

Complete Resolutions

Definition A complete resolution of a kG -module M is a commutative diagram

$$\begin{array}{ccccccccccc} \rightarrow & Q_{n+1} & \rightarrow & Q_n & \rightarrow & Q_{n-1} & \rightarrow & \cdots & \rightarrow & Q_i & \xrightarrow{d_i} & Q_0 & \xrightarrow{d_0} & Q_{-1} & \rightarrow & Q_{-2} & \rightarrow & \cdots \\ & \parallel & & \parallel & & \downarrow & & & & \downarrow & & \downarrow & & & & & & & \\ & P_{n+1} & \rightarrow & P_n & \rightarrow & P_{n-1} & \rightarrow & \cdots & \rightarrow & P_i & \rightarrow & P_0 & \rightarrow & M & & & & & \end{array}$$

where the P_i and Q_i are projective, P_0 is a projective resolution of M and Q is acyclic (i.e. exact).

n is called the coincidence index.

We also require that $\text{Hom}(Q_*, P)$ be acyclic for any projective P .

For $n=0$ this is the same as the definition used for the Tate cohomology of finite groups.

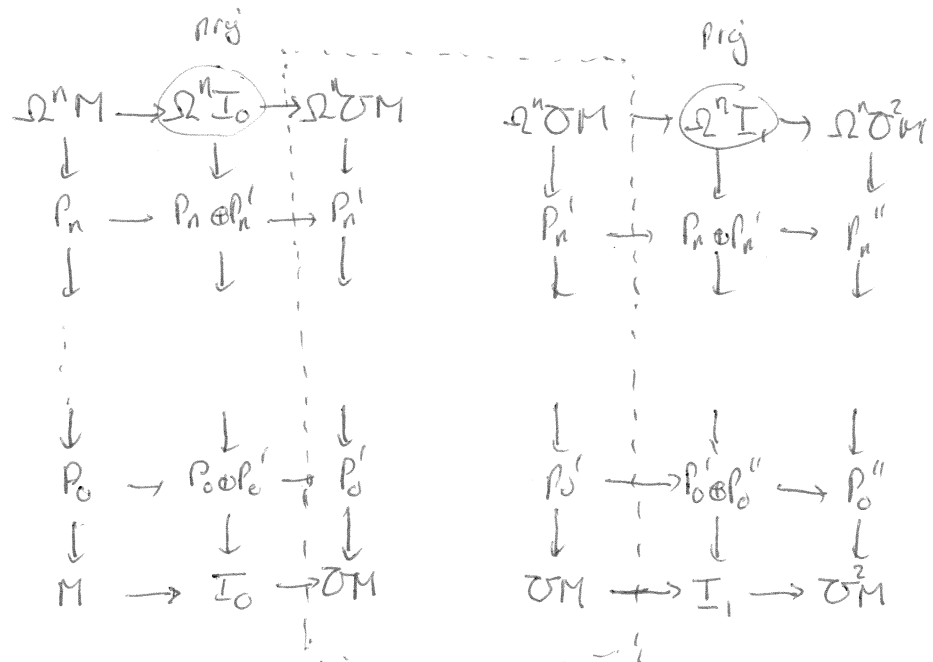
Theorem For groups G of type Φ , any kG -module has a complete resolution, there is one with coincidence index $\leq \text{findim} G$ and any two are chain homotopy equivalent.

Construction Given M , take an injective resolution

$$M \downarrow \\ I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

Use the Horseshoe Lemma

$n \geq \text{findom } G$



Splice \longrightarrow Cartan-Eilenberg resolution

Get

$$\Omega^n M \downarrow \\ \Omega^n I_0 \rightarrow \Omega^n I_1 \rightarrow \Omega^n I_2 \rightarrow \dots$$

a projective resolution (in the wrong direction)

Add a normal projective resolution of $\Omega^n M$.

For groups of type \mathbb{I} it is automatic that $\text{Hom}_{kG}(Q_0, P)$ is exact: since P has finite injective dimension we can prove this by induction on $\text{injdim} P$ (forgetting that P is projective).

When $\text{injdim} P = 0$, P is injective and $\text{Hom}_{kG}(Q_0, P)$ is exact by definition of injective.

Otherwise $P \rightarrow I \rightarrow \mathcal{U}P$, true for $\mathcal{U}P$ by induction and the long exact cohomology sequence for $\text{Hom}_{kG}(Q, -)$ proves it for P .

Note: $Q_1 \rightarrow Q_0 \xrightarrow{d_0} Q_{-1} \rightarrow \dots$
 $\downarrow \text{ind}_0 \quad \uparrow \dots$ exists because $\text{Hom}_{kG}(Q_0, P)$ is exact.
 $\downarrow \quad \downarrow \quad \downarrow$
 $\downarrow P$ projective

This allows us to fill in the extra morphisms in the definition of a complete resolution. It also allows us to show that any two complete resolutions of the same module are chain homotopy equivalent and that ind_i is determined up to projective summand (ex).

If $\cdots \rightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_2} Q_{-1} \rightarrow \cdots$

is a complete resolution of M we define $\Omega^i M = \ker d_i$.

It is well defined up to projective summands

$$\Omega^i \Omega^j M = \Omega^{i+j} M \quad \text{etc.}$$

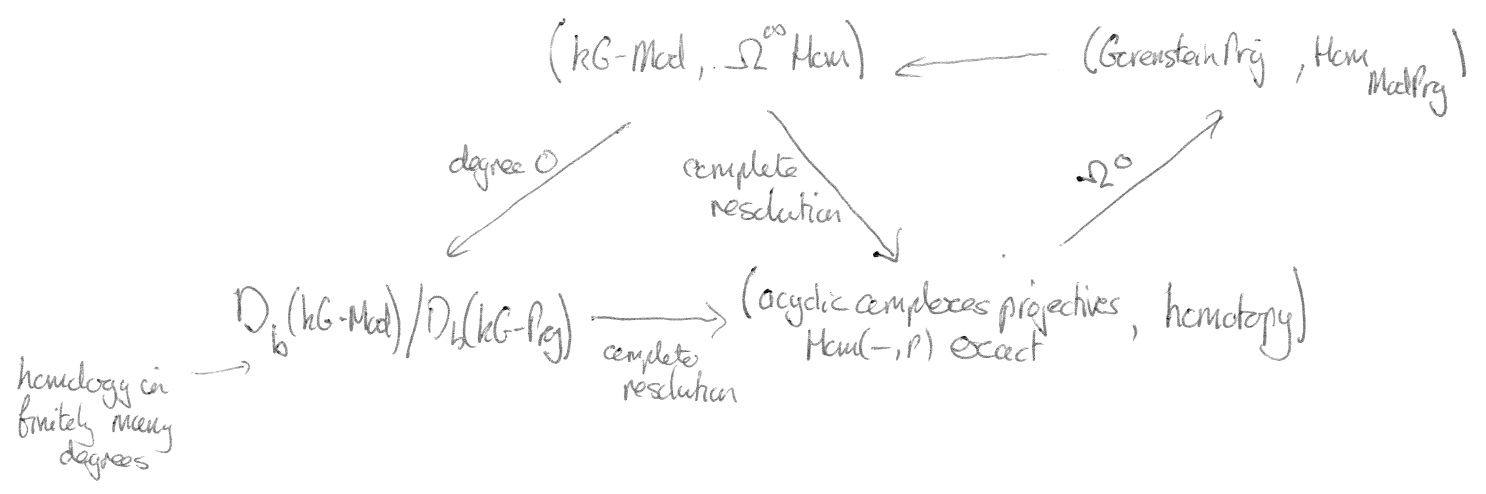
The modules that can occur as kernels in an acyclic complex of projectives (with $\text{Hom}(-, P)$ exact) are called Gorenstein projectives. They have many nice properties. (ex).

For any M we have a natural map $\Omega^0 M \rightarrow M$.

It is a stable isomorphism, which we sometimes write $\tilde{M} \xrightarrow{\cong} M$ and M is Gorenstein projective.

of the role of CW complexes in homotopy theory.

Theorem TFAE



These categories are triangulated in the usual way (analogously to the structures on the stable category for a finite group or on $D(kG)$).

In a triangulated category, $X \xrightarrow{f} Y \rightarrow Z$ triangle
 $f \text{ iso} \iff Z = 0$

Using this we can rephrase property \mathbb{F} :

Lemma $f: X \rightarrow Y$ is a stable isomorphism if and only if $f|_p: X|_p \rightarrow Y|_p$ is a stable isomorphism for every finite (p) -subgroup.