# Complete Resolutions

<u>Definition</u> A complete resolution of a $kG$-module $M$ is a commutative diagram

$$\begin{tikzcd}
\arrow[r] & Q_{n+1} \arrow[r] \arrow[d, equal] & Q_n \arrow[r] \arrow[d, equal] & Q_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_1 \arrow[r, "d_1"] \arrow[d] & Q_0 \arrow[r, "d_0"] \arrow[d] & Q_{-1} \arrow[r] & Q_{-2} \arrow[r] & \cdots \\
& P_{n+1} \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M & &
\end{tikzcd}$$

where the $P_i$ and $Q_i$ are projective, $P_\bullet$ is a projective resolution of $M$ and $Q_\bullet$ is acyclic (i.e. exact).

$n$ is called the coincidence index.

We also require that $\mathrm{Hom}(Q_*, P)$ be acyclic for any projective $P$.

For $n=0$ this is the same as the definition used for the Tate cohomology of finite groups.

<u>Theorem</u> For groups $G$ of type $\Phi$, any $kG$-module has a complete resolution, there is one with coincidence index $\leq \mathrm{findim}\, G$ and any two are chain homotopy equivalent.

**Construction** Given $M$, take an injective resolution

$$\begin{tikzcd} M \arrow[d] \\ I_0 \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \cdots \end{tikzcd}$$

Use the Horseshoe lemma

$n \geq \operatorname{findim} G$

$$\begin{tikzcd}[column sep=small]
\Omega^n M \arrow[r] \arrow[d] & \overset{\text{proj}}{\Omega^n I_0} \arrow[r] \arrow[d] & \Omega^n \mho M \arrow[d] & \Omega^n \mho M \arrow[r] \arrow[d] & \overset{\text{proj}}{\Omega^n I_1} \arrow[r] \arrow[d] & \Omega^n \mho^2 M \arrow[d] \\
P_n \arrow[r] \arrow[d] & P_n \oplus P_n' \arrow[r] \arrow[d] & P_n' \arrow[d] & P_n' \arrow[r] \arrow[d] & P_n \oplus P_n' \arrow[r] \arrow[d] & P_n'' \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
P_0 \arrow[r] \arrow[d] & P_0 \oplus P_0' \arrow[r] \arrow[d] & P_0' \arrow[d] & P_0' \arrow[r] \arrow[d] & P_0' \oplus P_0'' \arrow[r] \arrow[d] & P_0'' \arrow[d] \\
M \arrow[r] & I_0 \arrow[r] & \mho M & \mho M \arrow[r] & I_1 \arrow[r] & \mho^2 M
\end{tikzcd}$$

Splice $\longrightarrow$ Cartan-Eilenberg resolution

Get

$$\begin{tikzcd} \Omega^n M \arrow[d] \\ \Omega^n I_0 \arrow[r] & \Omega^n I_1 \arrow[r] & \Omega^n I_2 \arrow[r] & \cdots \end{tikzcd}$$

a projective resolution (in the wrong direction)

Add a normal projective resolution of $\Omega^n M$.

For groups of type $\Phi$ it is automatic that $\mathrm{Hom}_{kG}(Q_\bullet, P)$ is exact: since $P$ has finite injective dimension we can prove this by induction on $\mathrm{injdim}\, P$ (forgetting that $P$ is projective).

When $\mathrm{injdim}\, P = 0$, $P$ is injective and $\mathrm{Hom}_{kG}(Q_\bullet, P)$ is exact by definition of injective.
Otherwise $P \to I \to \mho P$, true for $\mho P$ by induction and the long exact cohomology sequence for $\mathrm{Hom}_{kG}(Q, -)$ proves it for $P$.

Note:

$$\begin{tikzcd}
Q_1 \arrow[r] & Q_0 \arrow[rr, "d_0"] \arrow[dr] & & Q_{-1} \arrow[r] \arrow[ddl, dotted] & {} \\
& & \mathrm{im}\, d_0 \arrow[ur] \arrow[d] & & \\
& & P & &
\end{tikzcd}$$

exists because $\mathrm{Hom}_{kG}(Q_\bullet, P)$ is exact.

projective

This allows us to fill in the extra morphisms in the definition of a complete resolution. It also allows us to show that any two complete resolutions of the same module are chain homotopy equivalent and that $\mathrm{im}\, d_i$ is determined up to projective summand (ex).

If $\cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_2} Q_{-1} \longrightarrow \cdots$

is a complete resolution of $M$ we define $\Omega^i M = \ker d_i$.
It is well defined up to projective summands
$\Omega^i \Omega^j M = \Omega^{i+j} M$ etc.

The modules that can occur as kernels in an acyclic complex of projectives (with $\mathrm{Hom}(-,P)$ exact) are called Gorenstein projectives. They have many nice properties. (ex).

For any $M$ we have a natural map $\Omega^0 M \to M$. It is a stable isomorphism, which we sometimes write $\tilde{M} \xrightarrow{\varepsilon} M$ and $\tilde{M}$ is Gorenstein projective.
cf the role of CW complexes in homotopy theory.

# Theorem TFAE

$$
\begin{tikzcd}
 & (kG\text{-Mod}, \Omega^\infty \mathrm{Hom}) \arrow[dl, "\text{degree } 0"'] \arrow[dr, "\substack{\text{complete}\\\text{resolution}}"] & & (\text{GorensteinProj}, \mathrm{Hom}_{\text{ModProj}}) \arrow[lll] \\
D_b(kG\text{-Mod})/D_b(kG\text{-Proj}) \arrow[rr, "\substack{\text{complete}\\\text{resolution}}"'] & & \left(\substack{\text{acyclic complexes projectives}\\ \mathrm{Hom}(-,P)\text{ exact}}, \text{homotopy}\right) \arrow[ur, "\Omega^0"] &
\end{tikzcd}
$$

homology in finitely many degrees → $D_b$

These categories are triangulated in the usual way (analogously to the structures on the stable category for a finite group or on $D(kG)$.

In a triangulated category, $X \xrightarrow{f} Y \longrightarrow Z$ triangle
$f$ iso $\Leftrightarrow$ $Z = 0$

Using this we can rephrase property $\Phi$:

<u>Lemma</u> $f: X \longrightarrow Y$ is a stable isomorphism if and only if $f\downarrow_P : X\downarrow_P \to Y\downarrow_P$ is a stable isomorphism for every finite $(p)$-subgroup.