PROBLEMS: COLEMAN INTEGRATION AND p-ADIC HEIGHTS

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(1) Let E be the elliptic curve $y^2 = x^3 - 55x + 157$ over $\mathbb{Q}$.
  (a) Compute the rank of $E(\mathbb{Q})$ and its Tamagawa numbers.
  (b) Let $P_1 = (4, 1)$ and $P_2 = (12, 35)$, let $h(P_i)$ denote the cyclotomic 7-adic height, and let $\int$ denote 7-adic Coleman integration. Compute the following ratios:
$$\frac{h(P_i)}{(\int_\infty^{P_i} \frac{dx}{2y})^2}, \quad i = 1, 2.$$
(2) Let X be the hyperelliptic curve $y^2 = x^5 + 1$. Compute the 7-adic matrix of Frobenius on $H^1_{\mathrm{dR}}(X)$.
(3) Let $X/\mathbb{Q}$ be a nice curve, $\omega \in H^0(X, \Omega^1)$, and $Q_i, Q_i' \in X(\mathbb{Q}_p)$. Suppose that $\sum_i (Q_i - Q_i')$ is the divisor of a rational function. Prove that
$$\sum_i \int_{Q_i'}^{Q_i} \omega = 0.$$
(4) Let $X/\mathbb{Q}$ be a hyperelliptic curve and $\omega \in H^0(X, \Omega^1)$. Show that for Weierstrass points $W_1, W_2 \in X(\mathbb{Q}_p)$, we have $\int_{W_1}^{W_2} \omega = 0$.
(5) Let X be the hyperelliptic curve $y^2 = x^5 + \frac{33}{16}x^4 + \frac{3}{4}x^3 + \frac{3}{8}x^2 - \frac{1}{4}x + \frac{1}{16}$ and let $P_1 = (0, \frac{1}{4}), P_2 = (-1, 1)$. Compute the 7-adic Coleman integrals
$$\int_{P_1}^{P_2} \frac{dx}{2y}, \int_{P_1}^{P_2} \frac{x\,dx}{2y}, \int_{P_1}^{P_2} \frac{x^2\,dx}{2y}, \int_{P_1}^{P_2} \frac{x^3\,dx}{2y}.$$

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