Worksheet 1

PRIMA Summer School on Brauer Classes

August 2, 2021

This is much more than you can do in half an hour, and not everyone will have the background to do every problem. Maybe start with #0 and #1, and then if you have more time, go to the problem that interests you the most.

0. **Introductions.**

   What are your colleagues' names? What university are they at? What is their favorite algebraic geometry book?

1. **An explicit $\mathbb{P}^1$ bundle with no relative $\mathcal{O}(1)$.**

   Work over $\mathbb{C}$. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^5$ be the "universal conic," cut out by the equation
   $$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$
   where x, y, z are coordinates on $\mathbb{P}^2$ and a, b, c, d, e, f are coordinates on $\mathbb{P}^5$. Let
   $$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 & & \mathbb{P}^5 \end{tikzcd}$$
   be the two projections, and let
   $$\mathcal{O}_X(m,n) := \pi_1^*\mathcal{O}_{\mathbb{P}^2}(m) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(n).$$

   (a) Observe that the fibers of π_2 are conics. Let $U \subset \mathbb{P}^5$ be the open set where the discriminant
   $$ae^2 + b^2f + cd^2 - bde - 4acf$$
   does not vanish, and let $V = \pi_2^{-1}(U)$; then $\pi_2 \colon V \to U$ is a *smooth* conic bundle, so it is a $\mathbb{P}^1$ bundle in the analytic topology, or the étale topology if you prefer that.

   (b) Observe that $\pi_1 \colon X \to \mathbb{P}^2$ is a $\mathbb{P}^4$ bundle, and that $\mathcal{O}_X(0,1)$ is a relative $\mathcal{O}(1)$. Thus every line bundle on X is of the form $\mathcal{O}_X(m,n)$.

(c) If we take a smooth fiber of π_2 and identify it with $\mathbb{P}^1$, then the restriction of $\mathcal{O}_X(m,n)$ is $\mathcal{O}_{\mathbb{P}^1}(2m)$. Convince yourselves that this implies that there is no relative $\mathcal{O}(1)$ for $V \to U$, or equivalently that $V \to U$ has no rational section, so it is not a $\mathbb{P}^1$ bundle in the Zariski topology.

(d) The $\mathbb{P}^1$ bundle $V \to U$ does not extend to a $\mathbb{P}^1$ bundle over $\mathbb{P}^5$, because the Brauer group of $\mathbb{P}^5$ is trivial. To see this you could use the exponential sequence as in problem 3 below, or you could use the Kummer sequence

$$\begin{gathered} 0 \to \mathbb{Z}/n \to \mathcal{O}^*_{\mathbb{P}^5} \to \mathcal{O}^*_{\mathbb{P}^5} \to 0 \\ f \mapsto f^n \end{gathered}$$

to see that $H^2(\mathcal{O}^*_{\mathbb{P}^5})$ has no n-torsion for any n.

(e) Optional: You can get sections of $V \to U$ on an explicit étale cover as follows. Let $U' \subset V$ be cut out by the equations

$$z = 0 \qquad \text{and} \qquad b^2 - 4ac \neq 0.$$

Convince yourselves that $\pi_2 \colon U' \to U$ is étale, meaning smooth of relative dimension 0, and that the pullback bundle $V \times_U U' \to U'$ has a section. Then for any line $L \subset \mathbb{P}^2$, we can replace $z = 0$ with the equation of the line, and $b^2 - 4ac \neq 0$ with an equation that says that the conic is not tangent to the line, to get a similar map $U'_L \to U$, and by letting L vary, we get an étale cover of U.

2. **Quillen bundles.**

In lecture we claimed that if $\alpha \in H^2(\mathcal{O}^*_X)$ is the Brauer class of a $\mathbb{P}^n$-bundle $\pi \colon P \to X$, then $\pi^*\alpha = 0$. And clearly the pullback bundle $\pi^*P = P \times_X P \to P$ has a section, given by the diagonal. So there must be a natural vector bundle F on P such that $\mathbb{P}F = P \times_X P$. It is sometimes called the *Quillen bundle*, and its restriction to a $\mathbb{P}^n$ fiber will be $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$, even though there need not be a line bundle on P whose restriction to a fiber is $\mathcal{O}(1)$.

(a) We have seen that if E is a vector bundle and $E' = E \otimes L$ for some line bundle L, then $\mathbb{P}E \cong \mathbb{P}E'$. (If you were uncomfortable with this before, now is a good time to think it through.) But the tautological bundles $\mathcal{O}_{\mathbb{P}E}(-1)$ and $\mathcal{O}_{\mathbb{P}E'}(-1)$ are different: convince yourselves that $\mathcal{O}_{\mathbb{P}E'}(-1) = \mathcal{O}_{\mathbb{P}E}(-1) \otimes \pi^*L$, by thinking about 1-dimensional subspaces of the fibers of E' and E.

Thus the vector bundles

$$\pi^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \qquad \text{and} \qquad \pi^*E' \otimes \mathcal{O}_{\mathbb{P}E'}(1)$$

are naturally isomorphic.

So when we have a $\mathbb{P}^n$ bundle $\pi\colon P \to X$ that is locally but not globally the projectivization of a vector bundle, the bundles $\pi^* E \otimes \mathcal{O}_{\mathbb{P}E}(1)$ glue together to give a global bundle F on P, whose restriction to a $\mathbb{P}^n$ fiber is $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$.

(b) Here is another perspective. The tangent bundle $T_{\mathbb{P}^n}$ has

$$\operatorname{Ext}^1(T_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) = H^1(\Omega^1_{\mathbb{P}^n}) = H^{1,1}(\mathbb{P}^n) = \mathbb{C}$$

canonically, with a basis given by the inclusion

$$\mathbb{Z} \cong H^2(\mathbb{P}^n, \mathbb{Z}) \hookrightarrow H^{1,1}(\mathbb{P}^n).$$

The preferred extension is given by the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}} \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

For the projectivization of a vector bundle E, the family version of the Euler sequence is

$$0 \to \mathcal{O}_{\mathbb{P}E} \to \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* E \to T_{\mathbb{P}E/X} \to 0.$$

But even for a $\mathbb{P}^n$ bundle $P \to X$ that is not the projectivization of a vector bundle, there is still a preferred extension

$$0 \to \mathcal{O}_P \to F \to T_{P/X} \to 0,$$

and the restriction of F to a $\mathbb{P}^n$ fiber is $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$. Discuss any questions you have about this.

(c) Optional, if you like Azumaya algebras: Notice that the vector bundles $\pi^* E \otimes \mathcal{O}_{\mathbb{P}E}(1)$ from part (a) satisfy

$$\pi_*(\pi^* E \otimes \mathcal{O}(1)) = E \otimes E^* = \operatorname{End}(E).$$

Convince yourselves that in general, π_* of the Quillen bundle will be an algebra that is locally but not globally a matrix algebra, that is, Azumaya algebra.

(d) Also optional: Let $C \subset \mathbb{P}^2$ be a smooth conic over a field, but allow the possibility that the field is not algebraically closed and the conic has no rational points, hence is not isomorphic to $\mathbb{P}^1$. For example, C could be the conic $x^2 + y^2 + z^2 = 0$ over $\mathbb{R}$. Argue that $F := T\mathbb{P}^2(-1)|_C$ is a Quillen bundle.

Generalize this description of the Quillen bundle to the case of a $\mathbb{P}^1$ bundle $\pi\colon P \to X$ defined over $\mathbb{C}$, but allow the possibility that the Brauer doesn't vanish. (If you haven't seen this analogy before, a conic without points over a non-closed field is like a family of conics without a rational section over a closed field. And this analogy works for any kind of variety – doesn't have to be conics.)

3. **Finding** $\mathrm{Br}(X)$ **with the exponential sequence.**

   (a) Work over $\mathbb{C}$, in the analytic topology. Then we have exponential sequence

   $$\begin{array}{c} 0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \\ \qquad\qquad f \mapsto e^{2\pi i f} \end{array}$$

   Write out the associated long exact sequence up to $H^3(\mathcal{O}_X)$.

   (b) Conclude that if X is a smooth complex variety with $H^{0,2}(X) = 0$ and $H^3(X, \mathbb{Z})$ torsion-free, then $\mathrm{Br}(X) = 0$. This includes $\mathbb{P}^n$, complete intersections of dimension ≥ 3, rational surfaces, and curves.

   On the other hand, if $H^{0,2}(X) \neq 0$ then $\mathrm{Br}(X)$ is infinite. This includes K3 and Abelian surfaces.

   Enriques surfaces are another interesting case, with $H^{0,2} = 0$ and $\pi_1 = \mathbb{Z}/2$, hence $\mathrm{Br} = \mathbb{Z}/2$.

   (c) Optional: If we retell the whole story about

   $$H^1(X, \mathrm{PGL}_r) \to H^2(\mathcal{O}_X^*)$$

   and the exponential sequence using smooth or continuous functions rather than holomorphic ones, then everything works the same, but now $\mathcal{O}_X$ has no higher cohomology, essentially because of partitions of unity. Convince yourselves that if the Brauer class of a $\mathbb{P}^{r-1}$ bundle maps to 0 in $H^3(X, \mathbb{Z})$, then it is the projectivization of a smooth vector bundle, though perhaps not a holomorphic one.

4. **If you really like transgression in spectral sequences.**

   Let X be a complex projective variety, and let $\pi\colon P \to X$ be a $\mathbb{P}^{r-1}$ bundle in the analytic topology. Then the E_2 page of the Leray spectral sequence

   $$E_2^{p,q} = H^p(R^q p_* \mathcal{O}_P^*) \ \Rightarrow \ H^{p+q}(\mathcal{O}_P^*)$$

   looks like

   $$\begin{tikzcd} \vdots & & & \\ 0 & \cdots & & \\ \mathbb{Z} \arrow[drr, "d"] & \cdots & & \\ \mathbb{C}^* & \mathrm{Pic}(X) & H^2(\mathcal{O}_X^*) & \cdots \end{tikzcd}$$

   and the exact sequence of low degree terms looks like

   $$0 \to \mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(P) \to \mathbb{Z} \xrightarrow{d} H^2(\mathcal{O}_X^*) \xrightarrow{\pi^*} H^2(\mathcal{O}_P^*).$$

Get your hands dirty with the map $d = d_2^{0,1}$ and convince yourselves that $d(1)$, or maybe $d(-1)$, is the Brauer class $\alpha \in H^2(\mathcal{O}_X^*)$ associated to P. Recall that we defined α using the connecting homomorphism

$$H^1(X, \mathrm{PGL}_r(\mathbb{C})) \to H^2(\mathcal{O}_X^*)$$

in Čech cohomology coming from the short exact sequence of coefficient groups

$$1 \to \mathbb{C}^* \to \mathrm{GL}_r(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C}) \to 1,$$

so you'll need some Čech description of d.

5. **If you like big diagrams of exact sequences.**

Let $\pi\colon P \to X$ be a $\mathbb{P}^{r-1}$ bundle, and let $\alpha \in H^2(\mathcal{O}_X^*)$ be its Brauer class.

(a) We saw in lecture that $r \cdot \alpha = 0$. The more usual way to see this is as follows. Let $\mathrm{SL}_r(\mathbb{C})$ be the special linear group, whose center is the scalar matrices with an r^{th} root of unity down the diagonal, and let $\mathrm{PSL}_r(\mathbb{C})$ be the quotient by this center. Then we get a diagram of exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/r \arrow[r] \arrow[d, hook] & \mathrm{SL}_r(\mathbb{C}) \arrow[r] \arrow[d, hook] & \mathrm{PSL}_r(\mathbb{C}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathrm{GL}_r(\mathbb{C}) \arrow[r] & \mathrm{PGL}_r(\mathbb{C}) \arrow[r] & 1 \end{tikzcd} \tag{1}$$

Convince yourselves that the right-hand vertical map is an isomorphism. Thus in the long exact sequences we get

$$\begin{tikzcd} H^1(X, \mathrm{PSL}_r(\mathbb{C})) \arrow[r] \arrow[d, equal] & H^2(X, \mathbb{Z}/r) \arrow[d] \\ H^1(X, \mathrm{PGL}_r(\mathbb{C})) \arrow[r] & H^2(\mathcal{O}_X^*), \end{tikzcd} \tag{2}$$

so the image of the bottom horizontal map is r-torsion.

(b) In lecture we interpreted α as the obstruction to the existince of a relative $\mathcal{O}(1)$ on $\pi\colon P \to X$, and we said that $r \cdot \alpha = 0$ because $\omega_{P/X}^*$ provides a relative $\mathcal{O}(r)$. So it seems like α might be an obstruction to taking an r^{th} root of $\omega_{P/X}$ on P, in some sense… Let's explore this idea.

First, extend the diagram (1) to a diagram with exact rows and columns as shown:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{Z}/r \arrow[r] \arrow[d] & \mathrm{SL}_r(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{PSL}_r(\mathbb{C}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, "z^r"] & \mathrm{GL}_r(\mathbb{C}) \arrow[r] \arrow[d, "\det"] & \mathrm{PGL}_r(\mathbb{C}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r, equal] \arrow[d] & \mathbb{C}^* \arrow[r] \arrow[d] & 1 & \\
 & 1 & 1 & &
\end{tikzcd}$$

So in the left-hand column we have the Kummer sequence, and you should convince yourselves that everything is ok.

Next, these short exact sequences give long exact sequences that extend the diagram (2) to the black parts of

$$\begin{tikzcd}
 & \mathrm{Pic}(X) \arrow[r, "\pi^*"] \arrow[d, "\cdot r"] & \mathrm{Pic}(P) \arrow[d, "\cdot r"] \\
 & \mathrm{Pic}(X) \arrow[r, "\pi^*"] \arrow[d] & \mathrm{Pic}(P) \arrow[d] \\
H^1(X, \mathrm{PSL}_r(\mathbb{C})) \arrow[r] \arrow[d, equal] & H^2(X, \mathbb{Z}/r) \arrow[r, "\pi^*"] \arrow[d] & H^2(P, \mathbb{Z}/r) \arrow[d] \\
H^1(X, \mathrm{PGL}_r(\mathbb{C})) \arrow[r] \arrow[d] & H^2(\mathcal{O}_X^*) \arrow[r, "\pi^*"] \arrow[d, "\cdot r"] & H^2(\mathcal{O}_{P^*}) \arrow[d, "\cdot r"] \\
0 \arrow[r] & H^2(\mathcal{O}_X^*) \arrow[r, "\pi^*"] & H^2(\mathcal{O}_{P^*}).
\end{tikzcd}$$

To this we add the blue parts coming from π^* of the Kummer sequence. Notice that the maps from the second row to the third give obstructions to taking r^{th} roots of line bundles.

Now take $[P]$ in the left-hand column, which maps to $\alpha \in H^2(\mathcal{O}_X^*)$ and then $0 \in H^2(\mathcal{O}_P^*)$. It has a preferred lift $\tilde{\alpha} \in H^2(X, \mathbb{Z}/r)$. If $\alpha \neq 0$ then $\tilde{\alpha}$ is not the obstruction to taking an r^{th} root of anything in $\mathrm{Pic}(X)$. But $\pi^*\tilde{\alpha} \in H^2(P, \mathbb{Z}/r)$ is probably the obstruction to taking an r^{th} root of $\omega_{P/X}$, or maybe $\omega_{P/X}^*$. And if $\alpha = 0$ and $P = \mathbb{P}E$ then $\tilde{\alpha}$ is probably the obstruction to taking an r^{th} root of $\det(E)$. Think about all this and see if you can say anything precise, or illuminating, or both.