Worksheet 2

PRIMA Summer School on Brauer Classes

August 3, 2021

Again this is much more than you can do in half an hour. Start with #0 and #1, then maybe look at another problem that interests you, or look at them later on your own.

0. **Introductions.**

   What are your colleagues' names? What university are they at? Have they done any travelling this summer?

1. **The Brauer class in a down-to-earth example.**

   Let X be a smooth projective curve of genus 2 over $\mathbb{C}$.

   (a) For any positive integer d there is a natural map from the symmetric power Sym^d_X to the Picard scheme Pic^d_X, sending a d-tuple of (unordered, not necessarily distinct) points $p_1, \cdots, p_d$ to the line bundle $\mathcal{O}_X(p_1 + \ldots + p_d)$. Convince yourselves that the fiber over $L \in \mathrm{Pic}^d_X$ is $\mathbb{P}H^0(L)$.

   (b) Suppose that $d \geq 3$. Use the Riemann–Roch formula and Serre duality to see that for any line bundle $L \in \mathrm{Pic}^d_X$ we have $h^0(L) = d-1$ and $h^1(L) = 0$. Thus the map $\mathrm{Sym}^d_X \to \mathrm{Pic}^d_X$ is a $\mathbb{P}^{d-2}$ bundle.

   (c) In fact the Brauer class of this $\mathbb{P}^{d-2}$ bundle vanishes, as follows. Choose a point $x \in X$. Consider map $\mathrm{Sym}^{d-1}_X \to \mathrm{Sym}^d_X$ given by

   $$p_1, \ldots, p_{d-1} \;\mapsto\; p_1, \ldots, p_{d-1}, x,$$

   and convince yourselves that the image is a divisor that gives a relative $\mathcal{O}(1)$. Or consider the map $\mathrm{Sym}^2_C \to \mathrm{Sym}^d_C$ given by

   $$p_1, p_2 \;\mapsto\; p_1, p_2, \overbrace{x, \ldots, x}^{d-2 \text{ times}},$$

   and convince yourselves that its image gives a rational section of $\mathrm{Sym}^d_C \to \mathrm{Pic}^d_C$. Or observe that

   $$\chi(L) = d - 1 \qquad\qquad \chi(L \otimes \mathcal{O}_X(x)) = d,$$

   and $\gcd(d-1, d) = 1$, so we get a relative $\mathcal{O}(1)$ by an argument from the end of today's lecture.

(d) But tomorrow we will want to work over a field k that is not algebraically closed – especially $k = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Q}_p$ – and then the Brauer class need not vanish. But if X has a k-rational point then the same argument shows that the Brauer class vanishes. In fact it is enough to have a point over an extension field of degree relatively prime to $d-1$, or indeed a 0-cycle of degree relatively prime to $d-1$. Notice that the canonical bundle gives a 0-cycle of degree 2.

(e) Optional: Generalize to curves of genus g. It may be helpful to know that the degree of the canonical bundle is $2g-2$.

2. **Issues with compactified Picard schemes of reducible schemes.**

In lecture we saw that rank-1 torsion-free sheaves on an integral scheme are stable with respect to any embedding in $\mathbb{P}^N$, so we can get a well-behaved compactification of the Picard scheme by taking its closure in the moduli space of stable sheaves. But here we will see that on a scheme with more than one irreducible component, some line bundles may be semi-stable or unstable.

Let X be the "banana curve," the union of two $\mathbb{P}^1$s meeting at two points, still over $\mathbb{C}$. Write $X = A \cup B$.

(a) Since A and B are isomorphic to $\mathbb{P}^1$, we can talk about the line bundles $\mathcal{O}_A(n)$ and $\mathcal{O}_B(n)$ for any $n \in \mathbb{Z}$. Convince yourselves that there are exact sequences

$$0 \to \mathcal{O}_B(-2) \to \mathcal{O}_X \to \mathcal{O}_A \to 0 \quad (1)$$
$$0 \to \mathcal{O}_A(-2) \to \mathcal{O}_X \to \mathcal{O}_B \to 0. \quad (2)$$

(The point is that the ideal sheaf of A in $A \cup B = X$ is the same as the ideal sheaf of $A \cap B$ in B.)
Also convince yourselves that there is a "Mayer–Vietoris sequence"

$$0 \to \mathcal{O}_X \to \mathcal{O}_A \oplus \mathcal{O}_B \to \mathcal{O}_{A\cap B} \to 0. \quad (3)$$

(b) Suppose we embed X into $\mathbb{P}^N$ in such a way that $\mathcal{O}_X(1)$ has degree 2 on each component: so, perhaps confusingly, we have $\mathcal{O}_X(1)|_A = \mathcal{O}_A(2)$ and $\mathcal{O}_X(1)|_B = \mathcal{O}_B(2)$. Let L be a line bundle on X whose restriction to A has degree a, and whose restriction to B has degree b. Using (3), check that the the Hilbert polynomial of X is

$$\chi(L(t)) = 4t + a + b.$$

Hint: Recall that $\chi(\mathcal{O}_{\mathbb{P}^1}(n)) = n + 1$.

(c) Use the exact sequence (1) to see that L is unstable if $b > a + 2$. Similarly, use (2) to see that L is unstable if $b < a - 2$. Convince yourselves that L is semi-stable if $b = a + 2$ or $a - 2$, and stable if $b = a + 1$, a, or $a - 1$.

So if we fix the Hilbert polynomial of L then there are either two values of a and b that give line stable bundles, or one value that gives stable bundles and two that give semi-stables. (It might help to graph these inequalities in the ab-plane.)

(d) Suppose we embed X into a different $\mathbb{P}^N$ so that $\mathcal{O}_X(1)$ has degree 1 on A and degree 2 on B. For example, X might be the union of a line A and a conic B in $\mathbb{P}^2$.

Now the Hilbert polynomial of L is

$$\chi(L(t)) = 3t + a + b,$$

L is unstable if $b > 2a - 1$ or $b < 2a - 3$, semi-stable if $b = 2a - 1$ or $b = 2a - 3$, and stable if $b = 2a + 2$.

So for a fixed Hilbert polynomial there exactly one value of a and b that give stable line bundles, and zero or one values that give semi-stables.

(e) Optional: Convince yourselves that for a fixed $a, b \in \mathbb{Z}$, there is a $\mathbb{C}^*$ worth of line bundles L with degree a on A and degree b on B. Or to put it another way, there is a $\mathbb{C}^*$ worth of ways to glue $\mathcal{O}_A(a)$ and $\mathcal{O}_B(b)$ together to get a line bundle on X.

Then let's sketch how these $\mathbb{C}^*$s get compactified by allowing line bundles to degenerate to torsion-free sheaves. If $x \in A \setminus B$, then ideal sheaf of x in X is a line bundle on X with degree -1 on A and degree 0 on B. If x wanders into $A \cap B$, then the ideal sheaf is torsion-free but no longer a line bundle. If x continues wandering into $B \setminus A$, then the ideal sheaf becomes a line bundle with degree 0 on A and degree -1 on B. Think about this and discuss any questions that arise.

3. **Details of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ degenerating to $\mathcal{O}(2) \oplus \mathcal{O}$ on $\mathbb{P}^1$.**

(a) Recall, or convince yourselves, that $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(2), \mathcal{O})$ is 1-dimensional, and that the non-split extension is given by the exact sequence

$$0 \to \mathcal{O} \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} \mathcal{O}(1)^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} \mathcal{O}(2) \to 0, \tag{4}$$

sometimes called the Euler sequence. Here x and y are homogeneous coordinates on $\mathbb{P}^1$.

We seek a vector bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ where for every $t \in \mathbb{A}^1$, the restriction to $\mathbb{P}^1 \times \{t\}$ is the extension of $\mathcal{O}(2)$ by $\mathcal{O}$ corresponding to $t \in \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O})$: thus for $t = 0$ we want $\mathcal{O}(2) \oplus \mathcal{O}$, and for all other t we want $\mathcal{O}(1)^2$. You might not be surprised to hear that such a "universal extension" exists for general reasons, but let's construct it explicitly in this case.

Let f be the projection $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$, and consider the map of vector bundles

$$f^*\mathcal{O}(-1)^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y \\ 0 & x \\ 0 & ty \end{pmatrix}} f^*\mathcal{O}^4. \tag{5}$$

The cokernel will be the bundle that we want.

(b) If we restrict to $t = 0$, then the cokernel of (5) is $\mathcal{O}(2) \oplus \mathcal{O}$.
Hint: Convince yourselves that the image of the top 3×2 block is the kernel of the surjection

$$\mathcal{O}^3 \xrightarrow{\left(\begin{smallmatrix} x^2 & xy & y^2 \end{smallmatrix}\right)} \mathcal{O}(2).$$

(c) If we restrict to some $t \neq 0$, then the cokernel of (5) is $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

Hint: Do some row operations to the 4×2 matrix so it is block diagonal with two 2×1 blocks; this won't change the image, so it won't change the cokernel. Then recognize the cokernel of each block using the exact sequence (4) twisted by -1.

(d) Optional: Think about how this relates to Hirzebruch surfaces.

By projectivizing $\mathcal{O} \oplus \mathcal{O}(2)$, we get the Hirzebruch surface $\mathbb{F}_2$, which can also be described as the blow-up of the cone $x^2 + y^2 = z^2$ in $\mathbb{P}^3$ at its singular point. By projectivizing $\mathcal{O}(1) \oplus \mathcal{O}(1)$ we get $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. From our work above we see that $\mathbb{F}_2$ is deformation-equivalent and thus diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, although it is not isomorphic as a complex manifold or algebraic variety. This was Hirzebruch's original point in introducing his surfaces: the even ones $\mathbb{F}_0, \mathbb{F}_2, \mathbb{F}_4, \dots$ are all diffeomorphic, as are the odd ones $\mathbb{F}_1, \mathbb{F}_3, \mathbb{F}_5, \dots$, and all of them are birational to one another, but none is isomorphic to any other, as can be seen by looking at the cone of curves.

(One place to read about Hirzebruch surfaces is Barth, Hulek, Peters, and van de Ven, *Compact complex surfaces*, Chapter V §4.)

4. **Checking stability in practice.**

It sounds hard to check that for *every* subsheaf $G \subset F$, the reduced Hilbert polynomial $p_G(t)$ is smaller than $p_F(t)$. But in many cases of interest, we can get away with checking much less.

Let X be smooth and connected, and let F be a vector bundle of rank 2. Work through the following sketch to prove that if every line bundle $L \subset F$ satisfies $p_L(t) < p_F(t)$, then F is stable.

(a) Suppose that $G \subset F$. Because F is torsion-free, we cannot have $\operatorname{rank}(G) = 0$.

(b) If $\operatorname{rank}(G) = 2$ then F/G is torsion, so either $G = F$ or $p_G(t) < p_F(t)$. (This is similar to what we saw in lecture with line bundles.)

(c) If $G \subset F$ has rank 1, take double duals to get a commutative square

$$\begin{array}{ccc} G & \longrightarrow & F \\ \downarrow & & \downarrow \\ G^{**} & \longrightarrow & F^{**}. \end{array}$$

Because F is a vector bundle, the right-hand vertical map is an isomorphism. Because the top horizontal map is injective, and the left-hand vertical map is generically an isomorphism, the bottom horizontal map is generically injective. Hence it is actually injective: otherwise the kernel would be a torsion sheaf. Next, because $\operatorname{rank}(G) = 1$ and X is smooth, G^{**} is a line bundle; this is the heart of the proof, and you might have to Google it. Lastly, the quotient G^{**}/G is torsion, so $p_{G^{**}}(t) \geq p_G(t)$, so if G was destabilizing for F then G^{**} would be at least as bad.

(If you want to study the intersection of two quadrics in $\mathbb{P}^5$ mentioned in lecture, this is a good way to prove stability.)