Sparse Approximation of PDEs based on Compressed Sensing

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Introduction

We address the following question

Can we employ Compressed Sensing to solve a PDE?

In particular, we consider the **weak formulation** of a PDE

$$\text{find } u \in U : \quad a(u,v) = \mathcal{F}(v), \quad \forall v \in V,$$

focusing on the **Petrov-Galerkin** (**PG**) discretization method [Aziz and Babuška, 1972].

Motivation:

- **reduce the computational cost** associated with a classical PG discretization;
- situations with a **limited budget** of evaluations of $\mathcal{F}(\cdot)$;
- better **theoretical understanding** of the PG method.

Case study:

- ✎ **Advection-diffusion-reaction (ADR) equation**, with $U = V = H_0^1(\Omega)$, $\Omega = [0,1]^d$, and

$$a(u,v) = (\eta \nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v) + (\rho u, v), \quad \mathcal{F}(v) = (f, v).$$

Compressed Sensing

CORSING (COmpRessed SolvING)

A theoretical study of CORSING

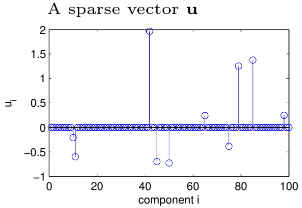
Compressed Sensing (CS)

[D. Donoho, 2006; E. Candès, J. Romberg, and T. Tao, 2006]

Consider a signal $\mathbf{s} \in \mathbb{C}^N$, **sparse** w.r.t. $\mathbf{\Psi} \in \mathbb{C}^{N\times N}$:

$$\mathbf{s} = \mathbf{\Psi}\mathbf{u} \quad \text{and} \quad \|\mathbf{u}\|_0 =: s \ll N,$$

where $\|\mathbf{u}\|_0 := \#\{i : \mathbf{u}_i \neq 0\}$.



It can be acquired by means of $m \ll N$ **linear** and **non-adaptive** measurements

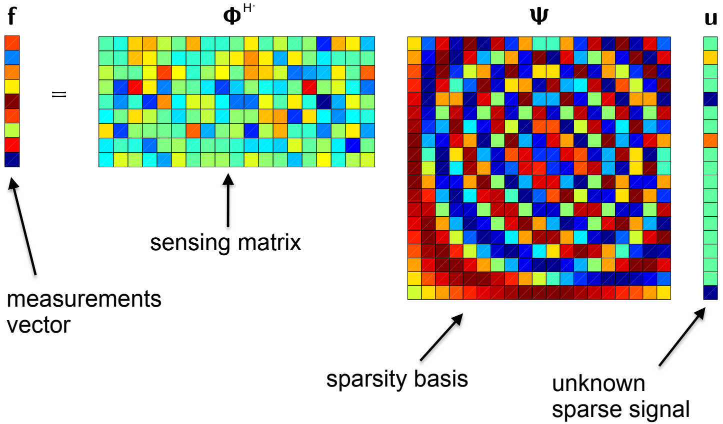
$$\langle \mathbf{s}, \boldsymbol{\varphi}_i \rangle =: f_i, \quad \text{for } i = 1, \ldots, m.$$

If we consider the matrix $\mathbf{\Phi} = [\boldsymbol{\varphi}_i] \in \mathbb{C}^{N\times m}$, we have

$$\mathbf{A}\mathbf{u} = \mathbf{f},$$

where $\mathbf{A} = \mathbf{\Phi}^{\mathsf{H}}\mathbf{\Psi} \in \mathbb{C}^{m\times N}$ and $\mathbf{f} \in \mathbb{C}^m$.

Sensing phase



Since $m \ll N$, the system $\mathbf{Au} = \mathbf{f}$ is highly underdetermined. How to recover the right $\mathbf{u}$ among its infinite solutions?

Recovery: finding a needle in a haystack

Thanks to the sparsity hypothesis, we can resort to **sparse recovery techniques**. We aim at approximating the solution to

$$(\mathrm{P}_0) \quad \min_{\mathbf{u}\in\mathbb{C}^N} \|\mathbf{u}\|_0, \quad \text{s.t. } \mathbf{A}\mathbf{u} = \mathbf{f}.$$

- ☹ Unfortunately, in general (P_0) is a **NP-hard** problem...
- ☺ There are **computationally tractable** strategies to approximate it!

In particular, we employ the **greedy algorithm Orthogonal Matching Pursuit** (**OMP**) to approximate

$$(\mathrm{P}_0^{\varepsilon}) \quad \min_{\mathbf{u}\in\mathbb{C}^N} \|\mathbf{u}\|_0 \quad \text{s.t. } \|\mathbf{A}\mathbf{u} - \mathbf{f}\|_2 \leq \varepsilon$$

or

$$(\mathrm{P}_0^{s}) \quad \min_{\mathbf{u}\in\mathbb{C}^N} \|\mathbf{A}\mathbf{u} - \mathbf{f}\|_2 \quad \text{s.t. } \|\mathbf{u}\|_0 \leq s.$$

Another valuable option is **convex relaxation** (not discussed here)

$$(\mathrm{P}_1) \quad \min_{\mathbf{u}\in\mathbb{C}^N} \|\mathbf{u}\|_1, \quad \text{s.t. } \mathbf{A}\mathbf{u} = \mathbf{f}.$$

Orthogonal Matching Pursuit (OMP)

Input:

Matrix $\mathbf{A} \in \mathbb{C}^{m \times N}$, with ℓ^2-normalized columns

Vector $\mathbf{f} \in \mathbb{C}^m$

Tolerance on the residual $\varepsilon > 0$ (or else, sparsity $s \in [N]$)

Output:

Approximate solution $\mathbf{u}$ to $(\mathrm{P}_0^\varepsilon)$ (or else, (P_0^s))

Procedure:

1: $\mathcal{S} \leftarrow \emptyset$ ▷ Initialization
2: $\mathbf{u} \leftarrow \mathbf{0}$
3: **while** $\|\mathbf{Au} - \mathbf{f}\|_2 > \varepsilon$ (or else, $\|\mathbf{u}\|_0 < s$) **do**
4: $\quad \bar{j} \leftarrow \arg\max_{j \in [N]} |[\mathbf{A}^{\mathsf{H}}(\mathbf{Au} - \mathbf{f})]_j|$ ▷ Select new index
5: $\quad \mathcal{S} \leftarrow \mathcal{S} \cup \{\bar{j}\}$ ▷ Enlarge support
6: $\quad \mathbf{u} \leftarrow \arg\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{Az} - \mathbf{f}\|_2$ s.t. $\mathrm{supp}(\mathbf{z}) \subseteq \mathcal{S}$ ▷ Minimize residual
7: **end while**
8: **return u**

- The computational cost for the (P_0^s) formulation is in general $\mathcal{O}(smN)$.

Recovery results based on the RIP

Many important recovery results in CS are based on the **Restricted Isometry Property** (**RIP**).

Definition (RIP)

A matrix $\mathbf{A} \in \mathbb{C}^{m \times N}$ satisfies the RIP(s, δ) iff

$$(1-\delta)\|\mathbf{u}\|_2^2 \leq \|\mathbf{A}\mathbf{u}\|_2^2 \leq (1+\delta)\|\mathbf{u}\|_2^2, \quad \forall \mathbf{u} \in \Sigma_s^N := \{\mathbf{v} \in \mathbb{C}^N : \|\mathbf{v}\|_0 \leq s\}.$$

Among many others, the **RIP** implies the following recovery result for OMP. [T. Zhang, 2011; A. Cohen, W. Dahmen, R. DeVore, 2015]

Theorem (RIP $\Rightarrow$ OMP recovery)

There exist $K \in \mathbb{N}$, $C > 0$ and $\delta \in (0,1)$ s.t. for every $s \in \mathbb{N}$, the following holds: if

$$\mathbf{A} \in RIP((K+1)s, \delta),$$

then, for any $\mathbf{f} \in \mathbb{C}^m$, the OMP algorithm computes in Ks iterations a solution $\mathbf{u}$ that fulfills

$$\|\mathbf{A}\mathbf{u} - \mathbf{f}\|_2 \leq C \inf_{\mathbf{w} \in \Sigma_s^N} \|\mathbf{A}\mathbf{w} - \mathbf{f}\|_2.$$

>

Compressed Sensing

CORSING (COmpRessed SolvING)

A theoretical study of CORSING

The reference problem

Given two Hilbert spaces U, V, consider the following problem

$$\text{find } u \in U : a(u,v) = \mathcal{F}(v), \quad \forall v \in V, \tag{1}$$

where $a : U \times V \to \mathbb{R}$ is a bilinear form and $\mathcal{F} \in V^*$. We will assume $a(\cdot,\cdot)$ to fulfill

$$\exists \alpha > 0 : \quad \inf_{u \in U} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \geq \alpha, \tag{2}$$

$$\exists \beta > 0 : \quad \sup_{u \in U} \sup_{v \in V} \frac{|a(u,v)|}{\|u\|_U \|v\|_V} \leq \beta, \tag{3}$$

$$\sup_{u \in U} a(u,v) > 0, \quad \forall v \in V \setminus \{0\}. \tag{4}$$

- ✎ (2) + (3) + (4) $\Longrightarrow \exists!$ solution to (1). [J. Nečas, 1962]
- ✎ We will focus on advection-diffusion-reaction (ADR) equations.

The Petrov-Galerkin method

Given $\Omega \subseteq \mathbb{R}^d$, consider the weak formulation of an **ADR equation**:

$$\text{find } u \in H_0^1(\Omega) : \underbrace{(\eta \nabla u, \nabla v) + (\mathbf{b}\nabla u, v) + (\rho u, v)}_{a(u,v)} = \underbrace{(f, v)}_{\mathcal{F}(v)}, \ \forall v \in H_0^1(\Omega). \tag{ADR}$$

Choose $U^N \subseteq H_0^1(\Omega)$ and $V^M \subseteq H_0^1(\Omega)$ with

$$U^N = \text{span}\{\underbrace{\psi_1, \ldots, \psi_N}_{\text{trials}}\}, \quad V^M = \text{span}\{\underbrace{\varphi_1, \ldots, \varphi_M}_{\text{tests}}\}$$

Then we can discretize (ADR) as

$$\mathbf{A}\widehat{\mathbf{u}} = \mathbf{f}, \quad A_{ij} = a(\psi_j, \varphi_i), \quad f_i = \mathcal{F}(\varphi_i)$$

with $\mathbf{A} \in \mathbb{C}^{M \times N}$, $\mathbf{f} \in \mathbb{C}^M$.

A common choice is $M = N$.

- Examples of Petrov-Galerkin methods: **Finite elements**, **spectral methods**, **collocation methods**, etc.

The main analogy

A fundamental **analogy** guided us through the development of our method...

Petrov-Galerkin method:		*Sampling:*
solution of a PDE	$\Longleftrightarrow$	signal
tests (bilinear form)		**measurements** (inner product)

Reference:

Compressed solving: a numerical approximation technique for elliptic PDEs based on compressed sensing
S. B., S. Micheletti, S. Perotto
Comput. Math. Appl. 2015; 70(6):1306-1335

Related literature

Ancestors: PDE solvers based on ℓ^1-minimization

1988 [J. Lavery, 1988; J. Lavery, 1989]
Inviscid Burgers' equation, conservation laws

2004 [J.-L. Guermond, 2004; J.-L. Guermond and B. Popov, 2009]
Hamilton-Jacobi, transport equation

CS techniques for PDEs

2010 [S. Jokar, V. Mehrmann, M. Pfetsch, and H. Yserentant, 2010]
Recursive mesh refinement based on CS (Poisson equation)

2011 [A. Doostan and H. Owhadi, 2011; K. Sargsyan et al., 2014; H. Rauhut and C. Schwab, 2014; J.-L. Bouchot et al., 2015]

Application of CS to **parametric PDEs** and **Uncertainty Quantification**

2015 [S. B., S. Micheletti, S. Perotto, 2015;
S. B., F. Nobile, S. Micheletti, S. Perotto, 2016]
CORSING for ADR problems

CORSING (COmpRessed SolvING)

Assembly phase

① Choose two sets of N independent elements of U and V:

$$\text{trials} \rightarrow \{\psi_1, \ldots, \psi_N\}, \quad \{\varphi_1, \ldots, \varphi_N\} \leftarrow \text{tests};$$

② choose $m \ll N$ tests $\{\varphi_{\tau_1}, \ldots, \varphi_{\tau_m}\}$:

DETERMINISTICALLY ↓ D-CORSING

how? ⇄

randomly ↓ R-CORSING

③ build $\mathbf{A} \in \mathbb{C}^{m \times N}$ and $\mathbf{f} \in \mathbb{C}^m$ as

$$[\mathbf{A}]_{ij} := a(\psi_j, \varphi_{\tau_i}) \quad [\mathbf{f}]_i := \mathcal{F}(\varphi_{\tau_i}).$$

Recovery phase

Find a compressed solution $\mathbf{u}_m^N$ to $\mathbf{Au} = \mathbf{f}$, via **sparse recovery**.

CORSING (COmpRessed SolvING)

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Recovery phase

Find a compressed solution $\mathbf{u}_m^N$ to $\mathbf{Au} = \mathbf{f}$, via **sparse recovery**.

Classical case: square matrices

When dealing with Petrov-Galerkin discretizations, one usually ends up with a **big square** matrix.

$$
\begin{array}{c}
\\ \\
\varphi_1 \rightarrow \\ \varphi_2 \rightarrow \\ \varphi_3 \rightarrow \\ \varphi_4 \rightarrow \\ \varphi_5 \rightarrow \\ \varphi_6 \rightarrow \\ \varphi_7 \rightarrow
\end{array}
\underbrace{
\begin{array}{c}
\begin{array}{ccccccc} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
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\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times
\end{bmatrix}
\end{array}
}_{a(\psi_j, \varphi_i)}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}
=
\begin{bmatrix} \mathcal{F}(\varphi_1) \\ \mathcal{F}(\varphi_2) \\ \mathcal{F}(\varphi_3) \\ \mathcal{F}(\varphi_4) \\ \mathcal{F}(\varphi_5) \\ \mathcal{F}(\varphi_6) \\ \mathcal{F}(\varphi_7) \end{bmatrix}
$$

"Compressing" the discretization

We would like to use only m random tests instead of N, with $m \ll N$...

$$
\begin{array}{c}
\\
\\
\varphi_1 \rightarrow \\
\varphi_2 \rightarrow \\
\varphi_3 \rightarrow \\
\varphi_4 \rightarrow \\
\varphi_5 \rightarrow \\
\varphi_6 \rightarrow \\
\varphi_7 \rightarrow
\end{array}
\underbrace{
\begin{array}{c}
\begin{matrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{matrix} \\
\begin{bmatrix}
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times \\
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\times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times
\end{bmatrix}
\end{array}
}_{a(\psi_j, \varphi_i)}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}
=
\begin{bmatrix} \mathcal{F}(\varphi_1) \\ \mathcal{F}(\varphi_2) \\ \mathcal{F}(\varphi_3) \\ \mathcal{F}(\varphi_4) \\ \mathcal{F}(\varphi_5) \\ \mathcal{F}(\varphi_6) \\ \mathcal{F}(\varphi_7) \end{bmatrix}
$$

Sparse recovery

...in order to obtain a **reduced discretization**.

$$\begin{array}{c} \\ \varphi_2 \rightarrow \\ \varphi_5 \rightarrow \end{array} \underbrace{\begin{array}{ccccccc} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \end{array}}_{a(\psi_j, \varphi_i)} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} \mathcal{F}(\varphi_2) \\ \mathcal{F}(\varphi_5) \end{bmatrix}$$

The solution is then computed using **sparse recovery** techniques.

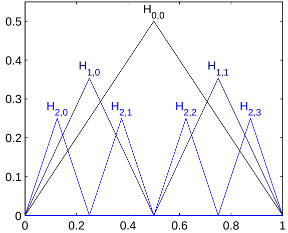
How to choose $\{\psi_j\}$ and $\{\varphi_i\}$?

How to choose $\{\psi_j\}$ and $\{\varphi_i\}$?

One heuristic criterion commonly used in CS is to choose one basis sparse in **space**, and the other in **frequency**.

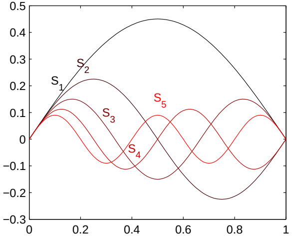
Hierarchical hat functions

[O. Zienkiewicz et al., 1982]



$\mathcal{H}$

Sine functions



$\mathcal{S}$

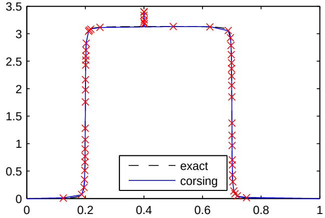
We name the corresponding strategies CORSING $\mathcal{HS}$ and $\mathcal{SH}$.

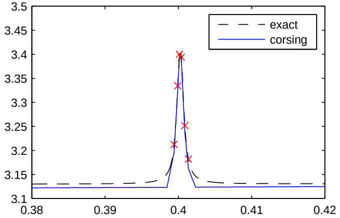
A 1D example

We test CORSING $\mathcal{HS}$ on the homogeneous 1D Poisson problem $(a(u,v) = (u', v'))$:

- Trial space dimension $N = 8191$
- Solution sparsity $s = 50$
- Selected random tests $m = 1200$

$$\textbf{Test Savings}: \quad \text{TS} := \frac{N - m}{N} \cdot 100\% \approx 85\%$$



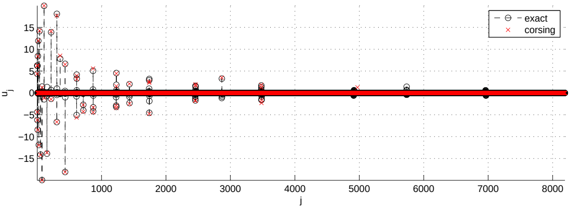


$\times$ = hat functions selected by OMP after solving the program

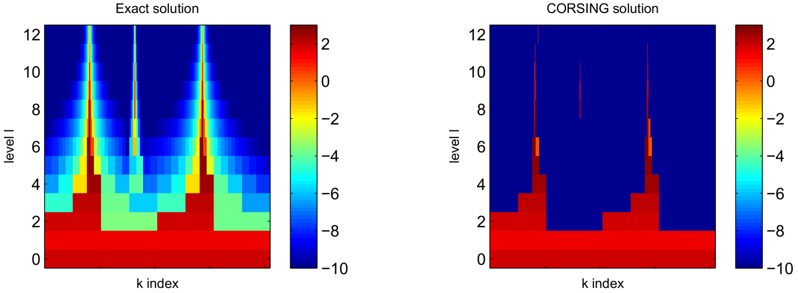
$$\min \|\mathbf{Au} - \mathbf{f}\|_2, \quad \text{s.t.} \|\mathbf{u}\|_0 \leq 50$$

A glance at the space of coefficients...

Lexicographic ordering



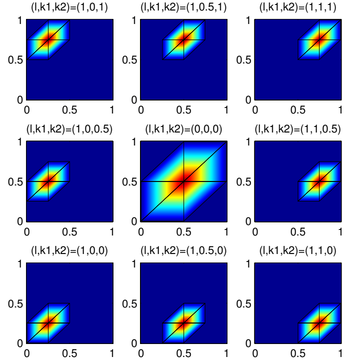
Level-based ordering $(\log_{10} |\widehat{u}_{\ell,k}|)$



Generalization to the 2D case (space domain)

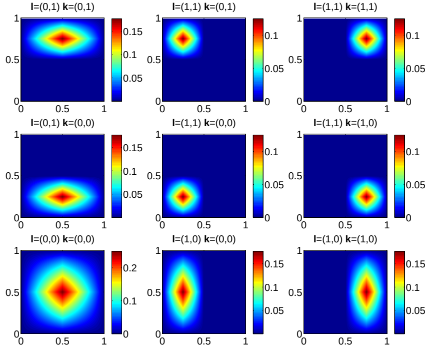
Hierarchical Pyramids

[H. Yserentant, 1986]



$\mathcal{P}$

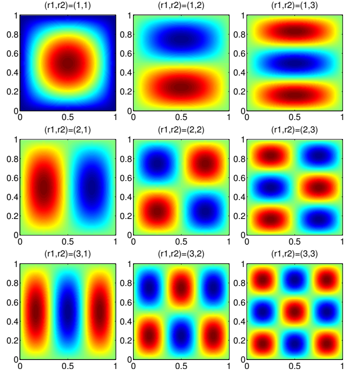
Tensor product of hat functions



$\mathcal{Q}$

The 2D case (frequency domain)

Tensor product of sine functions



$$\mathcal{S}$$

We have four strategies: CORSING $\mathcal{PS}$, $\mathcal{QS}$, $\mathcal{SP}$ and $\mathcal{SQ}$.

An advection-dominated example

We evaluate the CORSING performance on the following 2D advection-dominated problem

$$\begin{cases} -\mu\Delta u + \mathbf{b}\cdot\nabla u = f & \text{in } \Omega = (0,1)^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

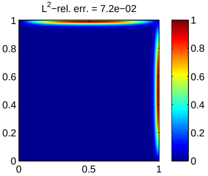
where $\mathbf{b} = [1,1]^\intercal$, $0 < \mu \ll 1$ and f s.t. the exact solution be

$$u_\mu^*(\mathbf{x}) = C_\mu(x_1 - x_1^2)(x_2 - x_2^2)(e^{x_1/\mu} + e^{x_2/\mu} - 2),$$

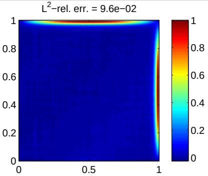
where $C_\mu > 0$ is chosen such that $\max_{\mathbf{x}\in\Omega} u_\mu^*(\mathbf{x}) = 1$.

- The function u_μ^* exhibits two **boundary layers** along the edges $\{x_1 = 1\}$ and $\{x_2 = 1\}$ of Ω.

$N = 16129$
TS $= 85\%$
ESP $= 1.00$
L^2-rel. err. $=$ 7.1e-02



$N = 16129$
TS $= 90\%$
ESP $= 0.94$
L^2-rel. err. $=$ 8.7e-02



Exact

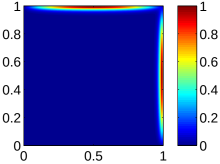


Figure: CORSING $\mathcal{SP}$, with $\mu = 0.01$: worst solution in the successful cluster (right). 50 random experiments are performed.

ESP = Empirical Success Probability

Cost reduction with respect to the full-PG (m=N)

We compare the assembly/recovery times of full-PG and CORSING.

full-PG			CORSING $\mathcal{SP}$			
A	**f**	t_{rec} (\\)	TS	**A**	**f**	t_{rec} (OMP)
2.5e+03	9.1e-01	7.1e+01	85%	3.8e+02	2.7e-01	8.1e+01
			90%	2.5e+02	2.0e-01	3.4e+01

- The assembly time reduction is proportional to TS.
- Also the RAM is reduced proportionally to TS.
- The recovery phase is cheaper for high TS rates.

The CORSING method can considerably reduce the computational cost associated with a full-PG discretization.

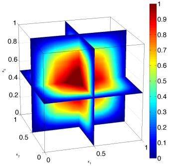
More challenging test cases

The CORSING technique has also been implemented for

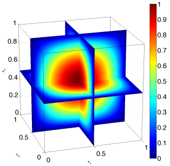
The 3D Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)^3 \\ u = 0 & \text{on} \partial\Omega \end{cases}$$

CORSING $\mathcal{QS}$
TS=85%



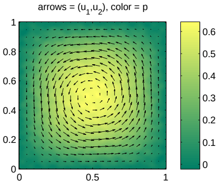
Exact solution



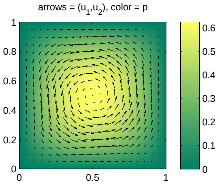
The 2D Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega = (0,1)^2 \\ \text{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on} \partial\Omega \end{cases}$$

CORSING $\mathcal{SP}$
TS=70%



Exact solution



Compressed Sensing

CORSING (COmpRessed SolvING)

A theoretical study of CORSING

A theoretical understanding of the method

Reference:
A theoretical study of COmpRessed SolvING for advection-diffusion-reaction problems
S.B., F. Nobile, S. Micheletti, S. Perotto
To appear in Mathematics of Computation

Some notation:

- Finite dimensional *trial* and *test* spaces

$$U^N := \text{span}\{\psi_j\}_{j\in[N]} \quad \text{and} \quad V^M := \text{span}\{\varphi_i\}_{i\in[M]},$$

  where $[k] := \{1, \dots, k\}$ for every $k \in \mathbb{N}$.

- The set of s-sparse elements of U^N

$$U_s^N := \left\{ \sum_{j\in[N]} u_j\psi_j : \|\mathbf{u}\|_0 \leq s \right\}$$

Simplification: Let us assume the bases $\{\psi_j\}_{j\in\mathbb{N}}$ and $\{\varphi_q\}_{q\in\mathbb{N}}$ to be **orthonormal**.

Local a-coherence

An important tool employed in the theoretical analysis is the **local a-coherence**, a generalization of the *local coherence* of CS.

Definition

Given $N \in \mathbb{N} \cup \{\infty\}$, the real-valued sequence $\boldsymbol{\mu}^N$ defined as

$$\mu_q^N := \sup_{j \in [N]} |a(\psi_j, \varphi_q)|^2, \quad \forall q \in \mathbb{N},$$

is called **local a-coherence** of $\{\psi_j\}_{j \in [N]}$ with respect to $\{\varphi_q\}_{q \in \mathbb{N}}$.

- Following [F. Krahmer and R. Ward, 2014], we define a computable **upper bound** $\boldsymbol{\nu}^N$ to $\boldsymbol{\mu}^N$:

$$\mu_q^N \leq \nu_q^N, \quad \forall q \in \mathbb{N}.$$

  Moreover, for every $M \in \mathbb{N}$, we define

$$\boldsymbol{\nu}^{N,M} := [\nu_1^N, \ldots, \nu_M^N]^\intercal \in \mathbb{R}^M.$$

Formalization of the CORSING procedure

PROCEDURE $\widehat{u}$ = CORSING $(N, s, \boldsymbol{\nu}^N, \widehat{\gamma}, \overline{\gamma})$

1. **[Definition of M and m]**

$$M \sim s^{\widehat{\gamma}} N; \quad m \sim s^{\overline{\gamma}} \|\boldsymbol{\nu}^{N,M}\|_1 \log(N/s);$$

2. **[Test selection]** Draw $\tau_1, \ldots, \tau_m$ *independently* at random from $[M]$ according to the probability

$$\mathbf{p} := \boldsymbol{\nu}^{N,M} / \|\boldsymbol{\nu}^{N,M}\|_1;$$

3. **[Assembly]** Build $\mathbf{A} \in \mathbb{R}^{m \times N}$, $\mathbf{f} \in \mathbb{R}^m$ and $\mathbf{D} \in \mathbb{R}^{m \times m}$, defined as:

$$A_{ij} := a(\psi_j, \varphi_{\tau_i}), \quad f_i := \mathcal{F}(\varphi_{\tau_i}), \quad D_{ik} := \frac{\delta_{ik}}{\sqrt{m p_{\tau_i}}}.$$

4. **[Recovery]**

> Find an approximate solution $\widehat{\mathbf{u}}$ to $\min_{\mathbf{u} \in \mathbb{R}^N} \|\mathbf{D}(\mathbf{A}\mathbf{u} - \mathbf{f})\|_2^2$, s.t. $\|\mathbf{u}\|_0 \leq s$;

> $\widehat{u} \leftarrow \sum_{j=1}^{N} \widehat{u}_j \psi_j$.

Main tools of the analysis

The theoretical analysis is based on **three main tools**:

1. the concept of **local a-coherence** between two bases;
2. **Chernoff's bounds** for the sum of random matrices [H. Chernoff,1952; R. Ahlswede and A. Winter, 2002; J. Tropp, 2012];
3. a variant of the classical inf-sup property, that we called **restricted inf-sup property (RISP)**, i.e.,

$$\inf_{\mathbf{u}\in\Sigma_s^N} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}\mathbf{u}}{\|\mathbf{u}\|_2\|\mathbf{v}\|_2} > \widetilde{\alpha} > 0,$$

   where $\Sigma_s^N := \{\mathbf{u} \in \mathbb{R}^N : \|\mathbf{u}\|_0 \leq s\}$.

From the ∞-dimensional problem to CORSING

While moving from the ∞-dimensional weak problem to the CORSING reduced formulation we will **track the inf-sup constant**:

	# of tests	inf-sup constant
Weak problem	∞	α
PG discretization	$M < \infty$	$\alpha(1-\widehat{\delta})^{\frac{1}{2}}$
CORSING	$m \ll M$	$\alpha(1-\widehat{\delta})^{\frac{1}{2}}(1-\overline{\delta})^{\frac{1}{2}}$

This will guarantee the stability of our method and will imply recovery error estimates for the CORSING technique.

$$\inf_{u \in U_s^N} \sup_{v \in V} \frac{a(u,v)}{\|u\|_U \|v\|_V} \rightsquigarrow \inf_{u \in U_s^N} \sup_{v \in V^M} \frac{a(u,v)}{\|u\|_U \|v\|_V} \rightsquigarrow \inf_{\mathbf{u} \in \Sigma_s^N} \sup_{\mathbf{v} \in \mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D} \mathbf{A} \mathbf{u}}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2}$$

Uniform RISP

Theorem

For every $s \in \mathbb{N}$, *given* $\widehat{\delta} \in (0,1)$, *choose* $M \in \mathbb{N}$ *such that*

$$\sum_{q>M} \mu_q^N \leq \frac{\alpha^2 \widehat{\delta}}{s}.$$

Then, for every $\varepsilon > 0$ *and* $\overline{\delta} \in (0,1)$, *provided*

$$m \gtrsim \overline{\delta}^{-2} \|\boldsymbol{\nu}^{N,M}\|_1 [s^2 \log(eN/s) + s \log(s/\varepsilon)],$$

the following uniform RISP holds with probability $\geq 1 - \varepsilon$

$$\inf_{\mathbf{u} \in \Sigma_s^N} \sup_{\mathbf{v} \in \mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D} \mathbf{A} \mathbf{u}}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2} > \widetilde{\alpha} > 0,$$

where $\widetilde{\alpha} := (1 - \widehat{\delta})^{\frac{1}{2}} (1 - \overline{\delta})^{\frac{1}{2}} \alpha$.

Non-uniform RISP: sketch of the proof (1/2)

The proof can be organized as follows:

1. Fix $\mathcal{S} \subseteq [N]$, with $|\mathcal{S}| = s$, and notice that

$$\inf_{\mathbf{u}\in\mathbb{R}^s} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}_\mathcal{S}\mathbf{u}}{\|\mathbf{u}\|_2\|\mathbf{v}\|_2} = [\lambda_{\min}(\mathbf{A}_\mathcal{S}^\intercal \mathbf{D}^2\mathbf{A}_\mathcal{S})]^{\frac{1}{2}} = [\lambda_{\min}(\overline{\mathbf{X}})]^{\frac{1}{2}}.$$

   Indeed, $\mathbf{A}_\mathcal{S}^\intercal \mathbf{D}^2\mathbf{A}_\mathcal{S}$ is the sample mean of random matrices

$$(\mathbf{A}_\mathcal{S}^\intercal \mathbf{D}^2\mathbf{A}_\mathcal{S})_{jk} = \frac{1}{m}\sum_{i=1}^{m} \underbrace{\frac{1}{p_{\tau_i}} a(\psi_{\sigma_j},\varphi_{\tau_i})a(\psi_{\sigma_k},\varphi_{\tau_i})}_{=:\mathbf{X}^{\tau_i}_{jk}}.$$

2. The minimum eigenvalue of $\mathbf{X}^{\tau_i}$ can be controlled **in expectation**:

$$\sum_{q>M}\mu_q^N \leq \frac{\widehat{\delta}\alpha^2}{s} \Longrightarrow \lambda_{\min}(\mathbb{E}[\mathbf{X}^{\tau_i}])^{\frac{1}{2}} = \inf_{u\in U_\mathcal{S}^N} \sup_{v\in V^M} \frac{a(u,v)}{\|u\|_U\|v\|_V} \geq (1-\widehat{\delta})^{\frac{1}{2}}\alpha$$

3. The thesis is proved by resorting to the **matrix Chernoff bounds**.

Non-uniform RISP: sketch of the proof (2/2)

Theorem (Matrix Chernoff bounds)

Consider a finite sequence of i.i.d. random, symmetric $s \times s$ real matrices $\mathbf{M}^1, \dots, \mathbf{M}^m$ such that

$$0 \leq \lambda_{\min}(\mathbf{M}^i) \text{ and } \lambda_{\max}(\mathbf{M}^i) \leq R \quad \text{almost surely}, \quad \forall i \in [m].$$

Define $\overline{\mathbf{M}} := \frac{1}{m}\sum_{i=1}^m \mathbf{M}^i$ *and* $\lambda_* := \lambda_{\min}(\mathbb{E}[\mathbf{M}^i])$. *Then,*

$$\mathbb{P}\{\lambda_{\min}(\overline{\mathbf{M}}) \leq (1-\delta)\lambda_*\} \lesssim s \exp\left(-\frac{m\delta^2\lambda_*}{R}\right), \quad \forall \delta \in [0,1].$$

- After choosing $\mathbf{M}^i = \mathbf{X}^{\tau_i}$, direct computations show that

$$0 \leq \lambda_{\min}(\mathbf{X}^{\tau_i}) \quad \text{and} \quad \lambda_{\max}(\mathbf{X}^{\tau_i}) \leq s\|\boldsymbol{\nu}^{N,M}\|_1.$$

- Finally, we consider the inf-sup over U_s^N employing a **union bound**.

□

Recovery error analysis

Our aim is to compare the recovery error $\|\widehat{u} - u\|_U$ with the best s-term approximation error of the exact solution u in U^N, i.e. the quantity $\|u^s - u\|_U$, where

$$u^s := \arg\min_{w \in U_s^N} \|w - u\|_U.$$

A key quantity is the following preconditioned random residual

$$\mathcal{R}(u^s) := \left[\frac{1}{m} \sum_{i=1}^{m} \frac{1}{p_{\tau_i}} [a(u^s, \varphi_{\tau_i}) - \mathcal{F}(\varphi_{\tau_i})]^2 \right]^{\frac{1}{2}} = \|\mathbf{D}(\mathbf{A}\mathbf{u}^s - \mathbf{f})\|_2.$$

Assumption: we assume that $\widehat{\mathbf{u}}$ solves the problem

$$\min_{\mathbf{u} \in \mathbb{R}^N} \|\mathbf{D}(\mathbf{A}\mathbf{u} - \mathbf{f})\|_2^2, \text{ s.t. } \|\mathbf{u}\|_0 \leq s$$

exactly (even if, in reality, OMP can only *approximate* its solution).

Two lemmas about $\mathcal{R}(u^s)$

✎ An argument analogous to **Cea's lemma** shows the following

Lemma

If the following uniform 2s-sparse RISP holds

$$\inf_{\mathbf{u}\in\Sigma_{2s}^N} \sup_{\mathbf{v}\in\mathbb{R}^m} \frac{\mathbf{v}^\intercal \mathbf{D}\mathbf{A}\mathbf{u}}{\|\mathbf{u}\|_2\|\mathbf{v}\|_2} > \widetilde{\alpha} > 0,$$

then the CORSING *procedure computes a solution* $\widehat{u}$ *such that*

$$\|\widehat{u} - u^s\|_U < \frac{2}{\widetilde{\alpha}}\mathcal{R}(u^s).$$

✎ Moreover, this mysterious residual behaves nicely *in expectation*!

Lemma

$$\mathbb{E}[\mathcal{R}(u^s)^2] \leq \beta^2\|u^s - u\|_U^2,$$

where β *is the continuity constant of* $a(\cdot,\cdot)$.

Error estimate in expectation

Theorem (CORSING recovery in expectation)

Let $s \leq N$ *and* $\mathcal{K} > 0$ *be such that* $\|u\|_U \leq \mathcal{K}$ *and* $\widehat{\delta}, \overline{\delta} \in (0,1)$. *Choose* $M \in \mathbb{N}$ *such that the following truncation condition is fulfilled*

$$\sum_{q>M} \mu_q^N \leq \frac{\alpha^2 \widehat{\delta}}{s}.$$

Then, for every $\varepsilon > 0$, *provided*

$$m \gtrsim \overline{\delta}^{-2} \|\boldsymbol{\nu}^{N,M}\|_1 [s^2 \log(N/s) + s \log(s/\varepsilon)],$$

the truncated CORSING *solution* $\mathcal{T}_{\mathcal{K}} \widehat{u}$ *fulfills*

$$\mathbb{E}[\|\mathcal{T}_{\mathcal{K}} \widehat{u} - u\|_U] \leq \left(1 + \frac{2\beta}{\widetilde{\alpha}}\right) \|u^s - u\|_U + 2\mathcal{K}\varepsilon,$$

where $\widetilde{\alpha} = (1-\widehat{\delta})^{\frac{1}{2}}(1-\overline{\delta})^{\frac{1}{2}}\alpha$ *and* $\mathcal{T}_{\mathcal{K}}(w) := \max(1, \mathcal{K}/\|w\|_U)w$.

Remarks:

- A possible choice for $\mathcal{K}$ is $\|\mathcal{F}\|_{V^*}/\alpha$.
- An analogous result holds **in probability**.

Application to the 1D Poisson problem

Proposition (CORSING $\mathcal{HS}$ recovery)

Fix a maximum hierarchical level $L \in \mathbb{N}$, corresponding to $N = 2^{L+1} - 1$. Then, for every $\varepsilon \in (0, 2^{-1/3}]$ and $s \leq 2N/e$, provided

$$M \gtrsim sN, \quad m \gtrsim \log M[s^2 \log(N/s) + s \log(s/\varepsilon)]$$

and chosen the upper bound $\boldsymbol{\nu}^N$ as

$$\nu_q^N \sim \frac{1}{q}, \quad \forall q \in \mathbb{N},$$

the CORSING $\mathcal{HS}$ *solution to the homogeneous 1D Poisson problem fulfills*

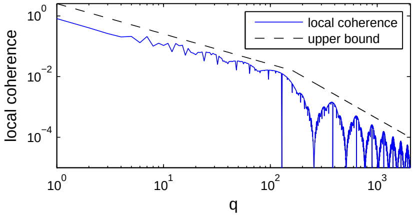
$$\mathbb{E}[|\mathcal{T}_{\mathcal{K}}\widehat{u} - u|_{H^1}] \leq 5|u^s - u|_{H^1} + 2\mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $|u|_{H^1} \leq \mathcal{K}$.

Sketch of the proof

For the 1D Poisson problem we have the following bound

$$\mu_q^N \lesssim \min\left\{\frac{N}{q^2}, \frac{1}{q}\right\}.$$



Then, we have

$$\sum_{q>M} \mu_q^N \lesssim N \sum_{q>M} \frac{1}{q^2} \sim \frac{N}{M}, \text{ required to be } \lesssim \frac{1}{s}.$$

Moreover, choosing $\nu_q^N \sim 1/q$ yields

$$\|\boldsymbol{\nu}^{N,M}\|_1 \sim \sum_{q=1}^{M} \frac{1}{q} \sim \log M.$$

$\square$

Application to 1D ADR problems

Consider the problem

$$\text{find } u \in H_0^1(\Omega): \ (u', v') + b(u', v) + \rho(u, v) = (f, v), \ \forall v \in H_0^1(\Omega) \quad \text{(ADR)}$$

with $b, \rho \in \mathbb{R}, \rho > 0$ and $\Omega = (0, 1)$. Let $H_0^1(\Omega)$ be endowed with $|\cdot|_{H^1(\Omega)}$.

Proposition (CORSING $\mathcal{HS}$ for 1D ADR)

Fix $N \in \mathbb{N}$. Then, for every $\varepsilon \in (0, 2^{-1/3}]$ and $s \leq 2N/e$, provided that

$$M \gtrsim sN, \quad |b|/M \lesssim 1, \quad |\rho|/M^2 \lesssim 1,$$

$$m \gtrsim (\log M + |b|^2 + |\rho|^2)[s^2 \log(N/s) + s \log(s/\varepsilon)],$$

and chosen the upper bound $\boldsymbol{\nu}^N$ such that

$$\nu_q^N \sim \frac{1}{q} + \frac{|b|^2}{q^3} + \frac{|\rho|^2}{q^5}, \quad \forall q \in \mathbb{N},$$

the CORSING $\mathcal{HS}$ *solution to* (ADR) *fulfills*

$$\mathbb{E}[|\mathcal{T}_{\mathcal{K}}\widehat{u} - u|_{H^1(\Omega)}] \lesssim (1 + |b| + |\rho|)|u^s - u|_{H^1(\Omega)} + \mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $|u|_{H^1(\Omega)} \leq \mathcal{K}$.

Application to the 1D diffusion equation

Let $\Omega = (0,1)$ and consider the problem

$$\text{find } u \in H_0^1(\Omega): \quad (\eta u', v') = (f, v), \quad \forall v \in H_0^1(\Omega). \qquad \text{(DIF)}$$

Proposition

Let $\eta \in L^\infty(\Omega)$ be such that

- *there exists $\eta_{\min} > 0$ so that $\eta(x) \geq \eta_{\min}$, for almost every $x \in \Omega$;*
- *there exists a finite set $\mathcal{P} \subseteq \overline{\Omega}$ such that $\eta \in \mathcal{C}^2(\Omega \setminus \mathcal{P})$;*
- $\sup_{x \in \Omega \setminus \mathcal{P}} |\eta^{(k)}(x)| < \infty$, *for $k = 1, 2$.*

Fix $L \in \mathbb{N}$ and put $N = 2^{L+1} - 1$. Then, provided

$$\nu_q^N \sim 1/q, \quad \forall q \in \mathbb{N},$$

and

$$M \gtrsim sN, \quad m \gtrsim \log M [s^2 \log(N/s) + s \log(s/\varepsilon)],$$

the CORSING $\mathcal{HS}$ *solution $\widehat{u}$ to* (DIF) *fulfills*

$$\mathbb{E}[|\mathcal{T}_\mathcal{K} \widehat{u} - u|_{H^1(\Omega)}] \leq \left(1 + \frac{4\|\eta\|_{L^\infty}}{\eta_{\min}}\right) |u^s - u|_{H^1(\Omega)} + 2\mathcal{K}\varepsilon,$$

for every $\mathcal{K} > 0$ such that $|u|_{H^1(\Omega)} \leq \mathcal{K}$.

A RIP theorem for CORSING

(with **S. Dirksen**, **H.C. Jung**, **H. Rauhut**, *RWTH Aachen*)

Theorem (RIP for CORSING)

Let $s, N \in \mathbb{N}$, *with* $s < N$, *and* $\widehat{\delta} \in (0,1)$. *Suppose the truncation condition*

$$\sum_{q>M} \mu_q^N \leq \frac{\alpha^2 \widehat{\delta}}{s}.$$

to be fulfilled. Then, provided $\delta \in \left(1 - (1-\widehat{\delta})\frac{\alpha^2}{\beta^2}, 1\right)$, *and*

$$m \gtrsim \delta^{-2} \|\boldsymbol{\nu}^{N,M}\|_1 s \log^3(s) \log(N),$$

it holds

$$\mathbb{P}\{\beta^{-1}\mathbf{DA} \in RIP(s,\delta)\} \geq 1 - N^{-\log^3(s)},.$$

where β *is the continuity constant of* $a(\cdot,\cdot)$.

CORSING computes the best s-term approximation to u in $\mathcal{O}(smN)$ flops.

Further results

- The previous results hold in the case of **nonorthogonal** trial and test functions. Indeed, they suffice to be **Riesz bases**, i.e.,

$$\|\sum_{j\in\mathbb{N}} u_j\psi_j\|_U \sim \|\mathbf{u}\|_2, \quad \forall \mathbf{u}\in U^N.$$

- We checked the theoretical hypotheses on the local a-coherence for the **2D and 3D ADR equations** *numerically*.

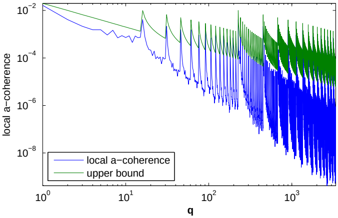


Figure: The plot shows that

$$\boldsymbol{\nu}^N_{\mathbf{q}} \sim \frac{1}{q_1 q_2 q_3}$$

is a local a-coherence upper bound for the 3D Poisson problem (CORSING $\mathcal{QS}$).

Wrap up: main results

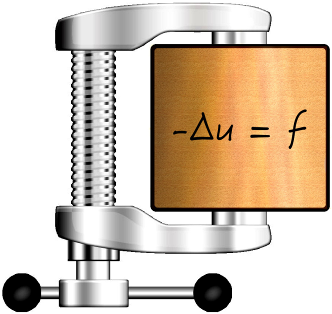
✓ CS can be successfully applied to solve PDEs, such as 1D, 2D, and 3D ADR problems, or the 2D Stokes problem;

✓ CORSING can considerably reduce the computational cost associated with a full-PG discretization;

✓ the local a-coherence is crucial to understand the behavior of the method theoretically;

Future directions

- Speed-up the recovery phase (get rid of the "N" in the cost $\mathcal{O}(smN)$);
- Investigate other trial/test combinations: e.g., **biorthogonal wavelets**, instead of hierarchical basis (**ongoing**);
- 2D and 3D theory (**ongoing**);
- apply CORSING to more challenging benchmarks, such as Navier-Stokes, or nonlocal problems;
- adapt the CORSING technique to the case of parametric PDEs.



Thank you for your attention!

...questions?