Abelian surfaces with everywhere good reduction

Lecture 1: Hilbert modular forms

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June 16, 2016

PIMS Summer School in Explicit Methods for Abelian Varieties

Introduction

In these notes, we describe methods for computing abelian surfaces with everywhere good reduction.

Two essential ingredients to our approach:

1. Eichler-Shimura conjecture;
2. Explicit equations for Hilbert modular surfaces.

Hilbert modular forms

Basic notations:

1. F: totally real field of narrow class number one and degree d;
2. $\mathcal{O}_F$: ring of integers of F;
3. $\mathfrak{d}_F$ the different of F.
4. For each $i = 1, \ldots, d$, let $a \mapsto a^{(i)}$ denote the i-th embedding of F into $\mathbf{R}$, so that we have an identification $F \otimes \mathbf{R} \simeq \mathbf{R}^d$.
5. F_+: set of totally positive elements in F, i.e. the inverse image of $(\mathbf{R}_+)^d$, and $\mathcal{O}_{F,+} = F_+ \cap \mathcal{O}_F$.
6. δ is a totally positive generator of $\mathfrak{d}_F$.

Basic definitions and properties

Let $\mathfrak{H}$ be the Poincaré upper half plane. The Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathfrak{H}^d$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, \ldots, z_d) = \left(\frac{a^{(i)} z_i + b^{(i)}}{c^{(i)} z_i + d^{(i)}} \right)_{i=1,\ldots,d}.$$

Let $k \geq 2$ be an even integer. The action above induces an action of the Hilbert modular group on the set of functions $f : \mathfrak{H}^d \to \mathbb{C}$ by

$$(f|_k\gamma)(z) = \left(\prod_{i=1}^{d} (c^{(i)} z_i + d^{(i)}) \right)^{-k} f(\gamma z),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F).$$

Let $\mathfrak{N}$ be an integral ideal, and set

$$\Gamma_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) : c \in \mathfrak{N} \right\}.$$

Basic definitions and properties

Definition

A Hilbert modular form of weight k and level $\mathfrak{N}$ is a holomorphic function $f : \mathfrak{H}^d \to \mathbb{C}$ such that

$$f|_k\gamma = f \text{ for all } \gamma \in \Gamma_0(\mathfrak{N}).$$

Equivalently, this means that

$$f(\gamma z) = \left(\prod_{i=1}^{d}(c^{(i)}z_i + d^{(i)})\right)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N}).$$

We denote by $M_k(\mathfrak{N})$ the space of all Hilbert modular forms of weight k and level $\mathfrak{N}$.

Hilbert modular forms

Basic definitions and properties

Let $f \in M_k(\mathfrak{N})$. Then f is invariant under the matrices $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ for $\mu \in \mathcal{O}_F$, which act as $z \mapsto z + \mu$. So it admits a q-expansion

$$f(z) = \sum_{\mu \in \mathcal{O}_F} a_\mu e^{2\pi i \operatorname{Tr}(\frac{\mu z}{\delta})},$$

where $\operatorname{Tr}(\nu z) = \nu^{(1)} z_1 + \cdots + \nu^{(d)} z_d$, for $\nu \in F$.

Lemma (Goetzky-Koecher's principle)

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. Then f admits a q-expansion of the form

$$f(z) = a_0 + \sum_{\mu \in \mathcal{O}_{F,+}} a_\mu e^{2\pi i \operatorname{Tr}(\frac{\mu z}{\delta})}.$$

In particular, f is holomorphic (at the cusps).

Basic definitions and properties

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. For every $\gamma \in \mathrm{SL}_2(F)$, we can write the q-expansion

$$(f|_k\gamma)(z) = a_0(f|_k\gamma) + \sum_{\mu\in\mathcal{O}_{F,+}} a_\mu(f|_k\gamma)e^{2\pi i\,\mathrm{Tr}(\frac{\mu z}{\delta})}.$$

This allows us to make the following definition.

Definition

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$. We say that f is a cusp form if $a_0(f|_k\gamma) = 0$ for all $\gamma \in \mathrm{SL}_2(F)$.

Hilbert modular forms

Basic definitions and properties

We denote by $S_k(\mathfrak{N})$ the space of all cusp forms of weight k and level $\mathfrak{N}$. Clearly $S_k(\mathfrak{N}) \subseteq M_k(\mathfrak{N})$.

Theorem

The spaces $S_k(\mathfrak{N})$ and $M_k(\mathfrak{N})$ are finite dimensional complex vector spaces.

Proof.

See Freitag's book. $\square$

Hilbert modular forms

Basic definitions and properties

Let $d\mu := \frac{dx_1 dy_1}{y_1^2} \cdots \frac{dx_d dy_d}{y_d^2}$ on $\mathfrak{H}^d$. One can show that μ is an $\mathrm{SL}_2(\mathbf{R})^d$-invariant measure on $\mathfrak{H}^d$.

For $f \in M_k(\mathfrak{N})$ and $g \in S_k(\mathfrak{N})$, one can show the following integral

$$\int_{\Gamma_0(\mathfrak{N})\backslash\mathfrak{H}^d} f(z)\overline{g(z)}(y_1 \cdots y_d)^k d\mu.$$

converges.

Definition

Let $f, g \in S_k(\mathfrak{N})$. We define the Petersson inner product of f and g by

$$\langle f, g\rangle := \int_{\Gamma_0(\mathfrak{N})\backslash\mathfrak{H}^d} f(z)\overline{g(z)}(y_1 \cdots y_d)^k d\mu.$$

Basic definitions and properties

Let $f \in S_k(\mathfrak{N})$ be a cusp form. Since f is invariant under the action of the matrices $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ for $\epsilon \in \mathcal{O}_F^\times$ in $\mathrm{SL}_2(\mathcal{O}_F)$, which act as $z \mapsto \epsilon^2 z$, we have

$$a_{\epsilon^2\mu} = a_\mu \text{ for all } \mu \in \mathcal{O}_{F,+} \text{ and } \epsilon \in \mathcal{O}_F^\times.$$

So, for every ideal $\mathfrak{m} \subseteq \mathcal{O}_F$, the quantity $a_{\mathfrak{m}}(f) = a_\mu$, where μ is a totally positive generator of $\mathfrak{m}$, is well-defined and depends only on $\mathfrak{m}$.

Hilbert modular forms

Basic definitions and properties

Definition

Let f be a cusp form of weight k and level $\mathfrak{N}$, and write its q-expansion

$$f(z) = \sum_{\mu \in \mathcal{O}_{F,+}} a_\mu e^{2\pi i \operatorname{Tr}(\frac{\mu z}{\delta})}.$$

For every integral ideal $\mathfrak{m}$, we define the Fourier coefficient of f at $\mathfrak{m}$ by

$$a_{\mathfrak{m}}(f) := a_\mu,$$

where $\mathfrak{m} = (\mu)$.

Hilbert modular forms

Hecke operators

Let f be a Hilbert modular form of weight k and level $\mathfrak{N}$.

For every prime $\mathfrak{p} \nmid \mathfrak{N}$, we define the function $T_{\mathfrak{p}} f : \mathfrak{H}^d \to \mathbb{C}$ as follows.

First, write $\mathfrak{p} = (\pi)$ where π is a totally positive, and then set

$$(T_{\mathfrak{p}} f)(z) := (f|_k \gamma_\infty)(z) + \sum_{a \in \mathcal{O}_F/\mathfrak{p}} (f|_k \gamma_a)(z),$$

where

$$\gamma_\infty := \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}, \ \gamma_a := \begin{pmatrix} 1 & a \\ 0 & \pi \end{pmatrix} \text{ for } a \in \mathcal{O}_F/\mathfrak{p}.$$

Hecke operators

Lemma

Let $\mathfrak{p} \nmid \mathfrak{N}$ be a prime. Then, the map

$$\begin{aligned} T_{\mathfrak{p}} : M_k(\mathfrak{N}) &\to M_k(\mathfrak{N}) \\ f &\mapsto T_{\mathfrak{p}} f \end{aligned}$$

is a linear operator which preserves $S_k(\mathfrak{N})$. We call $T_{\mathfrak{p}}$ the Hecke operator at $\mathfrak{p}$.

Proof.

Exercise. $\square$

Hilbert modular forms

Hecke operators

- Definition of Hecke operators extends (multiplicatively) to all integral ideals including those dividing the level $\mathfrak{N}$.
- From now on, if $\mathfrak{m}$ is an integral ideal, we let $T_{\mathfrak{m}}$ be the Hecke operator at $\mathfrak{m}$.
- The Hecke operators enjoy many beautiful and striking properties.

Theorem

Let $f, g \in S_k(\mathfrak{N})$ be cusp forms and $\mathfrak{m} \nmid \mathfrak{N}$. Then, we have

$$\langle T_{\mathfrak{m}} f, g \rangle = \langle f, T_{\mathfrak{m}} g \rangle.$$

Proof.

There is a proof of this result in the general setting in Shimura's article. $\square$

Hecke operators

Definition

The Hecke algebra of weight k and level $\mathfrak{N}$ acting on $S_k(\mathfrak{N})$ is the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_k(\mathfrak{N}))$ generated by the $T_{\mathfrak{m}}$ for all integral ideals $\mathfrak{m} \nmid \mathfrak{N}$. We denote it by $\mathbf{T}_k(\mathfrak{N})$.

Theorem (Shimura)

The Hecke algebra $\mathbf{T}_k(\mathfrak{N})$ is a finitely generated $\mathbf{Z}$-algebra which admits a basis of common eigenvectors for the action of $\mathbf{T}_k(\mathfrak{N})$ on $S_k(\mathfrak{N})$.

Proof.

We refer to Shimura's article. $\square$

Hecke operators

Definition

Let f be a cups form of weight k and level $\mathfrak{N}$. We say that f is an eigenform if f is a common eigenvector for the action of $\mathbf{T}_k(\mathfrak{N})$ on $S_k(\mathfrak{N})$. If in addition $a_{(1)}(f) = 1$, then we say that f is normalized.

Hilbert modular forms

Hecke operators

One of the most striking features of Hilbert modular forms is the deep connection between the eigenvalues of Hecke operators and the Fourier coefficients of eigenforms.

Theorem (Shimura)

Let $f \in S_k(\mathfrak{N})$ be a normalized eigenform. Then the followings are true.

1. *For every integral ideal $\mathfrak{m} \nmid \mathfrak{N}$, $a_{\mathfrak{m}}(f)$ is an algebraic integer such that*

$$T_{\mathfrak{m}} f = a_{\mathfrak{m}}(f) f.$$

2. *The field $K_f := \mathbf{Q}(a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F)$ is a number field, i.e. is a finite extension of $\mathbf{Q}$, which is totally real. We let $\mathcal{O}_f := \mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F]$ be the ring of integers of K_f.*

Hecke operators

Remark

We observe that Theorem 12 (b) is true because F has narrow class number one, and we only consider forms with trivial characters. In general, K_f will be a CM field. See Shimura's article for more details on this.

Hilbert modular forms

Old and new subspaces

For every integral ideal $\mathfrak{M}$ such that $\mathfrak{M} \mid \mathfrak{N}$, and for every divisor $\mathfrak{D}$ of $\mathfrak{N}\mathfrak{M}^{-1}$, let u be a totally positive generator of $\mathfrak{D}$.

Not hard to see that the map

$$\begin{aligned} \iota_{\mathfrak{D}} : S_k(\mathfrak{M}) &\to S_k(\mathfrak{N}) \\ f &\mapsto f_u, \end{aligned}$$

where $f_u(z) := f(uz)$, is independent of the choice of u and is an injection. We let

$$S_k(\mathfrak{N})^{\mathrm{old}} := \sum_{\substack{\mathfrak{M} \mid \mathfrak{N} \\ \mathfrak{D} \mid \mathfrak{N}\mathfrak{M}^{-1}}} \iota_{\mathfrak{D}}(S_k(\mathfrak{M}));$$

$$S_k(\mathfrak{N})^{\mathrm{new}} := \left(S_k(\mathfrak{N})^{\mathrm{old}}\right)^{\perp}.$$

Hilbert modular forms

Old and new subspaces

Call $S_k(\mathfrak{N})^{\mathrm{old}}$ (resp. $S_k(\mathfrak{N})^{\mathrm{new}}$) the old subspace (resp. new subspace) of $S_k(\mathfrak{N})$.

$S_k(\mathfrak{N})^{\mathrm{old}}$ and $S_k(\mathfrak{N})^{\mathrm{new}}$ are both stable under the Hecke action.

Definition

Let $f \in S_k(\mathfrak{N})$ be a normalized eigenform. We say that f is a newform if $f \in S_k(\mathfrak{N})^{\mathrm{new}}$.

Definition

Let f be a cusp form. We define the L-series of f by

$$L(f,s) := \sum_{\mathfrak{m} \subseteq \mathcal{O}_F} \frac{a_{\mathfrak{m}}(f)}{\mathrm{N}\mathfrak{m}^s}.$$

Old and new subspaces

Theorem (Shimura)

Let f be a cusp form. Then $L(f,s)$ is an entire function, i.e. is holomorphic on the whole complex plane. If f is a newform, then f admits an Euler product.

Proof.

The proof of this is essentially an adaption of what is known for $F = \mathbf{Q}$. So we refer to work of Shimura. □

Old and new subspaces

Theorem (Multipicity one)

Let f, g be two normalized eigenforms such that

$$a_{\mathfrak{m}}(f) = a_{\mathfrak{m}}(g) \text{ for all } \mathfrak{m} \nmid \mathfrak{N}.$$

Then, we have $f = g$.

Proof.

This follows from the relation between Hecke eigenvalues and Fourier coefficients, and the fact that f is determined by its q-expansion. $\square$

Old and new subspaces

We mentioned earlier that the definition of the Hecke operators can be extended to all integral ideals. With this in mind, we have the following result due to Miyake.

Theorem (Strong multiplicity one)

Let f be a newform. Then, we have

$$T_{\mathfrak{m}} f = a_{\mathfrak{m}}(f) f \text{ for all } \mathfrak{m} \subseteq \mathcal{O}_F.$$

Proof.

See Miyake's article. □

Old and new subspaces

Theorem 17 and Theorem 18 are both very powerful and extremely useful as they imply that every newform is uniquely determined by its Hecke eigenvalues or Fourier coefficients.

Theorem

Let $f \in S_k(\mathfrak{N})^{\mathrm{new}}$ be a newform, and let K_f be its field of Fourier coefficients. For each embedding $\tau : K_f \hookrightarrow \overline{\mathbf{Q}}$, there exists a newform $f^\tau \in S_k(\mathfrak{N})$ defined by

$$a_{\mathfrak{m}}(f^\tau) := \tau(a_{\mathfrak{m}}(f)), \text{ for all } \mathfrak{m} \subseteq \mathcal{O}_F.$$

The set $\{f^\tau : \tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})\}$ is called the Hecke orbit of f. We denote it by $[f]$.

Proof.

See Shimura's article. □

A quick summary

- There are several useful references in your notes: both theoretical and computational.
- Explicit computations use the Eichler-Jacquet-Langlands-Shimizu correspondence between Hilbert modular forms and quaternionic modular forms.
- Algorithms have been implemented in the Hilbert Modular Forms Package in `Magma`.

Eichler-Shimura and GL_2-type Modularity Conjectures

The Eichler-Shimura conjecture

Definition

Let A be an abelian variety defined over F. We say that A is of GL_2-type if there exists a number field K such that $\dim(A) = [K : \mathbf{Q}]$ and $\mathrm{End}_F(A) \otimes \mathbf{Q} \simeq K$.

If A/F is an abelian variety of GL_2-type with $\mathrm{End}_F(A) \otimes \mathbf{Q} \simeq K$ and $g = [K : \mathbf{Q}]$, then there exists an integral ideal $\mathfrak{N}$ such that the conductor of A is of the form

$$\mathrm{cond}(A) = \mathfrak{N}^g.$$

Eichler-Shimura and GL_2-type Modularity Conjectures

The Eichler-Shimura conjecture

The following statement relates Hilbert modular forms to abelian varieties of GL_2-types.

Conjecture (Eichler-Shimura)

Let F be a totally real number field of narrow class number one and $\mathfrak{N}$ an integral ideal of F. Let $f \in S_2(\mathfrak{N})$ be a newform. Then, there exists an abelian variety A_f/F of dimension $[K_f : \mathbf{Q}]$ with good reduction outside of $\mathfrak{N}$ and with $\mathcal{O}_f \hookrightarrow \mathrm{End}_F(A_f)$, such that

$$L(A_f, s) = \prod_{\tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})} L(f^\tau, s) = \prod_{g \in [f]} L(g, s).$$

Eichler-Shimura and GL_2-type Modularity Conjectures

The Eichler-Shimura conjecture

For $F = \mathbf{Q}$:

- Conjecture is a theorem, due to Eichler for prime level and Shimura in the general case.
- Converse is also known thanks to Khare-Wintenberger's work on Serre conjecture.
- Can make this construction explicit in many cases: See Cremona's work for elliptic curves, and Gonzalez-Guardia for abelian surfaces.

Eichler-Shimura and GL_2-type Modularity Conjectures

The Eichler-Shimura conjecture

For $[F : \mathbf{Q}] > 1$:

- Known cases of Eichler-Shimura conjecture exploit the cohomology of Shimura curves.
- For instance, the conjecture is known when $[F : \mathbf{Q}]$ is odd, or when $\mathfrak{N}$ is exactly divisible by a prime $\mathfrak{p}$ of $\mathcal{O}_F$.
- Simplest unknown case is when f is a newform of level (1) and weight 2 over a real quadratic field. In that case, the conjecture predicts that the associated abelian variety A_f has everywhere good reduction.

Eichler-Shimura and GL_2-type Modularity Conjectures

The GL_2-type modularity conjecture

The following is the converse statement to the Eichler-Shimura Conjecture.

Conjecture (GL_2-type Modularity)

Let F be a totally real number field of narrow class number one, and A an abelian variety of GL_2-type over F. Let $K := \mathrm{End}_F(A) \otimes \mathbf{Q}$ and write $\mathrm{cond}(A) = \mathfrak{N}^g$, with $g = [K : \mathbf{Q}]$. Then, there exists a newform $f \in S_2(\mathfrak{N})^{\mathrm{new}}$ such that

$$L(A, s) = \prod_{\tau \in \mathrm{Hom}(K_f, \overline{\mathbf{Q}})} L(f^\tau, s).$$

Eichler-Shimura and GL_2-type Modularity Conjectures

Table: Modularity of abelian varieties of GL_2-type

Hilbert newforms f/F with Hecke eigenvalues $\mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] \subseteq \mathcal{O}_K$ (weight 2, level $\mathfrak{N}$)	(Isogeny classes of) Abelian varieties A/F $\dim(A) = g$, $\mathrm{cond}(A) = \mathfrak{N}^g$ $\mathrm{End}_F(A) \otimes \mathbf{Q} = K$

—— Eichler-Shimura conjecture

—— GL_2-type Modularity conjecture

Eichler-Shimura and GL_2-type Modularity Conjectures

Example 1:

Let $F = \mathbf{Q}(\sqrt{5})$, $w = \frac{1+\sqrt{5}}{2}$ and $\mathfrak{N} = (5 + 2w)$. Then, $\mathfrak{N}$ is a prime of norm 31.

This is the smallest norm for which $\dim S_2(\mathfrak{N}) \neq 0$.

In that case, we have $\dim S_2(\mathfrak{N}) = 1$. So, there is a newform $f \in S_2(\mathfrak{N})$ with Fourier coefficients in $\mathbf{Z}$.

We proved that f corresponds to the elliptic curve

$$E:\ y^2 + xy + wy = x^3 - (1 + w)x^2,$$

In other words, we proved that E is modular and that

$$L(E, s) = L(f, s).$$

Eichler-Shimura and GL_2-type Modularity Conjectures

Example 1 (cont'd):

An alternate description of E:

Let D be the quaternion algebra over F which is ramified at $\mathfrak{N}$ and exactly one of the two real places.

Let $\mathcal{O}_D$ be a maximal order in D, and consider the Shimura curve $X_0^D(\mathfrak{N})$ obtained from $(\mathcal{O}_D)_1$, the units of norm 1 in $\mathcal{O}_D$.

Then $X_0^D(\mathfrak{N})$ is a curve of genus 1. Hence $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ is an elliptic curve.

By the Jacquet-Langlands correspondence, $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ and E are isogenous.

Eichler-Shimura and GL_2-type Modularity Conjectures

Example 2:

Again, we let $F = \mathbf{Q}(\sqrt{5})$, $w = \frac{1+\sqrt{5}}{2}$, and set $\mathfrak{N} = (7+3w)$.

Here $\mathfrak{N}$ is a prime of norm 61. This is the smallest norm such that $\dim S_2(\mathfrak{N}) = 2$.

There is a newform $f \in S_2(\mathfrak{N})$ such that $\mathcal{O}_f = \mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] = \mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

We showed that f corresponds to the Jacobian of the hyperelliptic curve $C:\ y^2 + g(x)y = f(x)$ given by

$$f := -wx^4 + (w-1)x^3 + (5w+4)x^2 + (6w+4)x + 2w + 1$$
$$g := x^3 + (w-1)x^2 + wx + 1.$$

(The curve C comes from the Brumer family of curves whose Jacobians have RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.) $\mathrm{Jac}(C)$ has real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

Eichler-Shimura and GL_2-type Modularity Conjectures

Example 2 (cont'd):

Alternatively:

Let D be the quaternion algebra over F which is ramified at $\mathfrak{N}$ and exactly one of the two real places.

Let $\mathcal{O}_D$ be a maximal order in D, and consider the Shimura curve $X_0^D(\mathfrak{N})$ obtained from $(\mathcal{O}_D)_1$, the units of norm 1 in $\mathcal{O}_D$.

Then $X_0^D(\mathfrak{N})$ is a curve of genus 2. Hence $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ is an abelian surface.

By the Jacquet-Langlands correspondence, $\mathrm{Jac}(X_0^D(\mathfrak{N}))$ has RM by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$ and is isogenous to $\mathrm{Jac}(C)$.

However, we do not know whether the curves $X_0^D(\mathfrak{N})$ and C themselves are isogenous.

Eichler-Shimura and GL_2-type Modularity Conjectures

Although many cases of the Eichler-Shimura and GL_2-type modularity conjectures are known, they still remain largely open.

We conclude this section with one of the most up-to-date results in this area.

Theorem (Freitas-Le Hung-Siksek)

Let F be a real quadratic field, and E an elliptic curve defined over F. Then E is modular.

Proof.

See Freitas-Le Hung-Siksek's work. □