Monday: $\mathbb{P}^n$ bundles and their Brauer classes. (Nick Addington)

Tuesday: Moduli spaces of stable sheaves and the Brauer class there (Sarah Frei)

Wednesday: Using that Brauer class to obstruct rational points (Nick)

Colliot-Thélène + Skorobogatov new book: "The Brauer-Grothendieck group"

Kollár's notes on geom. of Severi-Brauer varieties arXiv: 1606.04368

Work / $\mathbb{C}$

X = variety in $\mathbb{P}^N$, smooth if you want.

E = holo. vector bundle or loc. free sheaf of rank r.

favorites: · tangent + cotangent bundle

- line bundles $\mathcal{O}(d)$ on $\mathbb{P}^N$
- tautological bundles on Grassmannians
- syzygy bundles of subvarieties

take $Z \subset X$

$$0 \leftarrow I_Z \leftarrow \mathcal{O}(-1)^2 \oplus \mathcal{O}(-2)^3 \leftarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3)^9 \leftarrow \ker \leftarrow 0$$

a way to get cool vector bundles.

E is loc. triv

w/ an open cover $\{U_i\}$ of X

trivializations $E|_{U_i} \cong \mathcal{O}_{U_i}^r$

on overlaps $U_{ij} := U_i \cap U_j$

get transition maps $\psi_{ij} : \mathcal{O}_{U_{ij}}^r \to \mathcal{O}_{U_{ij}}^r$

equiv, a map $U_{ij} \to Gl_r(\mathbb{C})$

satisfy a cocycle condition

$$\psi_{jk} \circ \psi_{ij} = \psi_{ik}$$

different loc. trivializations

$\rightsquigarrow$ change ψ_{ij} by a coboundary

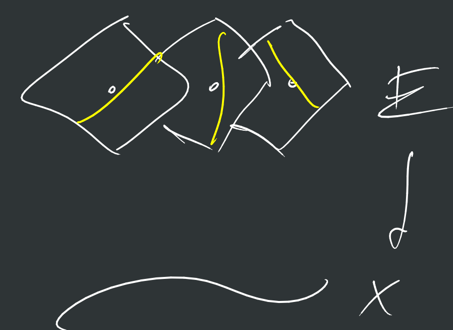
conversely, given a cocycle

can reconstruct E

So vector bundles on X

$= \check{H}^1(X, GL_r \mathbb{C})$

Projectivization of a v.b.



given a v.b. $E \to X$
of rank r

$\mathbb{P}E$ is a bundle of $\mathbb{P}^{r-1}$:

- $\mathbb{P}E = \{$1-dim'l subspaces of fibers of $E\}$

- or $\mathbb{P}E = (E \setminus \text{zero section}) / \mathbb{C}^*$

- or use the same trans. maps ψ_{ij}
  to glue $U_{ij} \times \mathbb{P}^{r-1}$
  rather than $U_{ij} \times \mathbb{C}^r$

but Gl_r doesn't act eff. on $\mathbb{P}^{r-1}$

center $\mathbb{C}^* =$ diagonal matrix
acts trivially.

in fact $\mathrm{Aut}(\mathbb{P}^{r-1}) = PGL_r\,\mathbb{C}$

$$:= Gl_r\,\mathbb{C}/\mathbb{C}^*.$$

$$\{\mathbb{P}^{r-1}\text{-bundles over } X\} = \check{H}^1(X, PGL_r\,\mathbb{C})$$

careful: analytic top. not Zariski

$$H^1(X, Gl_r) \longrightarrow H^1(X, PGL_r)$$

neither injective nor surjective in gen.

from $1 \to \mathbb{C}^* \to Gl_r \to PGL_r \to 1$

get a long exact seq.

$$H^1(\mathcal{O}_X^*) \to H^1(X, Gl_r) \to H^1(X, PGL_r) \to H^2(\mathcal{O}_X^*)$$

line bundles $= \mathrm{Pic}(X)$ vector bundles $\mathbb{P}^{r-1}$ bundles

Brauer classes live here.

if E is a v.b. and L is a l.b.

then $\mathbb{P}(E) = \mathbb{P}(E \otimes L)$

also: not every $\mathbb{P}^{r-1}$ bundle is $\mathbb{P}$(vector bundle)

obstruction $\alpha \in H^2(\mathcal{O}_X^*)$

is called the Brauer class of the

$\mathbb{P}^{r-1}$-bundle.

example in problem session.

Soon: α is r-torsion

Brauer group := all possible Brauer classes

of $\mathbb{P}^{r-1}$-bundles as r varies

$\subset$ torsion subgroup of $H^2(\mathcal{O}_X^*)$

Thm (Gabber + de Jong): it's the whole thing.

Let $\pi : P \to X$ be a $\mathbb{P}^{r-1}$ bundle

Def: a <u>relative $\mathcal{O}(d)$</u> is a line bundle L on P such that $L|_{\mathbb{P}^{r-1} \text{ fiber}} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(d)$

always, $\omega_{P/X}$ is a relative $\mathcal{O}(-r)$.

if $P = \mathbb{P}(E)$ then we get a rel. $\mathcal{O}(-1)$ and $\mathcal{O}(1)$

moreover, $\pi_* \mathcal{O}(1) = E^*$

(family version of the fact that $H^0(\mathcal{O}_{\mathbb{P}^{r-1}}(1)) =$ linear forms on $\mathbb{C}^r$

So conversely if $\exists$ a rel. $\mathcal{O}(1)$ for $\pi : P \to X$

take $E := \pi_*(\mathcal{O}(1))^*$

then $P = \mathbb{P}(E)$.

upshot: Brauer class = obstr. to having a rel. $\mathcal{O}(1)$.

let's see it with the Leray spectral seq.

$\pi : P \to X$ a $\mathbb{P}^{r-1}$ bundle

$$E_2^{p,q} = H^p(X, R^q\pi_* \mathcal{O}_P^*) \Rightarrow H^{p+q}(P, \mathcal{O}_P^*)$$

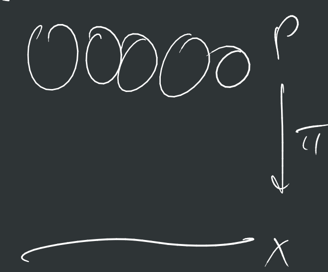
$R^q\pi_* \mathcal{O}_P^*$ = morally H^q of $\mathcal{O}_P^*$ along the fibers.

on $\mathbb{P}^{r-1}$, $H^0(\mathcal{O}^*) = \mathbb{C}^*$

$H^1(\mathcal{O}^*) = \mathrm{Pic} = \mathbb{Z}$

$H^2(\mathcal{O}^*) = 0$ (on worksheet)

higher... don't care.



in a family $P \to X$,

$R^0\pi_* \mathcal{O}_P^* = \mathcal{O}_X^*$

$R^1\pi_* \mathcal{O}_P^*$ = local system of $\mathbb{Z}$s

in fact trivial bec. $\omega_{P/X}$ gives a section of the loc. system.

$R^2\pi_* \mathcal{O}_P^* = 0$

$$E_2^{p,q} = H^p(X, R^q\pi_* \mathcal{O}_P^*) \Rightarrow H^{p+q}(P, \mathcal{O}_P^*)$$

E_2 page:

$$\begin{array}{c|cccc}
q & & & & \\
\vdots & \vdots & \vdots & \vdots & \\
q=2 & 0 & 0 & 0 & \cdots \\
q=1 & \mathbb{Z} & H^1(X,\mathbb{Z}) & H^2(X,\mathbb{Z}) & \cdots \\
q=0 & \mathbb{C}^* & \operatorname{Pic} X & H^2(\mathcal{O}_X^*) & \cdots \\
\hline
 & p=0 & p=1 & p=2 & p
\end{array}$$

$$d: \mathbb{Z} \to H^2(\mathcal{O}_X^*)$$

E_3 page:

$$\begin{array}{c|ccc}
q & & & \\
2 & 0 & 0 & 0 \\
1 & \ker d & ? & ? \\
0 & \mathbb{C}^* & \operatorname{Pic} X & \operatorname{coker} d \\
\hline
 & 0 & 1 & 2 \quad p
\end{array}$$

$p+q=1$

so $H^0(\mathcal{O}_P^*) = \mathbb{C}^*$

$$0 \to \operatorname{Pic} X \to H^1(\mathcal{O}_P^*) \to \ker d \to 0$$

$$0 \to \underline{\operatorname{coker} d} \to \underline{H^2(\mathcal{O}_P^*)} \to ?? \to 0$$

put them together:

$$1 \mapsto \pm\alpha$$

$$0 \to \operatorname{Pic} X \xrightarrow{\pi^*} \operatorname{Pic} P \to \mathbb{Z} \to H^2(\mathcal{O}_X^*) \xrightarrow{\pi^*} H^2(\mathcal{O}_P^*)$$

$L \mapsto$ deg of L on fiber of $\pi : P \to X$

- this describes $\mathrm{Pic}\, P$
- $r \in \mathbb{Z}$ is the image of $\omega_{P/X}^{\vee} \in \mathrm{Pic}\, P$

  so $r \cdot \alpha = 0$

- also: $\pi^* \alpha = 0$ in $\mathrm{Br}(P)$.

$$\begin{tikzcd} \pi^* P = P \times_X P \arrow[r] \arrow[d] & P \arrow[d] \\ P \arrow[r, "\pi"'] & X \end{tikzcd}$$

$\pi^* P$ is $\mathbb{P}(F)$ for some vector bundle F on P ??

Quillen bundle

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next goal: Brauer class of $\pi : P \to X$ vanishes iff it has a rational section

first: the dual $\mathbb{P}^{r-1}$ bundle P^*

if E is a vector bundle of rank r, with trans. maps $\varphi_{ij} : U_{ij} \to GL_r \mathbb{C}$

then for dual bundle $E^{\vee}$,

trans. maps are $((f_{ij})^T)^{-1}$

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, "z \mapsto z^{-1}"] & GL_r \arrow[r] \arrow[d, "A \mapsto A^{t-1}"] & PGL_r \arrow[r] \arrow[d, "\exists"] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL_r \arrow[r] & PGL_r \arrow[r] & 1
\end{tikzcd}$$

So for any $\mathbb{P}^{r-1}$ bundle $P \to X$
we can form $P^\vee$

points of $P^\vee$ = hyperplanes in fibers of P

Brauer class of $P^\vee$ is $-$ Brauer class of P

Prop: TFAE

1. $\pi: P \to X$ is loc. triv. in Zariski top.
2. it has a rational section
3. Brauer class vanishes
4. it's $\mathbb{P}(E)$ for some v.b. E.

(1) $\Rightarrow$ (2) Zar. open $U \subset X$

$$P|_U \cong U \times \mathbb{P}^{r-1}$$

has a section over U.

(2) $\Rightarrow$ (3) point of P = hyperplane in fibers of $P' \to X$

So a rational section of $P \to X$ gives a divisor in P' whose line bundle is a rel. $\mathcal{O}(1)$

So $\underline{-\alpha = 0}$ so $\alpha = 0$.

(3) $\Rightarrow$ (4) what we've been doing today.

(4) $\Rightarrow$ (1) a vector bundle is always Zar. loc. triv.