Abelian surfaces with everywhere good reduction

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Lassina Dembélé

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1 Introduction

In these notes, we describe methods for computing abelian surfaces with everywhere good reduction. There are two essential ingredients to our approach: the Eichler-Shimura conjecture for totally real number fields and explicit equations for Hilbert modular surfaces. This requires that we start with a review of the necessary background. For the most part, we follow our joint paper with A. Kumar [12].

2 Hilbert modular forms

Let F be a totally real field of narrow class number one and degree $d > 1$. We let $\mathcal{O}_F$ be the ring of integers of F, $\mathfrak{d}_F$ the different of F. For each $i = 1, \ldots, d$, let $a \mapsto a^{(i)}$ denote the i-th embedding of F into $\mathbf{R}$, so that we have an identification $F \otimes \mathbf{R} \simeq \mathbf{R}^d$. We let F_+ be the set of totally positive elements in F, i.e. the inverse image of $(\mathbf{R}_+)^d$, and $\mathcal{O}_{F,+} = F_+ \cap \mathcal{O}_F$. We fix a totally positive generator δ of $\mathfrak{d}_F$. (Note that every ideal has such a generator since F has narrow class number one.)

2.1 Basic definitions and properties

Let $\mathfrak{H}$ be the Poincaré upper half plane. The *Hilbert modular group* $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathfrak{H}^d$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, \ldots, z_d) = \left(\frac{a^{(i)} z_i + b^{(i)}}{c^{(i)} z_i + d^{(i)}} \right)_{i=1,\ldots,d}.$$

Let $k \geq 2$ be an *even* integer. The action above induces an action of the Hilbert modular group on the set of functions $f : \mathfrak{H}^d \to \mathbf{C}$ by

$$(f|_k\gamma)(z) = \Big(\prod_{i=1}^{d} (c^{(i)} z_i + d^{(i)}) \Big)^{-k} f(\gamma z),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F).$$

Let $\mathfrak{N}$ be an integral ideal, and set

$$\Gamma_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : c \in \mathfrak{N} \right\}.$$

Definition 2.1. *A* Hilbert modular form *of* weight k *and* level $\mathfrak{N}$ *is a holomorphic function* $f : \mathfrak{H}^d \to \mathbf{C}$ *such that*

$$f|_k\gamma = f \text{ for all } \gamma \in \Gamma_0(\mathfrak{N}).$$

Equivalently, this means that

$$f(\gamma z) = \Big(\prod_{i=1}^{d}(c^{(i)}z_i + d^{(i)})\Big)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N}).$$

We denote by $M_k(\mathfrak{N})$ the space of all Hilbert modular forms of weight k and level $\mathfrak{N}$.

Let $f \in M_k(\mathfrak{N})$. Then f is invariant under the matrices $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ for $\mu \in \mathcal{O}_F$, which act as $z \mapsto z + \mu$. So it admits a q-expansion

$$f(z) = \sum_{\mu \in \mathcal{O}_F} a_\mu e^{2\pi i \mathrm{Tr}(\frac{\mu z}{\delta})},$$

where $\mathrm{Tr}(\nu z) = \nu^{(1)}z_1 + \cdots + \nu^{(d)}z_d$, for $\nu \in F$. The following result is essentially a consequence of the Dirichlet unit theorem.

Lemma 2.2 (Goetzky-Koecher's principle)**.** *Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. Then f admits a q-expansion of the form*

$$f(z) = a_0 + \sum_{\mu \in \mathcal{O}_{F,+}} a_\mu e^{2\pi i \mathrm{Tr}(\frac{\mu z}{\delta})}.$$

In particular, f is holomorphic (at the cusps*).*

Proof. See Bruinier [3] or Goren [23]. □

Let $f \in M_k(\mathfrak{N})$ be a Hilbert modular form. Since f is invariant under the action of the matrices $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ for $\epsilon \in \mathcal{O}_F^\times$ in $\mathrm{SL}_2(\mathcal{O}_F)$, which act as $z \mapsto \epsilon^2 z$, we have

$$a_{\epsilon^2\mu} = a_\mu \text{ for all } \mu \in \mathcal{O}_{F,+} \text{ and } \epsilon \in \mathcal{O}_F^\times.$$

So, for every ideal $\mathfrak{m} \subseteq \mathcal{O}_F$, the quantity $a_\mathfrak{m}(f) = a_\mu$, where μ is a totally positive generator of $\mathfrak{m}$, is well-defined and depends only on $\mathfrak{m}$.

Definition 2.3. *Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$, and write its q-expansion*

$$f(z) = \sum_{\mu \in \mathcal{O}_{F,+}} a_\mu e^{2\pi i \mathrm{Tr}(\frac{\mu z}{\delta})}.$$

For every integral ideal $\mathfrak{m}$, we define the Fourier coefficient *of f at $\mathfrak{m}$ by*

$$a_\mathfrak{m}(f) := a_\mu,$$

where $\mathfrak{m} = (\mu)$.

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. For every $\gamma \in \mathrm{SL}_2(F)$, we can write the q-expansion

$$(f|_k\gamma)(z) = a_0(f|_k\gamma) + \sum_{\mu \in \mathcal{O}_{F,+}} a_\mu(f|_k\gamma) e^{2\pi i \mathrm{Tr}(\frac{\mu z}{\delta})}.$$

This allows us to make the following definition.

Definition 2.4. *Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. We say that f is a* cusp form *if $a_0(f|_k\gamma) = 0$ for all $\gamma \in \mathrm{SL}_2(F)$.*

We denote by $S_k(\mathfrak{N})$ the space of all cusp forms of weight k and level $\mathfrak{N}$. Clearly $S_k(\mathfrak{N}) \subseteq M_k(\mathfrak{N})$.

Theorem 2.5. *The spaces $S_k(\mathfrak{N})$ and $M_k(\mathfrak{N})$ are* finite *dimensional complex vector spaces.*

Proof. See [18]. □

Let $d\mu := \frac{dx_1 dy_1}{y_1^2} \cdots \frac{dx_d dy_d}{y_d^2}$ on $\mathfrak{H}^d$. We recall that $\mathrm{SL}_2(\mathbf{R})^d$ acts transitively on $\mathfrak{H}^d$, with the stabilizer of $\underline{i} = (\sqrt{-1}, \dots, \sqrt{-1})$ being $\mathrm{SO}(2)^d$. The measure $d\mu$ is the pushforward of the Haar measure dg on $\mathrm{SL}_2(\mathbf{R})^d$ via the bijection $\mathrm{SL}_2(\mathbf{R})^d / \mathrm{SO}(2)^d \simeq \mathfrak{H}^d$. So it is $\mathrm{SL}_2(\mathbf{R})^d$-invariant. For $f \in M_k(\mathfrak{N})$ and $g \in S_k(\mathfrak{N})$, one can show the following integral

$$\int_{\Gamma_0(\mathfrak{N})\backslash\mathfrak{H}^d} f(z)\overline{g(z)}(y_1 \cdots y_d)^k d\mu.$$

converges. For a proof of this, we refer to [3, 18].

Definition 2.6. *Let $f, g \in S_k(\mathfrak{N})$. We define the* Petersson inner product *of f and g by*

$$\langle f, g\rangle := \int_{\Gamma_0(\mathfrak{N})\backslash\mathfrak{H}^d} f(z)\overline{g(z)}(y_1 \cdots y_d)^k d\mu.$$

This means that $S_k(\mathfrak{N})$ is a finite dimensional vector space equipped with an inner product, namely the Petersson inner product. This fact is very useful both theoretically and algorithmically.

2.2 Hecke operators

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. For every prime $\mathfrak{p} \nmid \mathfrak{N}$, we define the function $T_\mathfrak{p} f : \mathfrak{H}^d \to \mathbf{C}$ as follows. First, write $\mathfrak{p} = (\pi)$ where π is a totally positive, and then set

$$(T_\mathfrak{p} f)(z) := (f|_k\gamma_\infty)(z) + \sum_{a \in \mathcal{O}_F/\mathfrak{p}} (f|_k\gamma_a)(z),$$

where

$$\gamma_\infty := \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, \ \gamma_a := \begin{pmatrix} 1 & a \\ 0 & \pi \end{pmatrix} \text{ for } a \in \mathcal{O}_F/\mathfrak{p}.$$

Lemma 2.7. *Let $\mathfrak{p} \nmid \mathfrak{N}$ be a prime. Then, the map*

$$\begin{aligned} T_\mathfrak{p} : M_k(\mathfrak{N}) &\to M_k(\mathfrak{N}) \\ f &\mapsto T_\mathfrak{p} f \end{aligned}$$

is a linear operator which preserves $S_k(\mathfrak{N})$. We call $T_\mathfrak{p}$ the Hecke operator *at $\mathfrak{p}$.*

Proof. Exercise. □

The definition of Hecke operators can be extended (multiplicatively) to all integral ideals including those dividing the level $\mathfrak{N}$. For an adelic treatment of this, we refer the reader to Shimura [40]. From now on, if $\mathfrak{m}$ is an integral ideal, we let $T_\mathfrak{m}$ be the Hecke operator at $\mathfrak{m}$. The Hecke operators enjoy many beautiful and striking properties. Among others, we have the following result.

Theorem 2.8. *Let $f, g \in S_k(\mathfrak{N})$ be cusp forms and $\mathfrak{m} \nmid \mathfrak{N}$. Then, we have*

$$\langle T_{\mathfrak{m}} f, g \rangle = \langle f, T_{\mathfrak{m}} g \rangle.$$

Proof. There is a proof of this result in the general setting in Shimura [40, §2]. □

Theorem 2.8 means that the Hecke operators $T_{\mathfrak{m}}$ for $\mathfrak{m} \nmid \mathfrak{N}$ are self-adjoint. Since the space $S_k(\mathfrak{N})$ is finite dimensional, it follows that such a $T_{\mathfrak{m}}$ is diagonalizable. In fact, more is true.

Definition 2.9. *The* Hecke algebra *of* weight k *and* level $\mathfrak{N}$ *acting on $S_k(\mathfrak{N})$ is the $\mathbf{Z}$-subalgebra of* $\mathrm{End}_{\mathbf{C}}(S_k(\mathfrak{N}))$ *generated by the $T_{\mathfrak{m}}$ for all integral ideals $\mathfrak{m} \nmid \mathfrak{N}$. We denote it by $\mathbf{T}_k(\mathfrak{N})$.*

Theorem 2.10 (Shimura)**.** *The Hecke algebra $\mathbf{T}_k(\mathfrak{N})$ is a* commutative finitely *generated $\mathbf{Z}$-algebra which admits a basis of common eigenvectors for the action of $\mathbf{T}_k(\mathfrak{N})$ on $S_k(\mathfrak{N})$.*

Proof. We refer to Shimura [40, §2]. □

Definition 2.11. *Let f be a cups form of weight k and level $\mathfrak{N}$. We say that f is an* eigenform *if f is a common eigenvector for the action of $\mathbf{T}_k(\mathfrak{N})$ on $S_k(\mathfrak{N})$. If in addition $a_{(1)}(f) = 1$, then we say that f is* normalized.

One of the most striking features of Hilbert modular forms is the deep connection between the eigenvalues of Hecke operators and the Fourier coefficients of eigenforms.

Theorem 2.12 (Shimura)**.** *Let $f \in S_k(\mathfrak{N})$ be a normalized eigenform. Then the followings hold.*

(a) For every integral ideal $\mathfrak{m} \nmid \mathfrak{N}$, $a_{\mathfrak{m}}(f)$ is an algebraic integer such that

$$T_{\mathfrak{m}} f = a_{\mathfrak{m}}(f) f.$$

(b) The field $K_f := \mathbf{Q}(a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F)$ is a number field, i.e. is a finite extension of $\mathbf{Q}$, which is totally real. We let $\mathcal{O}_f := \mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F]$. (This is a suborder of the ring of integers of K_f).

Remark 2.13. We observe that Theorem 2.12 (b) is true because F has narrow class number one, and we only consider forms with trivial characters. In general, K_f will be a CM field. See Shimura [40, §2] for more details on this.

2.3 Old and new subspaces

For every integral ideal $\mathfrak{M}$ such that $\mathfrak{M} \mid \mathfrak{N}$, and for every divisor $\mathfrak{D}$ of $\mathfrak{N}\mathfrak{M}^{-1}$, let u be a totally positive generator of $\mathfrak{D}$. Then, it is not hard to see that the map

$$\begin{aligned} \iota_{\mathfrak{D}} : S_k(\mathfrak{M}) &\to S_k(\mathfrak{N}) \\ f &\mapsto f_u, \end{aligned}$$

where $f_u(z) := f(uz)$, is independent of the choice of u and is an injection. We let

$$S_k(\mathfrak{N})^{\mathrm{old}} := \sum_{\substack{\mathfrak{M} \mid \mathfrak{N} \\ \mathfrak{D} \mid \mathfrak{N}\mathfrak{M}^{-1}}} \iota_{\mathfrak{D}}(S_k(\mathfrak{M}));$$

$$S_k(\mathfrak{N})^{\mathrm{new}} := \left(S_k(\mathfrak{N})^{\mathrm{old}} \right)^{\perp}.$$

We call $S_k(\mathfrak{N})^{\mathrm{old}}$ (resp. $S_k(\mathfrak{N})^{\mathrm{new}}$) the *old subspace* (resp. *new subspace*) of $S_k(\mathfrak{N})$. It can be showed that $S_k(\mathfrak{N})^{\mathrm{old}}$ and $S_k(\mathfrak{N})^{\mathrm{new}}$ are both stable under the Hecke action.

Definition 2.14. *Let $f \in S_k(\mathfrak{N})$ be a normalized eigenform. We say that f is a* newform *if $f \in S_k(\mathfrak{N})^{\mathrm{new}}$.*

Definition 2.15. *Let f be a cusp form. We define the L-*series *of f by*

$$L(f,s) := \sum_{\mathfrak{m} \subseteq \mathcal{O}_F} \frac{a_{\mathfrak{m}}(f)}{\mathrm{N}\mathfrak{m}^s}.$$

Theorem 2.16 (Shimura)**.** *Let f be a cusp form. Then $L(f,s)$ is an entire function, i.e. is holomorphic on the whole complex plane. If f is a newform, then $L(f,s)$ admits an* Euler product*.*

Proof. The proof of this is essentially an adaption of what is known for $F = \mathbf{Q}$. So we refer to [37, 40]. □

Theorem 2.17 (Multipicity one)**.** *Let f, g be two normalized eigenforms such that*

$$a_{\mathfrak{m}}(f) = a_{\mathfrak{m}}(g) \text{ for all } \mathfrak{m} \nmid \mathfrak{N}.$$

Then, we have $f = g$.

Proof. This follows from the relation between Hecke eigenvalues and Fourier coefficients, and the fact that f is determined by its q-expansion. □

We mentioned earlier that the definition of the Hecke operators can be extended to all integral ideals. With this in mind, we have the following result due to Miyake.

Theorem 2.18 (Strong multiplicity one)**.** *Let f be a newform. Then, we have*

$$T_{\mathfrak{m}} f = a_{\mathfrak{m}}(f) f \text{ for all } \mathfrak{m} \subseteq \mathcal{O}_F.$$

Proof. See [31]. □

Theorem 2.17 and Theorem 2.18 are both very powerful and extremely useful as they imply that every newform is uniquely determined by its Hecke eigenvalues or Fourier coefficients. Here is an immediate indication of their usefulness.

Theorem 2.19. *Let $f \in S_k(\mathfrak{N})^{\mathrm{new}}$ be a newform, and let K_f be its field of Fourier coefficients. For each embedding $\tau : K_f \hookrightarrow \overline{\mathbf{Q}}$, there exists a newform $f^\tau \in S_k(\mathfrak{N})$ defined by*

$$a_{\mathfrak{m}}(f^\tau) := \tau(a_{\mathfrak{m}}(f)), \text{ for all } \mathfrak{m} \subseteq \mathcal{O}_F.$$

The set $\{f^\tau : \tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})\}$ is called the Hecke orbit *of f. We denote it by $[f]$.*

Proof. See Shimura [40, §2]. □

For more background on Hilbert modular forms, see [3, 13, 18, 23, 40]. Here, we wish to point out some new techniques in the computation of Hilbert modular forms, which arise from the Eichler-Jacquet-Langlands-Shimizu correspondence between Hilbert modular forms and quaternionic modular forms. We will not go into details here, but instead refer the reader to [13] for a detailed description of these methods. The upshot is that it is possible to efficiently compute systems of Hecke eigenvalues for Hilbert modular cusp forms by instead computing modular forms on finite spaces or on Shimura curves. This will be crucial to the methods in these notes. The corresponding algorithms have been implemented in the Hilbert Modular Forms Package in `Magma` [2]).

3 Eichler-Shimura and GL_2-type Modularity Conjectures

3.1 The Eichler-Shimura conjecture

The following conjecture is instrumental for the method we develop in these notes. Before stating it, we need some definition.

Definition 3.1. *Let A be an abelian variety defined over F. We say that A is of* GL_2-type *if there exists a number field K such that $\dim(A) = [K : \mathbf{Q}]$ and $\mathrm{End}_F(A) \otimes \mathbf{Q} \simeq K$.*

If A/F is an abelian variety of GL_2-type with $\mathrm{End}_F(A) \otimes \mathbf{Q} \simeq K$ and $g = [K : \mathbf{Q}]$, then there exists an integral ideal $\mathfrak{N}$ such that the conductor of A is of the form

$$\mathrm{cond}(A) = \mathfrak{N}^g.$$

The following statement relates Hilbert modular forms to abelian varieties of GL_2-type.

Conjecture 3.2 (Eichler-Shimura)**.** *Let F be a totally real number field of narrow class number one and $\mathfrak{N}$ an integral ideal of F. Let $f \in S_2(\mathfrak{N})$ be a newform. Then, there exists an abelian variety A_f/F of dimension $[K_f : \mathbf{Q}]$ with good reduction outside of $\mathfrak{N}$ and with $\mathcal{O}_f \hookrightarrow \mathrm{End}_F(A_f)$, such that*

$$L(A_f, s) = \prod_{\tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})} L(f^\tau, s) = \prod_{g \in [f]} L(g, s).$$

When $F = \mathbf{Q}$, this conjecture is a theorem, due to Eichler for prime level and Shimura in the general case. The Eichler-Shimura construction can be summarized as follows. Let $N > 1$ be an integer, and let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$. This curve and its Jacobian $J_1(N)$ are defined over $\mathbf{Q}$. We recall that the space $S_2(\Gamma_1(N))$ of cusp forms of weight 2 and level $\Gamma_1(N)$ is a $\mathbf{T}$-module, where $\mathbf{T}$ is the Hecke algebra. Let $f \in S_2(\Gamma_1(N))$ be a newform, and let $I_f = \mathrm{Ann}_{\mathbf{T}}(f)$. Shimura [39] showed that the quotient

$$A_f := J_1(N)/I_f J_1(N)$$

is an abelian variety A_f of dimension $[K_f : \mathbf{Q}]$ defined over $\mathbf{Q}$ with endomorphisms by the order $\mathcal{O}_f = \mathbf{Z}[a_n(f) : n \geq 1]$ and that

$$L(A_f, s) = \prod_{g \in [f]} L(g, s),$$

where $[f]$ denotes the Galois orbit of f.

One of the main consequences of the proof of the Serre conjecture [35] by Khare-Wintenberger [27] is that the converse to Conjecture 3.2 is true when $F = \mathbf{Q}$. That is, an abelian variety of GL_2-type is isogenous to a $\mathbf{Q}$-simple factor of $J_1(N)$ for some N [28]. And so, this provides a theoretical construction of all abelian varieties of GL_2-type over $\mathbf{Q}$ with a prescribed conductor. In fact, one can make this explicit in many cases (see [8] for elliptic curves, and [22, 24] for abelian surfaces).

For $[F : \mathbf{Q}] > 1$, the known cases of Conjecture 3.2 exploit the cohomology of Shimura curves. For instance, the conjecture is known when $[F : \mathbf{Q}]$ is odd, or when $\mathfrak{N}$ is exactly divisible by a prime $\mathfrak{p}$ of $\mathcal{O}_F$ [46]. The simplest case in which Conjecture 3.2 is still unknown is when f is a newform of level (1) and weight 2 over a real quadratic field. In that case, the conjecture predicts that the associated abelian variety A_f has everywhere good reduction.

Table 1: Modularity of abelian varieties of GL_2-type

Hilbert newforms f/F with Hecke eigenvalues $\mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] \subseteq \mathcal{O}_K$ (weight 2, level $\mathfrak{N}$)	(Isogeny classes of) Abelian varieties A/F $\dim(A) = g$, $\mathrm{cond}(A) = \mathfrak{N}^g$ $\mathrm{End}_F(A) \otimes \mathbf{Q} = K$

- - - - Eichler-Shimura conjecture
- - - - GL_2-type Modularity conjecture

3.2 The GL_2-type modularity conjecture

The following is the converse statement to Conjecture 3.2.

Conjecture 3.3 (GL_2-type Modularity)**.** *Let F be a totally real number field of narrow class number one, and A an abelian variety of GL_2-type over F. Let $K := \mathrm{End}_F(A) \otimes \mathbf{Q}$ and write $\mathrm{cond}(A) = \mathfrak{N}^g$, with $g = [K : \mathbf{Q}]$. Then, there exists a newform $f \in S_2(\mathfrak{N})^{\mathrm{new}}$ such that*

$$L(A, s) = \prod_{\tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})} L(f^\tau, s).$$

Here are two of my favourite examples.

Example 3.4. Let $F = \mathbf{Q}(\sqrt{5})$, $w = \frac{1+\sqrt{5}}{2}$ and $\mathfrak{N} = (5 + 2w)$. Then, $\mathfrak{N}$ is a prime of norm 31. This is the smallest norm for which $S_2(\mathfrak{N}) \neq 0$. In that case, we have $\dim S_2(\mathfrak{N}) = 1$. So, there is a newform $f \in S_2(\mathfrak{N})$ with Fourier coefficients in $\mathbf{Z}$. In [10], we proved that f corresponds to the elliptic curve

$$E : y^2 + xy + wy = x^3 - (1 + w)x^2,$$

In other words, we proved that E is modular and that

$$L(E, s) = L(f, s).$$

An alternate description of E. Let D be the quaternion algebra over F which is ramified at $\mathfrak{N}$ and exactly one of the two real places. Let $\mathcal{O}_D$ be a maximal order in D, and consider the Shimura curve $X_0^D(\mathfrak{N})$ obtained from $(\mathcal{O}_D)_1$, the units of norm 1 in $\mathcal{O}_D$. Then $X_0^D(\mathfrak{N})$ is a curve of genus 1. Hence $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ is an elliptic curve. By the Jacquet-Langlands correspondence, $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ and E are isogenous.

Example 3.5. Again, we let $F = \mathbf{Q}(\sqrt{5})$, $w = \frac{1+\sqrt{5}}{2}$, and set $\mathfrak{N} = (7 + 3w)$. Here $\mathfrak{N}$ is a prime of norm 61. This is the smallest norm such that $\dim S_2(\mathfrak{N}) = 2$. There is a newform $f \in S_2(\mathfrak{N})$ such that $\mathcal{O}_f = \mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] = \mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. In [13], we show that f corresponds to the Jacobian of the hyperelliptic curve $C : y^2 + Q(x)y = P(x)$ given by

$$P(x) := -wx^4 + (w - 1)x^3 + (5w + 4)x^2 + (6w + 4)x + 2w + 1;$$
$$Q(x) := x^3 + (w - 1)x^2 + wx + 1.$$

(We have $\mathfrak{N}^2 = (\mathrm{disc}(C).)$ The curve C comes from the Brumer family described in [4]. So, its Jacobian $\mathrm{Jac}(C)$ has real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

Alternatively, let D be the quaternion algebra over F which is ramified at $\mathfrak{N}$ and exactly one of the two real places. Let $\mathcal{O}_D$ be a maximal order in D, and consider the Shimura curve $X_0^D(\mathfrak{N})$ obtained from $(\mathcal{O}_D)_1$, the units of norm 1 in $\mathcal{O}_D$. Then $X_0^D(\mathfrak{N})$ is a curve of genus 2. Hence $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ is an abelian surface. By the Jacquet-Langlands correspondence, $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$ and is isogenous to $\mathrm{Jac}(C)$. However, we do not know whether the curves $X_0^D(\mathfrak{N})$ and C themselves are isogenous.

In Table 1, we summarize the connection between the Eichler-Shimura and GL_2-type modularity conjectures. Although many cases of these conjectures are known, they still remain largely open. We conclude this section with one of the most up-to-date results in this area.

Theorem 3.6 (Freitas-Le Hung-Siksek)**.** *Let F be a real quadratic field, and E an elliptic curve defined over F. Then E is modular.*

Proof. See Freitas-Le Hung-Siksek [19]. □

4 Elliptic curves with everywhere good reduction

4.1 Historical note

To the best of our knowledge, the first example of an elliptic curve with everywhere good reduction was discovered by Tate. Namely, he showed that the curve E defined by

$$E:\ y^2 + xy + \epsilon^2 y = x^3,$$

where $\epsilon = \frac{5+\sqrt{29}}{2}$ is the fundamental unit in $F = \mathbf{Q}(\sqrt{29})$, has discriminant $\Delta = -\epsilon^{10}$. This curve is extensively studied by Serre in [34]. In [38], Shimura discusses similar examples, and proposes a general strategy for constructing higher dimension analogues. From the early 70s to the late 90s, a great deal of work went into finding more examples of elliptic curves with everywhere good reduction defined over quadratic fields. Here is a non exhaustive list of references on the subject [36, 44, 9, 7, 32, 25].

4.2 Elkies-Donnelly search method

Let F be a real quadratic field of narrow class number one, and let ϵ be the fundamental unit of $\mathcal{O}_F$. Let E be an elliptic curve with everywhere good reduction defined over F. Suppose that E is given by the extended Weierstrass equation

$$E:\ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with coefficients $a_i \in \mathcal{O}_F$ and discriminant Δ. Without loss of generality, we can assume that $\Delta = \pm\epsilon^m$ with $0 \leq m < 12$. A refinement of an argument of Stroeker [44] by Elkies [15] shows that we can in fact assume that $m \in \{1, 2, 3, 4, 5\}$. Recall that the pair (c_4, c_6) satisfies the equation

$$x^3 - y^2 = 1728\Delta. \tag{1}$$

Elkies heuristics assert that if E is an elliptic curve of discriminant Δ, then for each archmedian place v the quantities $|v(c_4^3)|$, $|v(c_6^2)|$ and $1728|v(\Delta)|$ are roughly of the same size. Otherwise, there must be a large amount of cancellation in (1). One expects this not to happen very often.

Given m (and hence Δ), set

$$\begin{aligned} H(x) &= \frac{(v_0(x)|v_0(\Delta)|^{-1/3})^2 + (v_1(x)|v_1(\Delta)|^{-1/3})^2}{\sqrt{D}} \\ &= \frac{\left(v_0(x)|v_1(\Delta)|^{1/3}\right)^2 + \left(v_1(x)|v_0(\Delta)|^{1/3}\right)^2}{\sqrt{D}}, \end{aligned}$$

where $v_0, v_1 : F \hookrightarrow \mathbf{R}$ are the real embeddings of F. Then, H becomes a positive definite quadratic form in x over $\mathcal{O}_F$. The normalizing factor $\sqrt{D}^{-1}$ ensures that this form has discriminant 1. One searches for points on (1) by running over small values of $x \in \mathcal{O}_F$. (The search can be further refined by weighing the form H depending of the height of Δ.)

The method has been refined by Steve Donnelly to remove the restriction that Δ is a unit, and extended to all totally real number fields (of narrow class number one). This is currently the algorithm used in `Magma` [2] to search for elliptic curves with prescribed conductor on such fields.

Example 4.1. Let $F = \mathbf{Q}(\sqrt{1997})$, and $w := \frac{1+\sqrt{1997}}{2}$. A search in `Magma` [2] using the algorithm described above returns six elliptic curves with trivial conductor over F. They are pairwise non-isogenous and determine three $\mathrm{Gal}(F/\mathbf{Q})$-conjugacy classes represented by

$$\begin{aligned} &E_1 : y^2 + wxy = x^3 + (w+1)x^2 + (111w + 5401)x + (2406w + 81112); \\ &E_2 : y^2 + wxy + (w+1)y = x^3 - x^2 + (9370w - 208733)x + (2697263w - 61535794); \\ &E_3 : y^2 + (w+1)xy + (w+1)y = x^3 - wx^2 + (19636w + 434383)x + (5730650w + 125261893). \end{aligned}$$

By Theorem 3.6 these curves are modular. By a `Magma` [2] computation, we check that there are exactly 6 Hilbert newforms of level (1) and weight 2 over F with integer Hecke eigenvalues. So, these are the only elliptic curves with everywhere good reduction over F.

Exercise 4.2. *Show that there are no elliptic curves with everywhere good reduction over $F = \mathbf{Q}(\sqrt{2017})$.*

5 Abelian surfaces with everywhere good reduction

5.1 Historical note

In contrast to the case of elliptic curves, which we described in Section 4, the only examples of abelian surfaces with everywhere good reduction in the literature before [12] were of the following kinds: surfaces with complex multiplication [11], or $\mathbf{Q}$-surfaces [5, 37] or products of elliptic curves). Furthermore, except for the latter, none of these examples is given by an explicit equation. This could possibly be explained by the fact that it is not easy to embed such surfaces into projective spaces. Another additional complication is that it can happen that a curve has bad reduction at a given prime when its Jacobian still has good reduction at the same prime.

5.2 Hilbert modular surfaces

Let K be a real quadratic field of discriminant D'. The Hilbert modular surface $Y_-(D')$ is a compactification of the coarse moduli space which parametrizes principally polarized abelian surfaces with real multiplication by the ring of integers $\mathcal{O}_K$ of K, i.e. pairs (A, ι), where $\iota : \mathcal{O}_K \to \mathrm{End}_{\overline{\mathbf{Q}}}(A)$ is a homomorphism. The surfaces $Y_-(D')$ have models over the integers, with good reduction away from primes dividing D'.

Recently, Elkies and Kumar [16] computed explicit birational models over $\mathbf{Q}$ for these Hilbert modular surfaces for all the fundamental discriminants D' less than 100. They describe such a $Y_-(D')$ as a double cover of $\mathbf{P}^2$, with equation $z^2 = f(r,s)$, where r, s are parameters on $\mathbf{P}^2$. They also give the map to $\mathcal{A}_2$, which is birational to $\mathcal{M}_2$, the moduli space of genus 2 curves. It is given by expressing the Igusa-Clebsch invariants of the image point as rational functions of r and s.

5.3 Our approach

Our strategy for producing abelian surfaces with everywhere good reduction combines the Eichler-Shimura conjecture with the explicit equations in [16]. To produce such a surface A, we proceed as follows:

(a) Find a Hilbert modular form of level (1) and weight 2 for a real quadratic field F, with coefficients in a real quadratic field K_f of discriminant D'.

(b) Find an F-rational point on the Hilbert modular surface $Y_-(D')$, for which the L-function of the associated abelian surface matches that of f at several Euler factors, up to twist.

(c) Compute the correct quadratic twist of the abelian surface, or the genus 2 curve.

(d) Check that the abelian surface has good reduction everywhere.

(e) Prove that the L-functions indeed match up, i.e. that A is modular.

5.4 Method 1: Point search on Hilbert modular surfaces

We illustrate this with the following example. The smallest discriminant for which we obtain a surface with everywhere good reduction is $D = 53$. The abelian surface A_f has real multiplication by (an order in) the field $\mathbf{Q}(\sqrt{2})$. In fact, we will see that it has real multiplication by the full ring of integers.

An equation for the Hilbert modular surface $Y_-(8)$ is given in [16]. As a double-cover of $\mathbf{P}^2_{r,s}$, it is given by

$$z^2 = 2(16rs^2 + 32r^2s - 40rs - s + 16r^3 + 24r^2 + 12r + 2).$$

It is a rational surface (over $\mathbf{Q}$) and therefore the rational points are dense. In particular, there is an abundance of rational points of small height. The Igusa-Clebsch invariants $(I_2 : I_4 : I_6 : I_{10}) \in \mathbf{P}^2_{(1:2:3:5)}$ are given by

$$\left(-\frac{24B_1}{A_1}, -12A, \frac{96AB_1 - 36A_1B}{A_1}, -4A_1B_2\right),$$

where

$$\begin{aligned}
A_1 &= 2rs^2, \\
A &= -(9rs + 4r^2 + 4r + 1)/3, \\
B_1 &= (rs^2(3s + 8r - 2))/3, \\
B &= -(54r^2s + 81rs - 16r^3 - 24r^2 - 12r - 2)/27, \\
B_2 &= r^2.
\end{aligned}$$

Table 2: The first few Hecke eigenvalues of a base change newform of level (1) and weight 2 over $\mathbf{Q}(\sqrt{53})$. Here $e = \sqrt{2}$.

$\mathrm{N}\mathfrak{p}$	$\mathfrak{p}$	$a_\mathfrak{p}(f)$	$s_\mathfrak{p}(f)$	$t_\mathfrak{p}(f)$
4	2	$e+1$	2	7
7	$-w-2$	$-e-2$	-4	16
7	$-w+3$	$-e-2$	-4	16
9	3	$-3e+1$	2	1
11	$w-2$	$3e$	0	4
11	$w+1$	$3e$	0	4
13	$w-1$	$-2e+1$	2	19
13	$-w$	$-2e+1$	2	19
17	$-w-5$	-3	-6	43
17	$w-6$	-3	-6	43
25	5	$2e+4$	8	58
29	$-w-6$	$3e-3$	-6	49
29	$w-7$	$3e-3$	-6	49

Recall that we expect to find a point of $Y_-(8)$ over $F = \mathbf{Q}(\sqrt{53})$, corresponding to the principally polarized abelian surface A which should match the Hilbert modular form f. The L-series of a surface A arising from our search is obtained by counting points on the residue fields $\mathbf{F}_\mathfrak{p} = \mathcal{O}_F/\mathfrak{p}$ as $\mathfrak{p}$ runs over the set of primes. On the other hand, the L-series of the conjectural surface A_f attached to f can be written as

$$L(A_f, s) = L(f, s)L(f^\tau, s) = \prod_{\mathfrak{p}} \frac{1}{Q_\mathfrak{p}(\mathrm{N}(\mathfrak{p})^{-s})},$$

where

$$\begin{aligned} Q_\mathfrak{p}(T) &:= (T^2 - a_\mathfrak{p}(f)T + \mathrm{N}(\mathfrak{p}))(T^2 - a_\mathfrak{p}(f)^\tau T + \mathrm{N}(\mathfrak{p})) \\ &= T^4 - s_\mathfrak{p}(f)T^3 + t_\mathfrak{p}(f)T^2 - \mathrm{N}(\mathfrak{p})s_\mathfrak{p}(f)T + \mathrm{N}(\mathfrak{p})^2. \end{aligned}$$

We would like the local factors of these two L-series to match.

We first make a list of all F-rational points of height up to a given bound B on the Hilbert modular surface. Next, for each of these rational points, we try to construct the corresponding genus 2 curve C over F, whose Jacobian corresponds to the moduli point (r, s) we have chosen, and check whether the characteristic polynomial of Frobenius on its first étale cohomology group matches up the polynomial $Q_\mathfrak{p}(T)$ giving the corresponding Euler factor of surface A_f attached to the Hilbert modular form. If a candidate point (r, s) passes this test for say the first 50 primes (ordered by norm) of F of good reduction for f and $A = \mathrm{Jac}(C)$, we can be reasonably convinced that it is the correct curve, and then try to prove that A is associated to f. (There are several subtleties in this process, and we refer the reader to [12] for details.)

In this particular example, a search of $Y_-(8)$ for all points of height ≤ 200 using [14] (implemented in `Sage`) gives the parameters

$$r = -\frac{24 + 10w}{11^2}, \quad s = \frac{136 - 24w}{11^2},$$

and the Igusa-Clebsch invariants

$$\begin{aligned} I_2 &= 208 + 88w, \\ I_4 &= -1660 - 588w, \\ I_6 &= -428792 - 135456w, \\ I_{10} &= 643072 + 204800w. \end{aligned}$$

By using Mestre's algorithm [30] which is implemented in `Magma`, we obtain a curve with the above invariants. We reduce this curve using the algorithm in [1] implemented in `Sage` [33] to get the curve

$$\begin{aligned} C' : y^2 = (-6w + 25)x^6 &+ (-60w + 246)x^5 + (-242w + 1017)x^4 \\ &+ (-534w + 2160)x^3 + (-626w + 2688)x^2 \\ &+ (-440w + 1724)x - 127w + 567. \end{aligned}$$

By further reducing the curve C' we get the following.

Theorem 5.1. *Let $C : y^2 + Q(x)y = P(x)$ be the curve over $F = \mathbf{Q}(\sqrt{53})$, where*

$$\begin{aligned} P &:= -4x^6 + (w - 17)x^5 + (12w - 27)x^4 + (5w - 122)x^3 + (45w - 25)x^2 \\ &\quad + (-9w - 137)x + 14w + 9, \\ Q &:= wx^3 + wx^2 + w + 1. \end{aligned}$$

Then

(a) The discriminant of this curve is $\Delta_C = -\epsilon^7$. Thus C has everywhere good reduction.

(b) The surface $A := \mathrm{Jac}(C)$ has real multiplication by $\mathbf{Z}[\sqrt{2}]$. It is modular and corresponds to the unique Hecke constituent $[f]$ in $S_2(1)$.

Proof. The surface A has a 7-torsion point defined over F, hence the mod 7 Galois representation is reducible. To prove that A is modular, we use Skinner-Wiles [42]. See [12] for details. □

5.5 Method 2: Splitting abelian varieties

We can use this method when the Hilbert newform f is a base change, i.e. when the Hecke eigenvalues of f satisfy

$$a_{\mathfrak{p}}(f) = a_{\sigma(\mathfrak{p})}(f) \text{ for all } \mathfrak{p},$$

where $\mathrm{Gal}(F/\mathbf{Q}) = \langle\sigma\rangle$. In this case f arises from a newform $g \in S_2(\Gamma_1(D))$. Since the level of f is (1), the form $g \in S_2(\Gamma_1(D), \chi_D)^{\mathrm{new}}$ by [29, Prop. 2, p.263], where χ_D is the fundamental character of the quadratic field $F = \mathbf{Q}(\sqrt{D})$. The coefficient field L_g of g is a quartic CM field which contains K_f. The non-trivial element of $\mathrm{Gal}(L_g/K_f)$, which we denote by $(x \mapsto \bar{x}, x \in L_g)$, extends to complex conjugation. The abelian variety B_g attached to the form g is a fourfold such that $\mathrm{End}_{\mathbf{Q}}(B_g) \otimes \mathbf{Q} \simeq L_g$.

Let w_D be the Atkin-Lehner involution on $S_2(\Gamma_1(D), \chi_D)^{\mathrm{new}}$. This induces an involution on B_g, which we still denote by w_D. Shimura [37, § 7.7] shows the followings:

(a) w_D is defined over F, and $w_D^\sigma = -w_D$;

(b) $w_D \cdot [x] = [\bar{x}] \cdot w_D$, where $[x]$ denotes the endomorphism induced on B_g by $x \in L_g$.

(c) The abelian surface $A_f := (1 + w_D)B_g$ is defined over F, and is isogenous to its Galois conjugate given by $A_f^\sigma := (1 - w_D)B_g$. Moreover, we have

$$B_g \otimes_{\mathbf{Q}} F \sim A_f \times A_f^\sigma.$$

This algebraic splitting is not very useful for our purpose. Instead, we produce an explicit equation for the surface A_f by writing down an analytic splitting of the fourfold B_g. For this, we assume that A_f and A_f^σ are principally polarizable.

To describe the method, we recall that by [6, Theorems 6.2.4 and 6.2.6], there exist newforms $g_1, g_2 \in S_2(\Gamma_1(D), \chi_D)^{\text{new}}$ such that $\{g_1, \bar{g}_1, g_2, \bar{g}_2\}$ is a basis of the Hecke constituent of g and

$$w_D(g_1) = \bar{\lambda}_D(g_1)\bar{g}_1,\ w_D(g_2) = \bar{\lambda}_D(g_2)\bar{g}_2,$$

where $a_D(g)$ is the Hecke eigenvalue of g at D and $\lambda_D(g) = \frac{a_D(g)}{\sqrt{D}}$, the pseudo-eigenvalue of w_D. The matrix of w_D in the basis $\{g_1, \bar{g}_1, g_2, \bar{g}_2\}$ is given by

$$W_D := \begin{bmatrix} 0 & \lambda_D(g_1) & 0 & 0 \\ \bar{\lambda}_D(g_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_D(g_2) \\ 0 & 0 & \bar{\lambda}_D(g_2) & 0 \end{bmatrix}$$

From this, we see that $W_D^\sigma = -W_D$. The following lemma is a simple adaptation of Cremona's [7, Lemma 5.6.2].

Lemma 5.2. *The set of forms $h_i^\pm := \frac{1}{2}(g_i \pm w_D(g_i))$, $i = 1, 2$, are bases for the $\pm$-eigenspaces of W_D, acting on the Hecke constituent of g, which give a decomposition of the space of differential 1-forms $H^0(B_g \otimes_{\mathbf{Q}} F, \Omega^1_{B_g \otimes_{\mathbf{Q}} F/F})$ according to the action of $\mathrm{Gal}(F/\mathbf{Q})$.*

We recall that $H_1(B_g, \mathbf{Z})$ is a Hecke module of rank 4 over $\mathbf{Z}$. So, the $\pm$-eigenspaces of w_D $H_1(B_g, \mathbf{Z})^\pm$ are free Hecke submodules of $\mathbf{Z}$-rank 2 each.

Lemma 5.3. *Let $\Lambda_g^\pm$ be the period lattices obtained by integrating the forms in Lemma 5.2 against $H_1(B_g, \mathbf{Z})^\pm$, and set $\Lambda_g = \Lambda_g^+ \oplus \Lambda_g^-$. Then, there exist an abelian fourfold B'_g defined over $\mathbf{Q}$, and an isogeny $\phi : B'_g \to B_g$ whose degree is a power of 2, such that $B'_g(\mathbf{C}) = \mathbf{C}^4/\Lambda_g$. Moreover, $B'_g = \mathrm{Res}_{F/\mathbf{Q}}(A_f)$ where A_f is an abelian surface defined over F.*

Proof. We first note that the complex tori $\mathbf{C}^2/\Lambda_g^\pm$ and $\mathbf{C}^4/\Lambda_g$ have canonical Riemann forms obtained by restriction of the intersection pairing $\langle \cdot, \cdot \rangle$ on B_g. Therefore, they are the complex points of some abelian varieties. Since $h_1^+, h_2^+, h_1^-, h_2^-$ is a basis of the Hecke constituent of g, [37, Theorem 7.14 and Proposition 7.19] imply that there exist a fourfold B'_g defined over $\mathbf{Q}$, and an isogeny $\phi :\ B'_g \to B_g$, such that $B'_g(\mathbf{C}) = \mathbf{C}^4/\Lambda_g$.

Next, let $x \in H_1(B_g, \mathbf{Z})$, then we have

$$2x = (x + w_D x) + (x - w_D x) = y_+ + y_- \in H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^-.$$

Hence the exponent of $H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^-$ inside $H_1(B_g, \mathbf{Z})$ divides 2. This implies that the degree of ϕ is a power of 2.

Since w_D is defined over F and $w_D^\sigma = -w_D$, the bases $\{h_1^+, h_2^+\}$ and $\{h_1^-, h_2^-\}$ are $\mathrm{Gal}(F/\mathbf{Q})$-conjugate. Therefore $\mathbf{C}^2/\Lambda_g^+$ and $\mathbf{C}^2/\Lambda_g^-$ are the complex points of some abelian surfaces defined over F that are Galois conjugate. Let A_f be the surface such that $A_f(\mathbf{C}) = \mathbf{C}^2/\Lambda_g^+$. Then, we see that $B'_g = \mathrm{Res}_{F/\mathbf{Q}} A_f$ by construction. □

In practice, we can replace B_g by B'_g, and hence assume that

$$H_1(B_g, \mathbf{Z}) = H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^- = H_1(A_f, \mathbf{Z}) \oplus H_1(A_f^\sigma, \mathbf{Z}).$$

The above integration then gives the period lattice decomposition

$$\Omega_{B_g} = \Omega_{A_f} \times \Omega_{A_f^\sigma} = (\Omega_1 \,|\, \Omega_2) \times (\Omega_1^\sigma \,|\, \Omega_2^\sigma).$$

Provided that the intersection pairing restricted to $H_1(A_f, \mathbf{Z})$ and $H_1(A_f^\sigma, \mathbf{Z})$ induces principal polarizations, we can compute the surfaces A_f and A_f^σ as Jacobians of curves C_f and C_f^σ (defined over F).

We illustrate this with an example at the discriminant $D = 73$. The abelian surface A_f has real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}])$.

A symplectic basis for $H_1(B_g, \mathbf{Z})$ is given by the modular symbols [43]

$$\begin{aligned}
\gamma_1 &:= 2\{-1/57, 0\} - \{-1/62, 0\} - \{-1/52, 0\} + 2\{-1/29, 0\} + \{-1/18, 0\},\\
\gamma_2 &= -\{-1/62, 0\} + 2\{-1/41, 0\} - \{-1/52, 0\} + 2\{-1/12, 0\} + 2\{-1/29, 0\}\\
&\quad + \{-1/18, 0\} - \{-1/36, 0\},\\
\gamma_3 &:= \{-1/57, 0\} - \{-1/41, 0\} - \{-1/18, 0\} + \{-1/36, 0\},\\
\gamma_4 &:= -\{-1/57, 0\} + \{-1/62, 0\} - \{-1/41, 0\} + \{-1/52, 0\} - \{-1/12, 0\}\\
&\quad - 2\{-1/29, 0\} - \{-1/18, 0\} + \{-1/24, 0\},\\
\gamma'_1 &:= \{-1/57, 0\} + \{-1/41, 0\} + \{-1/18, 0\} - \{-1/36, 0\},\\
\gamma'_2 &:= \{-1/57, 0\} + \{-1/62, 0\} + \{-1/41, 0\} - \{-1/52, 0\} - \{-1/12, 0\}\\
&\quad - \{-1/18, 0\} + \{-1/24, 0\},\\
\gamma'_3 &:= -\{-1/62, 0\} + \{-1/52, 0\} + \{-1/18, 0\},\\
\gamma'_4 &:= \{-1/62, 0\} - \{-1/52, 0\} - \{-1/18, 0\} + \{-1/36, 0\}.
\end{aligned}$$

We chose that basis so that $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4\}$ are integral bases for $H_1(B_g, \mathbf{Z})^+$ and $H_1(B_g, \mathbf{Z})^-$. Computing the matrix G of the intersection pairing in that basis, we see that B_g is principally polarized. We also see that $H_1(B_g, \mathbf{Z})^+$ and $H_1(B_g, \mathbf{Z})^-$ have the same polarization of type $(2, 2)$, meaning that A_f and A_f^σ are principally polarized. By integrating the bases of differential forms $\{h_1^+, h_2^+\}$ and $\{h_1^-, h_2^-\}$ from Lemma 5.2 against the Darboux bases $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4\}$, we obtain the Riemann period matrices Ω_{A_f} and $\Omega_{A_f^\sigma}$, where

$$\Omega_{B_g} = \Omega_{A_f} \times \Omega_{A_f^\sigma} = (\Omega_1 \,|\, \Omega_2) \times (\Omega_1^\sigma \,|\, \Omega_2^\sigma),$$

with

$$\begin{aligned}
\Omega_1 &:= \begin{pmatrix} 101.34000... - 7.5977...i & -2.6423... - 2.6129...i \\ 23.92200... - 47.37900...i & 11.19300... - 4.6090...i \end{pmatrix}\\
\Omega_2 &:= \begin{pmatrix} 38.70800... - 12.29300...i & -6.9177... + 1.6149...i \\ -62.63000... + 19.89100...i & -4.275400... + 0.99804...i \end{pmatrix}\\
\Omega_1^\sigma &:= \begin{pmatrix} 0.53699... - 3.7425...i & 3.6304... - 3.4371...i \\ 0.86887... - 6.0555...i & -2.2437... + 2.1243...i \end{pmatrix}\\
\Omega_2^\sigma &:= \begin{pmatrix} -1.4059... + 2.3130...i & -1.3867... - 5.5613...i \\ -1.4059... - 2.3130...i & -1.3867... + 5.5613...i \end{pmatrix}
\end{aligned}$$

This yields the normalized period matrices

$$Z := \begin{pmatrix} -0.50106... + 0.29103...i & 0.43700... - 0.012594...i \\ 0.43700... - 0.012594...i & 0.41383... + 0.18028...i \end{pmatrix}$$
$$Z^\sigma := \begin{pmatrix} -0.22570... + 0.80024...i & 0.54639... - 0.32080...i \\ 0.54639... - 0.32080...i & -0.67931... + 0.47944...i \end{pmatrix}$$

We compute the Igusa-Clebsch invariants I_2, I_4, I_6 and I_{10} to 200 decimal digits of precision using Z and Z^σ, and identify them as elements in F (using to Lemma 5.3). In the weighted projective space $\mathbf{P}^2_{(1:2:3:5)}$, this gives the point

$$(I_2 : I_4 : I_6 : I_{10}) = \left(1, \frac{-3080592b + 36303121}{3750827536}, \frac{-724297788520b + 811909152327}{229715681614784}, \frac{6808713659928b - 5817295179641}{673143675040422478040 8}\right),$$

where $b = \sqrt{73}$. By using Mestre's algorithm [30] which is implemented in `Magma`, we obtain a curve with the above invariants. We reduce this curve using the algorithm in [1] implemented in `Sage` [33] to get the curve

$$\begin{aligned} C' : y^2 = {} & (4w - 19)x^6 + (12w - 56)x^5 + (12w - 74)x^4 + (16w - 10)x^3 + (-12w - 63)x^2 \\ & + (12w + 46)x - 4w - 15. \end{aligned}$$

This yields a global minimal model, and we have the following theorem.

Theorem 5.4. *Let $C : y^2 + Q(x)y = P(x)$ be the curve over $F = \mathbf{Q}(\sqrt{73})$, where*

$$\begin{aligned} P &:= (w - 5)x^6 + (3w - 14)x^5 + (3w - 19)x^4 + (4w - 3)x^3 + (-3w - 16)x^2 + (3w + 11)x \\ &\quad + (-w - 4); \\ Q &:= x^3 + x + 1. \end{aligned}$$

Then

(a) The discriminant of this curve is $\Delta_C = -\epsilon^2$. Thus C has everywhere good reduction.

(b) The surface $A := \mathrm{Jac}(C)$ has real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. It is modular and corresponds to the unique Hecke constituent $[f]$ in $S_2(1)$.

Proof. Only the proof of modularity is different from what we did in the previous example. Here the prime 3 is inert in $\mathcal{O}_f = \mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. So we prove that the surface A is modular by combining arguments in [17] and [20, 21]. $\square$

In contrast to the examples in Theorems 5.1 and 5.4, there are curves whose Jacobians have everywhere good reduction while the curves themselves do not. We now discuss one such example, for the field $F = \mathbf{Q}(\sqrt{929})$, with Hecke eigenvalues in $\mathbf{Q}(\sqrt{13})$.

Theorem 5.5. *Let $C : y^2 + Q(x)y = P(x)$ be the curve over F, where*

$$\begin{aligned} P(x) &:= 23x^6 + (90w - 45)x^5 + 33601x^4 + (28707w - 14354)x^3 + 3192149x^2 \\ &\quad + (811953w - 405977)x + 19904990, \\ Q(x) &:= x^3 + x + 1. \end{aligned}$$

Then

Table 3: The first few Hecke eigenvalues of a non-base change newform of level (1) and weight 2 over $\mathbf{Q}(\sqrt{929})$. Here $e = (1+\sqrt{13})/2$.

$\mathrm{N}\mathfrak{p}$	$\mathfrak{p}$	$a_{\mathfrak{p}}(f)$	$s_{\mathfrak{p}}(f)$	$t_{\mathfrak{p}}(f)$
2	$561w - 8830$	$-e+1$	1	1
2	$561w + 8269$	e	1	1
5	$-4w - 59$	$-e+1$	1	7
5	$4w - 63$	e	1	7
9	3	3	6	27
11	$-8342w + 131301$	$2e - 3$	-4	13
11	$8342w + 122959$	$-2e - 1$	-4	13
19	$-50w - 737$	$e - 2$	-3	37
19	$50w - 787$	$-e - 1$	-3	37
23	$-42832w + 674165$	$4e - 4$	-4	-2
23	$42832w + 631333$	$-4e$	-4	-2
29	$-2w + 31$	$-2e + 6$	10	70
29	$2w + 29$	$2e + 4$	10	70

(a) The discriminant $\Delta_C = 3^{22}$, hence C has bad reduction at (3)*.*

(b) The surface $A := \mathrm{Jac}(C)$ has everywhere good reduction. It is modular and corresponds to the form f listed in Table 3.

Proof. To prove modularity, we use [20, 21, Theorem 1.1 in Erratum]. We refer to [12] for details. $\square$

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