Worksheet 3

PRIMA Summer School on Brauer Classes

August 4, 2021

0. **Introductions.**

   What are your colleagues' names? What university are they at? Do they have any pets?

1. **Pic^3 of a genus-2 curve has $\mathbb{R}$-points, even if the curve doesn't.**

   Let X be a smooth, projective curve of genus 2 defined over $\mathbb{R}$. The canonical bundle satisfies $h^0(K_X) = 2$, $h^1(K_X) = 1$, and $\deg(K_X) = 2$.

   (a) Argue that X has line bundles of odd degree if and only if it has $\mathbb{R}$-points. (The main step was essentially done in lecture: if $\deg L$ is odd, then any divisor associated to L is a formal linear combination of some number of $\mathbb{C}$-points and an odd number of $\mathbb{R}$-points.)

   (b) The canonical bundle determines a map $f\colon X \to \mathbb{P}^1$. Over $\mathbb{C}$, it is a 2-to-1 cover ramified at 6 points. We have $f^*\mathcal{O}_{\mathbb{P}^1}(1) = K_X$. If $A, B, C, D, E, F \in X(\mathbb{C})$ are the 6 ramification points, then

   $$\begin{aligned} K_X = \mathcal{O}_X(2A) = \mathcal{O}_X(2B) = \mathcal{O}_X(2C) \\ = \mathcal{O}_X(2D) = \mathcal{O}_X(2E) = \mathcal{O}_X(2F). \end{aligned}$$

   (c) On the other hand, the adjunction formula for a branched double cover gives

   $$K_X = f^*K_{\mathbb{P}^1} \otimes \mathcal{O}_X(A + B + C + D + E + F),$$

   and we have $K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, so

   $$K_X^{\otimes 3} = \mathcal{O}_X(A + B + C + D + E + F).$$

   (d) Thus $\mathcal{O}_X(A + B + C) = \mathcal{O}_X(D + E + F)$.

(e) If X has no $\mathbb{R}$-points, then complex conjugation puts the 6 ramification points A, B, C, D, E, F into pairs. Label them so that they are paired

$$A \leftrightarrow D \qquad\qquad B \leftrightarrow E \qquad\qquad C \leftrightarrow F.$$

Conclude that the line bundle $\mathcal{O}_X(A+B+C)$ gives a point of $\mathrm{Pic}^3_X(\mathbb{C})$ that is fixed by complex conjugation, hence gives a point of $\mathrm{Pic}^3_X(\mathbb{R})$.

(f) Optional: Think about this in terms of the $\mathbb{P}^1$ bundle

$$\mathrm{Sym}^3(X) \to \mathrm{Pic}^3(X)$$

that appeared on Tuesday's worksheet. Complex conjugation acts on the $\mathbb{C}$-points of both varieties, and there are no fixed points upstairs, but there are fixed points downstairs.

Again, if you like this kind of thing you should check out Gross and Harris's paper *Real algebraic curves.*

2. **Pic^2 of a genus-3 curve over $\mathbb{Q}_p$.**

This example comes from Poonen and Stoll's paper *The Cassels–Tate pairing on polarized abelian varieties*, Proposition 29.

Let p be a prime, and let X be the plane quartic curve

$$x^4 + py^4 + p^2z^4 = 0$$

over $\mathbb{Q}$. It is a smooth curve of genus 3.

(a) To warm up, argue that X has no $\mathbb{Q}_p$-points. If $(x : y : z)$ were such a point, clear denominators and common factors to get a point with values in $\mathbb{Z}_p$ whose coordinates are relatively prime. Now we must have $p \mid x^4$, so $p \mid x$, so $p^4 \mid x^4$, so $p^2 \mid py^4$, so $p \mid y$, so $p^4 \mid y^4$, so $p^4 \mid p^2z^4$, so $p \mid z$. But this contradicts the fact that the coordinates were relatively prime.

(b) Next we want to show that X has no line bundles of odd degree, or equivalently, no zero-cycles of odd degree. Convince yourselves that it is equivalent to show that if X has a rational point over a finite extension $K/\mathbb{Q}_p$, then the degree of the extension is even.

(c) Let $\mathcal{O}_K$ be the ring of integers in K, that is, the integral closure of $\mathbb{Z}_p$ in K. Then $\mathcal{O}_K$ is a discrete valuation ring, its residue field is $\mathbb{F}_{p^f}$ for some integer f, and we have $\deg(K/\mathbb{Q}_p) = ef$ where e is called the *ramification index*. The valuation of p changes from 1 in $\mathbb{Z}_p$ to e in $\mathcal{O}_K$.

Let $(x : y : z)$ be a K-point of X. Clear denominators and common factors to get an $\mathcal{O}_K$-point. If e is odd then x^4 and py^4 and p^2z^4 all have different valuations modulo 4, so they cannot add up to zero.

(d) Next we want to argue that if $p \equiv -1 \pmod 4$, then X has no line bundles of degree 2. (Notice that X does have line bundles of degree 4, including the canonical bundle, which is $\mathcal{O}_X(1)$.) Equivalently, we want to show that if X has a K-rational point then $\deg(K/\mathbb{Q}_p) \equiv 0 \pmod 4$. So if $e \equiv 0 \pmod 4$ then we're done, and if $e \equiv 2 \pmod 4$ then we want to show that f is even.

If $e \equiv 2 \pmod 4$, then py^4 has a different valuation from $x^4 + p^2z^4$, so we must have $py^4 = 0$ and $x^4 + p^2z^4 = 0$. Thus pz^2/x^2 is a square root of -1, so its image in the residue field $\mathbb{F}_{p^f}$ is a square root of -1 there, so in particular the group of units $\mathbb{F}_{p^f}^\times$ has an element of order 4, so $p^f \equiv 1 \pmod 4$. Because $p \equiv -1 \pmod 4$, we see that f must be even.

(e) Optional: Let $\alpha \in \mathrm{Br}(\mathrm{Pic}^2_X)$ be the Brauer class that obstructs the existence of a universal line bundle on $X \times \mathrm{Pic}^2_X$. The Hilbert polynomial of a degree-2 line bundle is $4t$, so $4 \cdot \alpha = 0$.

By a theorem of Lichtenbaum, Pic^2_X has $\mathbb{Q}_\ell$-points for every prime ℓ. We see that α cannot vanish at any $\mathbb{Q}_p$-point, because there are no actual line bundles of degree 2 defined over $\mathbb{Q}_p$. Poonen and Stoll argue that if $p \equiv -1 \pmod{16}$, for example $p = 31$, then X has a $\mathbb{Q}_2$-point; thus α vanishes at every $\mathbb{Q}_2$-point of Pic^2_X. And X is smooth over $\mathbb{F}_\ell$ for any prime $\ell \neq 2$ or p, so you can argue that X has a zero-cycle of degree 1; thus α vanishes at every $\mathbb{Q}_\ell$-point of Pic^2_X.

I suspect that if $p \equiv -1 \pmod{16}$ and w is any $\mathbb{Q}_p$-point of Pic^2_X, then $\alpha|_w \in \mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ has order 4, not order 2. But I don't know how to prove it directly. Do you have any ideas? (If you do, please email me: `adding@uoregon.edu`.)