Remarks on the history of the Entropy Conjecture and the Role of Rufus Bowen. by Mike Shub

I recommend that all students of dynamical systems read Smale's 1967 article: Smale,S, Differentiable Dynamical Systems, Bull.Amer.Math.Soc. (73) 1967, 747-817 which was reprinted with comments in : Smale,S., The Mathematics of Time: essays on dynamical systems, economic processes, and related topics. Springer,2012. The article is an amazing synthesis of subjects bringing together a diverse variety of subjects with a unifying vision which defines much of what we call dynamical systems until this day. I was fortunate to be a student of Smale's as this article was written. In the article Smale raises the question (se (2.6) in the article) of what homotopy classes of maps contain what we now call Morse-Smale diffeomorphisms. He notes that it follows from the Lefschetz trace formula that $|L(f^m)| < C$ where L is the Lefschetz trace and C is a constant independent of m. In 1971, I proved that more must be true. In fact, all the eigenvalues of f_* mapping the real homology of M to itself must be roots of unity[1]. We may rephrase this as $\ln|\lambda|=0$ for any eigenvalue λ of f_*. A lot more work has been done on the question of which isotopy classes of diffeomorphism contain Morse-Smale diffeomorphisms Much of it is referenced in Shub, M (ICM), All, Most, Some Differentiable Dynamical Systems, Proceedings of the International Congress of Mathematicians,Vol.3, Madrid Spain,2006,Eurpean Mathematical Society,99-120. But that discussion would take us to far afield. In the summer of 1971, Bob Williams and I were also wondering whether all isotopy classes of diffeomorphisms contained structurally stable ones. I mentioned the problem to Smale and he solved it the same afternoon[2]. Shortly after I proved that structurally stable systems are C^0 dense. The argument followed Smale's. Taking a fine triangulation, neighborhoods of the zero cells were contracted into themselves, neighborhoods of the one cells were stretched across themselves etc. From the construction I could easily see a relation between the matrices defining the symbolic dynamics of the sub shifts produced and the induced map on a chain complex computing the homology given by the relative homology of a neighborhood of the i-cells modulo the (i-1)-cells. It followed easily that the growth rate (lim sup 1/n ln) of the number of periodic points of period n, $GR(N_n(f))$, was greater than or equal to $\ln|\lambda|$ for any eigenvalue λ of f_*. I understood this as a potential generalization of my result for Morse-Smale systems which might be valid for Axiom A no cycle systems in general and perhaps somehow more generally. It was the academic year 1971-72. as I remember there was a gathering at Charles Pugh's Berkeley house in the evening. Present were Charles, Bob Williams, Rufus and perhaps others. We were discussing these ideas when Rufus pointed out that for Axiom A no cycle systems the growth rate of the number of periodic points is the entropy, which might be the concept I was looking for. **Thanks Rufus!** I went back to Santa Cruz where I was teaching that year and thought about it. By the summer of 1972, I was ready to conjecture that the topological entropy, h(f) is greater than or equal to $\ln|\lambda|$ for any eigenvalue λ of f_*. The conjecture had two parts. For general C^1 f and for axiom A no cycle f in particular. The conjecture was proven for Axiom A no cycle diffeomorphisms in the early 70's by Bob Williams and myself and Ruelle and Sullivan.

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[1]Shub,M., Morse-Smale Diffeomorphisms are Unipotent on Homology, in Dynamical Systems (Ed. M.M.Peixoto), Academic Press, New York, 1973, 489-492.

[2] Smale,S., Stability and isotopy in discrete dynamical systems, in Dynamical Systems, op. cit.

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In the mid 80's it was proven for C ^{infinity} maps by Yomdin. See Shub(ICM) for references..The case of finite differentiability remains open for any r bigger than equal to 1. Rufus also wrote at least two papers on the inequality,Bowen, R. (1974). Entropy versus homology for certain diffeomorphisms. *Topology*, *13*(1), 61-67 and Bowen, R. (1978). Entropy and the fundamental group. *The structure of attractors in dynamical systems*, 21-29.In the first he proved the conjecture for axiom a systems with zero dimensional omega and in the second he bounded the entropy from below by the growth rate of the induced map on the fundamental group. There are some relations of the entropy conjecture to other asymptotic relations between dynamics and algebraic topology which can be seen in Shub(ICM). These relations somehow form a whole in my mind and Yomdin's theorem gives hope for the others. I am fairly confident that Enrique Pujals, Yun Yang and I are resolving the question of the growth rate of the number of periodic points for smooth degree two maps of the two sphere in the positive which gives me even more hope that they may all be true.