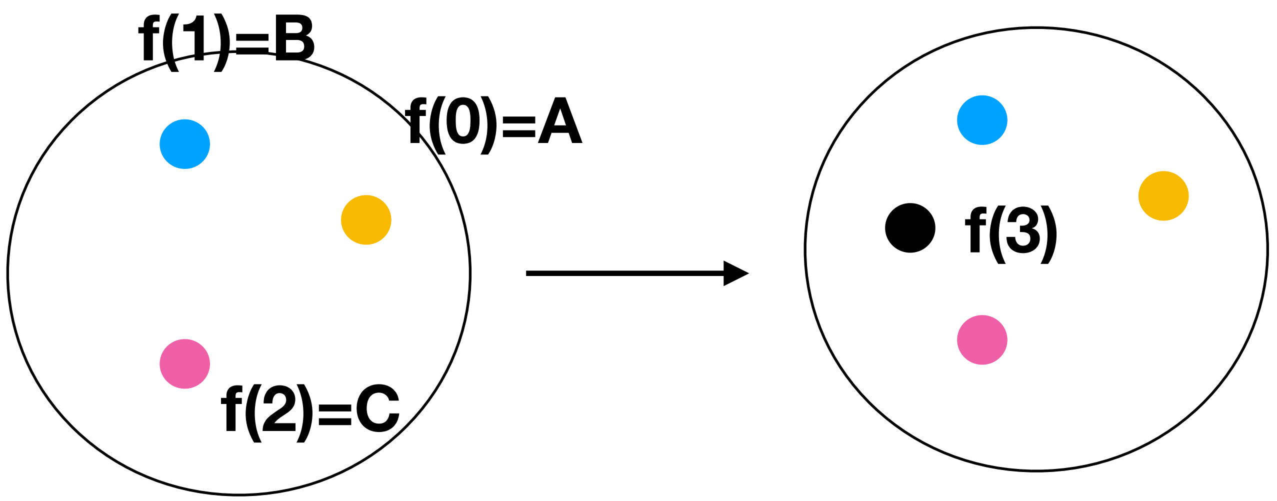

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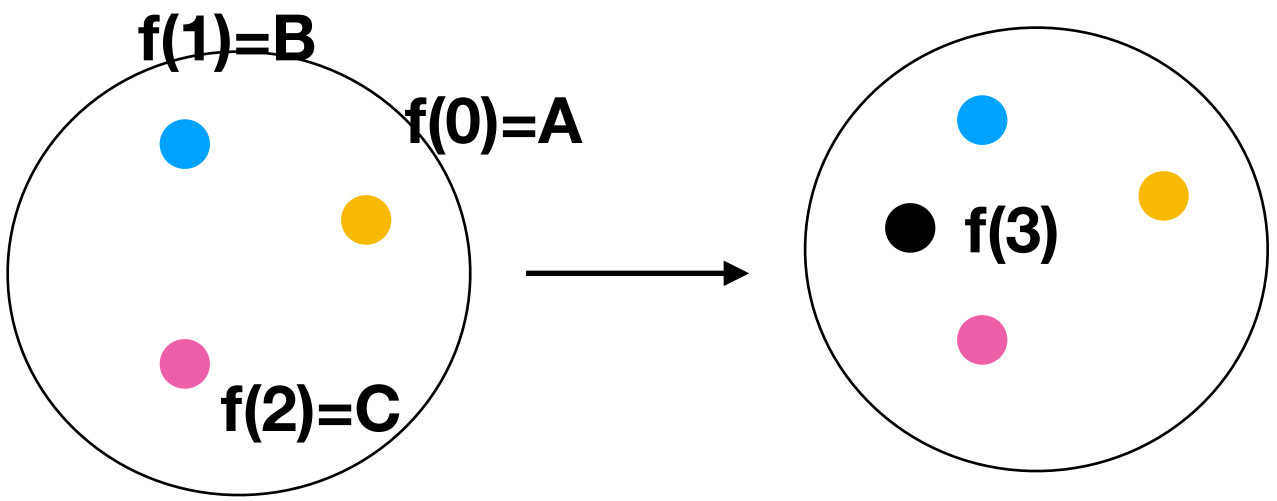

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Equivariant section:

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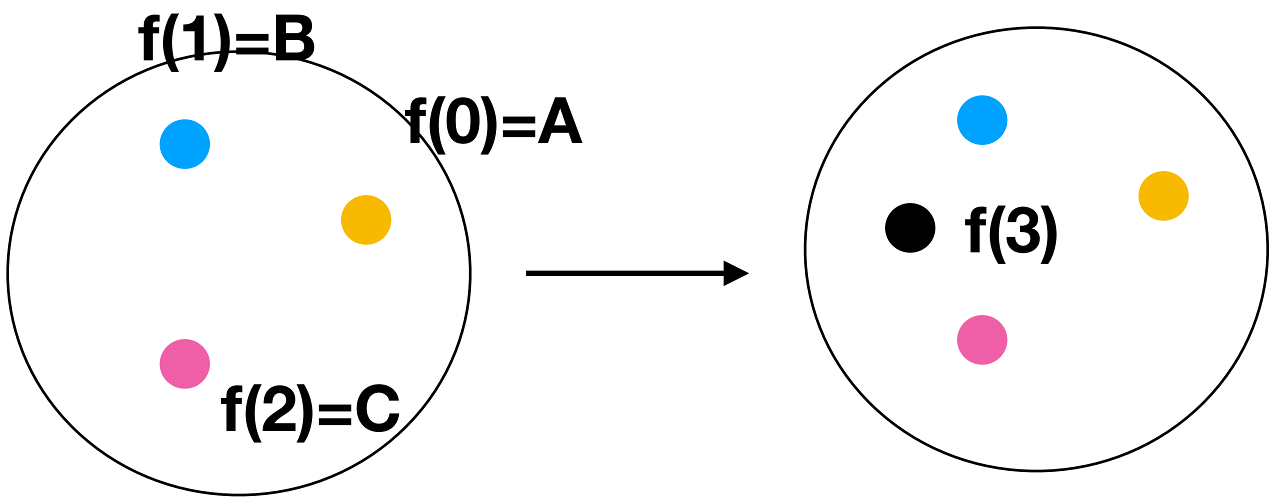

Unique Mobius map f: f(0)=A, f(1)=B, f(2)=C

Unordered case

Equivariant section:

Construct n(n-1)(n-2) many Mobius map using all order 3 numbers to arrange “add close by”

Proof techniques

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1. **Translate to a spherical braid group B_n problem**

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1. Translate to a spherical braid group B_n problem
2. Torsions in B_n are rigidity (Gives you GG result)
3. Canonical reduction system C of a <u>simple twist</u>
4. Decompose the sphere using C and consider the location of old points (which components)

Homological Stability for Selmer Spaces?

Aaron Landesman

Stanford University

Workshop on Arithmetic Topology
Vancouver, Canada

Slides available at http://www.web.stanford.edu/~aaronlan/slides/

Ranks of elliptic curves

Theorem (Mordell-Weil)

Let E be an elliptic curve over a global field K (such as $\mathbb{Q}$ or $\mathbb{F}_q(t)$). Then the group of K-rational points $E(K)$ is a finitely generated abelian group.

For E an elliptic curve over K, write $E(K) \simeq \mathbb{Z}^r \oplus T$ for T a finite group. Then, r is the **rank** of E.

Question

What is the average rank of an elliptic curve?

Motivation

Conjecture (Minimalist Conjecture)

The average rank of elliptic curves is $1/2$. Moreover,

- 50% of curves have rank 0,
- 50% have rank 1,
- 0% have rank more than 1.

Goal

Give three descriptions of certain Selmer spaces $\mathrm{Sel}^d_{n,\mathbb{F}_q}$, so that for n fixed, homological stability in d would imply the last part of the above conjecture over $\mathbb{F}_q(t)$.

What are the Selmer spaces?

Goal

Describe certain Selmer spaces $\mathrm{Sel}^d_{n,\mathbb{F}_q}$, so that for n fixed, homological stability in d would imply 0% of elliptic curves over $\mathbb{F}_q(t)$ have rank more than 1.

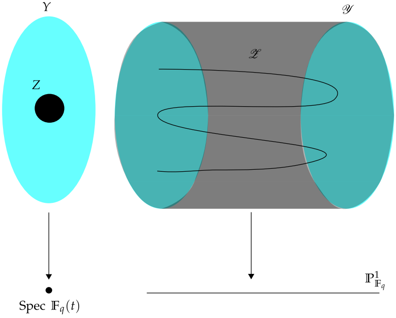

Figure: A point of Selmer space

Points of $\mathrm{Sel}^d_{n,\mathbb{F}_q}$ parameterize genus 1 curves Y of height d over $\mathbb{F}_q(t)$ with a degree n divisor Z. Alternatively, points parameterize genus 1 surfaces $\mathscr{Y}$ over $\mathbb{P}^1_{\mathbb{F}_q}$ of height d with a degree n divisor $\mathscr{Z}$.

Results on Selmer spaces

Theorem (L)

For $d \geq 2$, and char $k \neq 2$, dim $H_0(\mathrm{Sel}^d_{n,k}) = \sum_{m|n} m$. So the 0th homology of n-Selmer spaces stabilize in d, and stability is achieved once $d = 2$.

An elliptic curve over $\mathbb{F}_q(t)$ has height at most d if it can be written in the form

$$y^2 z = x^3 + A(s,t)xz^2 + B(s,t)z^3,$$

where $A(s,t)$ and $B(s,t)$ are homogeneous polynomials in $\mathbb{F}_q[s,t]$ of degrees $4d$ and $6d$.

Corollary

The proportion of elliptic curves of height at most d with rank ≥ 2 over $\mathbb{F}_q(t)$ tends to 0 as q tends to ∞.

Results on Selmer spaces

Corollary

The proportion of elliptic curves of height at most d with rank ≥ 2 over $\mathbb{F}_q(t)$ tends to 0 as q tends to ∞.

Question

Can one show 0% of elliptic curves of height d have rank ≥ 2 over a fixed $\mathbb{F}_q(t)$ in the limit that $d \to \infty$?

This question would likely be implied if one could show the higher homologies of Selmer spaces stabilize in d.

First description of Selmer spaces

Let $\mathscr{X}_n$ denote the algebraic stack parameterizing pairs (Y, D) where Y is a relative genus 1 curve and $D \subset Y$ is a flat degree n Cartier divisor, considered up to rational equivalence.
Then, the Selmer space is

$$\mathrm{Sel}^d_{n,k} = Hom_{12d}(\mathbb{P}^1_k, \mathscr{X}_n)$$

where Hom_{12d} denotes space of maps of degree $12d$.

Remark

One can also think of the above maps as relative genus 1 surfaces over $\mathbb{P}^1$ with a degree n divisor and with $12d$ singular fibers.

Second Description of Selmer spaces via $\mathscr{M}_{1,1}$

Let $\overline{\mathscr{M}}_{1,1}$ denote the moduli stack of semistable elliptic curves, $\mathscr{E} \to \overline{\mathscr{M}}_{1,1}$ denote the universal elliptic curve, and $\mathscr{E}[n] \subset \mathscr{E}$ denote the relative n-torsion.

$$\begin{array}{ccc} \mathscr{E}[n] & \longrightarrow & \mathscr{E} \\ & \searrow \quad \swarrow & \\ & \overline{\mathscr{M}}_{1,1} & \end{array}$$

Let $\mathscr{Y}_n := [\overline{\mathscr{M}}_{1,1}/\mathscr{E}[n]]$. Then, the Selmer space is

$$\mathrm{Sel}^d_{n,k} = Hom_{12d}(\mathbb{P}^1_k, \mathscr{Y}_n)$$

where Hom_{12d} denotes space of maps of degree $12d$.

Third Description of Selmer spaces via Hurwitz spaces

$$\begin{array}{ccc} \mathrm{Sel}^d_{n,\mathbb{C}} & \longrightarrow & \mathrm{CHur}^c_{\mathrm{ASL}_2(\mathbb{Z}/n\mathbb{Z}),12d} \\ \downarrow & & \downarrow \rho \\ \mathscr{W}^d_{\mathbb{C}} & \xrightarrow{f} & \mathrm{Conf}_{12d} \end{array} \tag{1}$$

is a fiber product where

Conf_{12d} is the space of $12d$ unordered points on $\mathbb{P}^1_{\mathbb{C}}$

$\mathscr{W}^d_{\mathbb{C}}$ is the space of height d elliptic curves over $\mathbb{C}(t)$ with squarefree discriminant

$\mathrm{CHur}^c_{\mathrm{ASL}_2(\mathbb{Z}/n\mathbb{Z}),12d}$ is a certain Hurwitz space of covers of $\mathbb{P}^1$

The map f sends the elliptic curve $y^2z = x^3 + A(s,t)xz^2 + B(s,t)z^3$, to the vanishing locus of its discriminant, $27A(s,t)^2 + 4B(s,t)^3$.

Summary

For given n and d over a fixed finite field k, there is a space $\mathrm{Sel}^d_{n,k}$ parameterizing "n-Selmer elements" for height d elliptic curves over $k(t)$.

$\vdots$						
$n = 3$	$\mathrm{Sel}^1_{3,k}$	$\mathrm{Sel}^2_{3,k}$	$\mathrm{Sel}^3_{3,k}$	$\mathrm{Sel}^4_{3,k}$	$\mathrm{Sel}^5_{3,k}$	$\mathrm{Sel}^6_{3,k}$
$n = 2$	$\mathrm{Sel}^1_{2,k}$	$\mathrm{Sel}^2_{2,k}$	$\mathrm{Sel}^3_{2,k}$	$\mathrm{Sel}^4_{2,k}$	$\mathrm{Sel}^5_{2,k}$	$\mathrm{Sel}^6_{2,k}$
$n = 1$	$\mathrm{Sel}^1_{1,k}$	$\mathrm{Sel}^2_{1,k}$	$\mathrm{Sel}^3_{1,k}$	$\mathrm{Sel}^4_{1,k}$	$\mathrm{Sel}^5_{1,k}$	$\mathrm{Sel}^6_{1,k}$
	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$

Homological stability in d for H_0 implies that 0% of elliptic curves over $\mathbb{F}_q$ have rank at least 2 in the large q limit. Homological stability in d for all H_i would likely imply that 0% of elliptic curves over a fixed finite field have rank at least 2.

Let G denote the group $\mathrm{ASL}_2(\mathbb{Z}/n\mathbb{Z})$ thought of as 3×3 matrices of the form

$$\begin{pmatrix} \alpha & \beta & * \\ \gamma & \delta & * \\ 0 & 0 & 1 \end{pmatrix}$$

where the upper 2×2 submatrix defines an element of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. Let $c \subset G$ denote the conjugacy class of the element

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Then, $\mathrm{CHur}^c_{\mathrm{ASL}_2(\mathbb{Z}/n\mathbb{Z}),r}$ denotes $\mathrm{ASL}_2(\mathbb{Z}/n\mathbb{Z})$ covers of $\mathbb{P}^1$ branched at r points, unbranched at $\infty \in \mathbb{P}^1$, with monodromy at those r points lying in c. Additionally, we require that the resulting cover is connected, and two covers are considered equivalent if they are related by translation by an element of G.

The original definition of the n-Selmer space

To construct the Selmer space, let $\mathscr{UW}_k^d$ be the universal family of Weierstrass models over $\mathscr{W}_k^d$. We have projections

$$\mathscr{UW}_k^d \xrightarrow{f} \mathbb{P}^1 \times \mathscr{W}_k^d \xrightarrow{g} \mathscr{W}_k^d.$$

Then,

$$\mathrm{Sel}_{n,k}^d := R^1 g_*(R^1 f_* \mu_n).$$