Lecture 2

Tuesday, August 3, 2021 10:35 AM

Moduli spaces of stable sheaves & the Brauer class

Huybrechts-Lehn "The geometry of moduli spaces of sheaves"

X proj. variety / field $k = \bar{k}$

Fix embedding $X \hookrightarrow \mathbb{P}^n \rightsquigarrow \mathcal{O}_X(1)$

goal: (quasi-) proj. variety param. vector bundles / coherent sheaves on X.

1st issue: infinitely many connected components $\Rightarrow$ not finite type

ex: Line bundles on $\mathbb{P}^1$

each $\mathcal{O}(n)$ gives different connected component, $n \in \mathbb{Z}$

Solution: fix some numerical invariants

e.g. Chern classes, or...

Def: For $F \in \mathrm{Coh}(X)$, the Hilbert polynomial of F is

$$P_F(t) := \chi(F(t))$$

$F(t) = F \otimes \mathcal{O}_X(t)$

depends on choice!

If $t \gg 0$, $h^i(F(t)) = 0 \quad \forall\, i > 0$

$$P_F(t) = h^0(F(t))$$

& is a polynomial in t

- compute use HRR
- $\deg P_F = \dim \mathrm{Supp}\, F$
- leading coeff. > 0
- constant in flat families

2nd issue: space param all sheaves w/ fixed Hilb. poly. is typically not separated.

ex: on $\mathbb{P}^1$ $\mathcal{O}(1)\oplus\mathcal{O}(1)$, $\mathcal{O}\oplus\mathcal{O}(2)$, $\mathcal{O}(-1)\oplus\mathcal{O}(3)$, ...

have Hilb poly $2t+4$

Can form family $\mathcal{F}$ over $\mathbb{A}^1$ w/

$$\mathcal{F}|_{\mathbb{P}^1\times\{z\}} = \begin{cases} \mathcal{O}(1)\oplus\mathcal{O}(1) & z\neq 0 \\ \mathcal{O}\oplus\mathcal{O}(2) & z=0 \end{cases}$$

$\rightsquigarrow$ map $\mathbb{A}^1 \to M$

$$z\neq 0 \mapsto \mathcal{O}(1)\oplus\mathcal{O}(1)$$
$$z=0 \mapsto \mathcal{O}\oplus\mathcal{O}(2)$$

$\Rightarrow$ M is not separated.

Solution: add a stability condition

Def: F is pure if $\forall\ 0\neq E\subseteq F$,

$$\dim \operatorname{Supp} E = \dim \operatorname{Supp} F$$

(X integral, $\operatorname{rk} F>0 \Rightarrow$ pure = torsion-free)

Def: The reduced Hilbert polynomial of F is

$$p_F(t) = \frac{P_F(t)}{\text{leading coeff. of } P_F} \quad \text{(monic)}$$

<u>Def</u>: F is <u>stable</u> (resp. <u>semi-stable</u>) if $\forall\ E \subsetneq F$, $p_E(t) < p_F(t)\ \forall\ t \gg 0$.

(resp. $\leq$)

<u>ex</u>:
$$\begin{array}{ll} F = \mathcal{O} \oplus \mathcal{O}(2) & P_F(t) = 2t+4 \\ \cup\!\!\!\mid & p_F(t) = t+2 \\ E = \mathcal{O}(2) & P_E(t) = t+3 = p_E(t) \end{array}$$

$\therefore F$ unstable

But $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is semi-stable
$$\begin{array}{ll} \cup\!\!\!\mid & \\ \mathcal{O}(d)\quad d \leq 1 & p_{\mathcal{O}(d)}(t) = t+(d+1) \\ & \leq t+2 \end{array}$$

In fact, its <u>polystable</u> = sum of stable sheaves

<u>Thm</u>: There exists a quasi-proj. variety param. stable sheaves w/ any given Hilb. poly., & a proj. variety param. polystable sheaves.

Note:

- If $k \neq \overline{k}$: change "stable" to "geometrically stable"
- can also work over $\operatorname{Spec}\mathbb{Z}$, $\operatorname{Spec}\mathbb{Z}_p$, ...

Examples:

① A connected component of Pic_X

Prop: If X is geometrically integral, all rank-1 torsion-free sheaves are stable wrt any embedding $X \hookrightarrow \mathbb{P}^n$.

Pf:

$$0 \neq E \subsetneq F^{\text{rk } 1 \text{ tors. free}} \Rightarrow \operatorname{rk} E = 1, \ \operatorname{rk}(F/E) = 0$$

$$0 \to E \to F \to F/E \to 0$$

$P_F(t) = at^n + \cdots$, $a > 0$, $n = \dim X$

$P_{F/E}(t) = bt^m + \cdots$, $b > 0$, $m = \dim \operatorname{Supp}(F/E) < n$

$$P_E(t) = P_F(t) - P_{F/E}(t)$$

divide by a to get $p_E(t) < p_F(t)$

□.

Gives natural compactification of Pic_X - component, & being line bundle is open condition

<u>Thm</u>: X smooth $\Rightarrow$ its also a closed condition

(2) $\mathrm{Hilb}^n X$ param ideal sheaves of 0-dim'l length n subsch. of X

(3) X = intersection of 2 quadrics in $\mathbb{P}^5$ / $\mathbb{C}$

$= \{f = g = 0\}$

$\rightsquigarrow$ pencil of quadrics

$$Q_{[a:b]} = \{af + bg = 0\} \quad [a:b] \in \mathbb{P}^1$$

If f, g generic, X smooth

$\Rightarrow Q_{[a:b]}$ smooth, except when

$$\det(a \underbrace{M_f}_{\text{symm. matrix of } f} + bM_g) = 0, \quad \text{i.e.}$$

except at 6 pts of $\mathbb{P}^1$

Each smooth $Q_{[a:b]} \cong Gr(2,4)$

$$0 \to \underset{\substack{\text{tautological} \\ \text{bundle}}}{S} \to \mathcal{O}^4 \to \underset{\substack{\text{quotient} \\ \text{bundle}}}{Q} \to 0$$

Consider $S^*|_X$, $Q|_X$ - rk 2 stable sheaves on X

moduli space param. these rk 2 stable sheaves on X V = double cover of $\mathbb{P}^1$ branched over those 6 pts.

The Brauer class:

M = mod. sp. of stable sheaves w/ fixed Hilb. poly

goal: When does $\exists$ a universal sheaf on $X \times M$, i.e. $\mathcal{U}$ s.t.

$$\mathcal{U}|_{X \times \{[F]\}} \cong F \,?$$

If $\mathcal{U}$ exists, then M fine, i.e. represents functor:

For family $\mathcal{F}$ of sh. on $X \times T$,

$\exists$ map $f: T \to M$ s.t.

$$(1 \times f)^* \mathcal{U} \otimes \pi_2^* L \cong \mathcal{F}$$

Line bundle on T

A universal sheaf always exists locally (analytic / étale), but $\exists$ a Brauer class that can obstruct it globally:

1^{st}, replace $F \in M$ w/ $F(n)$

for $n >> 0$

(boundedness $\Rightarrow \exists\ n$ that works $\forall\ F \in M$)

So assume F is globally generated

& $h^i(F) = 0\ \forall\ i > 0$.

Let $m = h^0(F) = \chi(F)$

$\mathcal{U}'$ univ. sheaf – only well-defined up to $\otimes$ line bundle

$$X \times M' \xrightarrow[\pi_{M'}]{} M' \xrightarrow[\text{étale}]{} M$$

$E := \pi_{M' *}\mathcal{U}'$ rk m vector bundle on M'

$P' := \mathbb{P}E \quad p : P' \to M'$

$\mathcal{U}'$ vs $\mathcal{U}' \otimes \pi_{M'}^* L$ also universal

E vs $E \otimes L$

$\mathcal{O}_{\mathbb{P}E}(1)$ vs $\mathcal{O}_{\mathbb{P}(E\otimes L)}(1) \cong \mathcal{O}_{\mathbb{P}E}(1) \otimes L^*$

on $X \times P'$, $(1\times p)^* \underline{\mathcal{U}'} \otimes \underline{\mathcal{O}_{\mathbb{P}E}(1)}$ is well-defined

$\Longrightarrow$ get $\mathbb{P}^{m-1}$-bundle $\pi : P \to M$,

$P|_{[F]} = \mathbb{P}H^0(F)$, w/ univ. sh. $\tilde{\mathcal{U}}$ on $X \times P$.

If $\exists$ relative $\mathcal{O}(1)$ for π, then

$$\underline{\tilde{\mathcal{U}} \otimes \pi_2^* \mathcal{O}(-1)}$$

$$\begin{tikzcd} & X \times P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & P \end{tikzcd}$$

descends to a univ. sheaf on $X \times M$.

Recall: relative $\mathcal{O}(1)$ exists

$\iff P = \mathbb{P}(\text{v.b.})$

$\iff$ Brauer class vanishes

In general, no relative $\mathcal{O}(1)$.

Call the Brauer class the obstruction to the existence of a univ. sheaf on $X \times M$.

When does $\exists$ relative $\mathcal{O}(1)$?

As above, $\pi_{2*} \tilde{\mathcal{U}}$ rk m v.b. on P, & restriction to any $\mathbb{P}^{m-1}$-fiber is $\mathcal{O}(1)^m$

$\Rightarrow \det(\pi_{2*}\tilde{\mathcal{U}})$ is a relative $\Theta(m)$

If E v.b. on X w/ $\chi(E \otimes F) = k$, $F \in M$

$\Rightarrow \det R\pi_{2*}(\tilde{\mathcal{U}} \otimes \pi_1^* E)$ is a relative $\Theta(k)$

If $\exists \{E_i\}$ w/ $\gcd(\chi(E_i \otimes F)) = 1$, then $\exists$ relative $\Theta(1)$, & hence universal sheaf on $X \times M$.