Monday: $\mathbb{P}^n$-bundles

$\pi: P \to X$ over $\mathbb{C}$

didn't talk much about
Azumaya algs. or
central simple algs.

$P = \mathbb{P}E$ loc. but not globally
E only well-def. up to $E \rightsquigarrow E \otimes L$
but $\underbrace{End(E)}_{\text{bundle of algebras}} = E^* \otimes E$ does globalize

$$End(E \otimes L) \cong End(E)$$

Conversely, given a bundle of algs
A on X, locally $M_r(\mathbb{C})$
can produce a $\mathbb{P}^{r-1}$-bundle
bec. $Aut(M_r(\mathbb{C})) = PGL_r \mathbb{C}$

worked / $\mathbb{C}$, but what about other fields k.

if $k = \bar{k}$, story is very similar.
use étale top rather than analytic.

over $\mathbb{C}$, $H^2_{ét}(\mathcal{O}_X^*)$ = tors. part of $H^2_{an}(\mathcal{O}_X^*)$

If $k \neq \bar{k}$ now "$\mathbb{P}^n$-bundle" should mean a smooth morphism $\pi : P \to X$ whose geom. fibers are $\mathbb{P}^n$.

now it's interesting even $X = \operatorname{Spec} k$

first example: $X = \operatorname{Spec} \mathbb{R}$
P = pointless conic
$x^2 + y^2 + z^2 = 0 \subset \mathbb{P}^2_{\mathbb{R}}$

a rational section of $P \to X$ would just be an $\mathbb{R}$-point of P but there is none.

a rel. $\mathcal{O}(1)$ would also give
an $\mathbb{R}$-point of P

but base change

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & P \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{R} = X \end{tikzcd}$$

this is an étale open cover of X.

the Azumaya alg. is the quaternions $(\mathbb{H})$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = M_2(\mathbb{C})$$

in fact $Br(\mathbb{R}) = \mathbb{Z}/2$

also: $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$

for $\mathrm{Br}(\mathbb{Q})$, exact seq.

$$0 \to \mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(\mathbb{R}) \oplus \bigoplus_p \mathrm{Br}(\mathbb{Q}_p)$$
$$\longrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

"class field theory"

not here to tell that story properly, but we'll use it later.

$\mathrm{Br}(\mathbb{F}_q) = 0$

but $\mathrm{Br}(\mathbb{F}_q((t))) = \mathbb{Q}/\mathbb{Z}$

$\mathrm{Br}(\mathbb{F}_q(t))$ is like $\mathrm{Br}(\mathbb{Q})$

On Tuesday, Sarah did moduli sp. of stab. sh. + their Brauer class

story works equally well over any field.

also said that line bundles on proj. geom. integral schemes are stable w.r.t. any embedding $X \hookrightarrow \mathbb{P}^N$

$\rightsquigarrow$ Pic_X has a Brauer class that obstructs existence of a univ. line bundle on $X \times \mathrm{Pic}_X$.

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let X = sm. proj. curve of genus g over an alg. closed field,

Pic^0_X is an ab. var of dim g.

points param deg-0 line bundles on X.

Pic^d_X : points param. deg-d line bundles.

$\mathrm{Pic}^0_X \circlearrowright \mathrm{Pic}^d_X$ freely + transitively.

so if we choose $M \in \mathrm{Pic}^d_X$

then $\mathrm{Pic}^0_X \to \mathrm{Pic}^d_X$ is an iso

$L \longmapsto L \otimes M$

but no preferred iso.

"torsor" or "principal homog. space" over Pic^0.

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if k not alg. closed

then Pic^0 and Pic^d are defined $/\mathbb{k}$

a line bundle L on X, def over k gives a point of Pic^d

but not every point of Pic^d comes from a line bundle!

Example 0: $k = \mathbb{R}$ X = conic $x^2+y^2+z^2=0$ in $\mathbb{R}$

no l.b. of deg 1 def$/\mathbb{R}$

OTOH, $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}$

$$\begin{array}{ccccccc}
\cdots & \mathcal{O}(-2) & \mathcal{O}(-1) & \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) & \cdots \\
 & = K_X & \circlearrowleft_{\sigma} & \circlearrowleft_{\sigma} & \circlearrowleft_{\sigma} & \circlearrowleft_{\sigma} & \\
 & \circlearrowleft_{\sigma} & & & & &
\end{array}$$

$\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ gen'd by complex conj. (call it σ)

given $x \in X(\mathbb{C})$

$\sigma(x) \neq x$

but $\mathcal{O}_X(x) \cong \sigma^* \mathcal{O}_X(x)$

so give an $\mathbb{R}$-point of Pic_X!

Example 1: (Poonen + Stoll "The Cassels-Tate pairing..." Prop. 26)

$k = \mathbb{R}$ $\quad X =$ hyperelliptic curve determined by $y^2 = \underbrace{-x^6 - x - 1}_{< 0 \ \forall x}$

genus $= 2$

so X has no $\mathbb{R}$-points.

claim: X has no line bundles of deg 3, def. over $\mathbb{R}$

pf: $\chi = 3 + 1 - 2 = 2 > 0$

so $H^0(\deg 3 \text{ line bundle}) \neq 0$

a section would cut out an eff. zero-cycle of deg 3

w/ either 3 $\mathbb{R}$-points

or an $\mathbb{R}$-point and a $\mathbb{C}$-point.

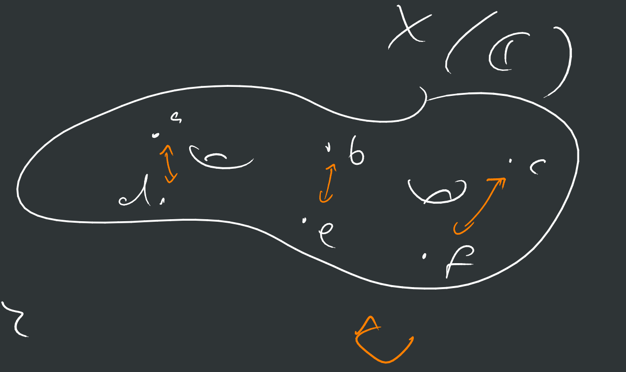
But Pic^3_X still has $\mathbb{R}$-points!

claim:

$\mathcal{O}_X(a+b+c)$

$\cong \mathcal{O}_X(d+e+f)$

pf: problem session



hyperelliptic involution $y \mapsto -y$ fixes 6 points

So they give a $\mathbb{C}$-point of Pic^3_X that's fixed by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$

$\rightsquigarrow$ an $\mathbb{R}$-point of Pic^3_X.

check out Gross + Harris "Real alg. curves"

Example 2 on worksheet:

genus-3 curve over $\mathbb{Q}_p$ with no line bundle of deg 2 but Pic^2_X has $\mathbb{Q}_p$-points.

Back to gen setup:

X = proj, geom. integral var / field k

Pic_X is an open subset of
the moduli sp. of stab. sh. on X

$\rightsquigarrow \exists\ \alpha \in \mathrm{Br}(\mathrm{Pic}_X)$ that obstructs
existence of a univ. line bundle

a point $p \in \mathrm{Pic}_X(k)$ represents an
actual line bundle iff $\alpha|_p = 0 \in \mathrm{Br}(k)$

Sarah said: order of α divides
$\gcd\{\chi(L \otimes E) \mid E$ any v.b. on X def. over $k\}$

if X has a k-point x
then $\chi(L \otimes \mathcal{O}(x)) = \chi(L) + 1$
so $\gcd = 1$ so $\alpha = 0$

More gen if X has a pt
over a deg-d extension of k
then $d \cdot \alpha = 0$

(or just a zero-cycle of deg d def/k)

traditional interpretation:
Hochschild-Serre spec. seq gives

$$0 \to \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_X(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X)$$
$$p \longmapsto \alpha|_p$$

a k-point of X would split this map

not obv from our perspective: $p \mapsto \alpha|_p$
is a group hom.

OTOH we can deal with compactified
$\overline{\mathrm{Pic}}_X$ just as easily as Pic_X
(if X is not smooth)

Brauer–Manin obstr. to $\mathbb{Q}$-points.

$k = \mathbb{Q}$ $\qquad\qquad\qquad \alpha \in Br(Pic^3_X)$

X = hyperell. curve determined by

$$y^2 = -x^6 - x - 1$$

Seen: $\quad X(\mathbb{R}) = \emptyset$

$Pic^3_X(\mathbb{R}) \neq \emptyset$

but $\forall\, p \in Pic^3_X(\mathbb{R})$

$$\alpha|_p \neq 0 \text{ in } Br(\mathbb{R}) = \mathbb{Z}/2$$

Thm of Lichtenbaum: $\quad Pic^3_X(\mathbb{Q}_p) \neq \emptyset \quad \forall p$

claim: $\underline{\alpha \text{ vanishes at all those points.}}$

by prev. comments enough to show that X has $\mathbb{Q}_p$-points

or at least 0-cycle of deg 1, $\forall p$

K_X has deg 2, so enough to get a point over an ext of odd deg.

Computer: X is smooth over $\mathbb{F}_p$

unless $p = 2$ or 101 or 431.

over $\mathbb{F}_2$, $(1,1)$ is a smooth pt

$\leadsto$ lifts to a $\mathbb{Q}_2$ point

by Hensel's lemma.

Sim we find smooth $\mathbb{F}_{101}$ and $\mathbb{F}_{431}$ points

if X is smooth $/\mathbb{F}_p$, by the

weil conjectures

X has $\mathbb{F}_{p^k}$ points $\forall\ k \gg 0$

take k odd

$\leadsto$ Hensel's lemma gives a point

over an ext $K/\mathbb{Q}_p$ of deg k.

$\boxtimes$

Last: $P_{12}{}^3_X$ has $\mathbb{R}$-points

and $\mathbb{Q}_p$-points $\forall p$

but no $\mathbb{Q}$-points.

$$\text{have} \quad 0 \to Br(\mathbb{Q}) \to \overset{\mathbb{Z}/2}{Br(\mathbb{R})} \oplus \bigoplus_p \overset{\mathbb{Q}/\mathbb{Z}}{Br(\mathbb{Q}_p)} \to \underline{\mathbb{Q}/\mathbb{Z}} \to 0$$

suppose $p \in P_{12}{}^3_X(\mathbb{Q})$

then $\alpha|_p \in Br(\mathbb{Q})$

maps to $1/2$ in $Br(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$

maps to 0 in $Br(\mathbb{Q}_p)$ $\forall p$

so $1/2$ in that last $\mathbb{Q}/\mathbb{Z}$

but $Br(\mathbb{Q}) =$ ker of the map $\to \mathbb{Q}/\mathbb{Z}$

$\to\leftarrow$.

$\blacksquare$