# Abelian surfaces with everywhere good reduction

## Lecture 2: Abelian surfaces with everywhere good reduction

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June 17, 2016

PIMS Summer School in Explicit Methods for Abelian Varieties
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## Lecture 2: The good, the bad, and the ugly

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# Eichler-Shimura and $\mathrm{GL}_2$-type Modularity Conjectures

**Table:** Modularity of abelian varieties of $\mathrm{GL}_2$-type

| | |
|---|---|
| Hilbert newforms $f/F$<br>with Hecke eigenvalues<br>$\mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] \subseteq \mathcal{O}_K$<br>(weight 2, level $\mathfrak{N}$) | (Isogeny classes of)<br>Abelian varieties $A/F$<br>$\dim(A) = g$, $\mathrm{cond}(A) = \mathfrak{N}^g$<br>$\mathrm{End}_F(A) \otimes \mathbf{Q} = K$ |

—— Eichler-Shimura conjecture

—— $\mathrm{GL}_2$-type Modularity conjecture

# Elliptic curves with everywhere good reduction

**Historical note**

First example of an elliptic curve with everywhere good reduction was discovered by Tate.

Namely, he showed that the curve $E$ defined by

$$E: \ y^2 + xy + \epsilon^2 y = x^3,$$

where $\epsilon = \frac{5+\sqrt{29}}{2}$ is the fundamental unit in $F = \mathbf{Q}(\sqrt{29})$, has discriminant $\Delta = -\epsilon^{10}$.

This curve is extensively studied by Serre.

Shimura discusses similar examples, and proposes a general strategy for constructing higher dimension analogues.

From the early 70s to the late 90s, a great deal of work went into finding more examples of elliptic curves with everywhere good reduction defined over quadratic fields.

# Elliptic curves with everywhere good reduction

## Elkies-Donnelly search method

Let $F$ be a real quadratic field of narrow class number one, and let $\epsilon$ be the fundamental unit of $\mathcal{O}_F$.

Let $E$ be an elliptic curve with everywhere good reduction defined over $F$.

Suppose that $E$ is given by the extended Weierstrass equation

$$E:\ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with coefficients $a_i \in \mathcal{O}_F$ and discriminant $\Delta$.

Without loss of generality, we can assume that $\Delta = \pm\epsilon^m$ with $0 \leq m < 12$.

A refinement of an argument of Stroeker by Elkies shows that we can in fact assume that $m \in \{1, 2, 3, 4, 5\}$.

# Elliptic curves with everywhere good reduction

## Elkies-Donnelly search method

Recall that

$$c_4^3 - c_6^2 = 1728\Delta,$$

and given $m$ (and hence $|\Delta|$), set

$$H = \frac{(v_0(c_4)|v_0(\Delta)|^{-1/3})^2 + (v_1(c_4)|v_1(\Delta)|^{-1/3})^2}{\sqrt{D}}$$
$$= \frac{\left(v_0(c_4)|v_1(\Delta)|^{1/3}\right)^2 + \left(v_1(c_4)|v_0(\Delta)|^{1/3}\right)^2}{\sqrt{D}},$$

where $v_0, v_1 : F \hookrightarrow \mathbf{R}$ are the real embeddings of $F$.

Then, $H$ becomes a positive definite quadratic form in $c_4$ on $\mathcal{O}_F$. The normalizing factor $\sqrt{D}^{-1}$ ensures that this form has discriminant 1.

# Elliptic curves with everywhere good reduction

## Elkies-Donnelly search method

Let $N > 0$ be a fixed bound, and consider the ellipse

$$\mathscr{R}_N := \{(x, y) \in \mathbf{R}^2 : H(x, y) \leq N\}.$$

There are roughly $\pi N$ elements $c_4$ in the intersection $\mathcal{O}_F \cap \mathscr{R}_N$.

Elkies heuristics show that as $H$ represents elements of $[1, N]$, a positive proportion of $c_4$ will give elliptic curves with unit discriminants.

So, for each $D$, the algorithm finds a reduced basis for the quadratic form $H$, and tries all candidates up to $N$.

Method was refined by Steve Donnelly:

1. Remove the restriction that $\Delta$ is a unit;
2. Extended to all totally real number fields (of narrow class number 1).

This is currently the algorithm used in `Magma` to search for elliptic curves with prescribed conductor of such fields.

# Elliptic curves with everywhere good reduction

## Elkies-Donnelly search method: Example

Let $F = \mathbf{Q}(\sqrt{1997})$, and $w := \frac{1+\sqrt{1997}}{2}$.

Then there are 6 elliptic curves over $F$ with everywhere good reduction.

They are pairwise non-isogenous and determines 3 $\mathrm{Gal}(F/\mathbf{Q})$-conjugacy classes.

The two conjugacy classes are represented by

$$\begin{aligned}
E_1 &: y^2 + wxy = x^3 + (w+1)x^2 + (111w + 5401)x + (2406w + 81112);\\
E_2 &: y^2 + wxy + (w+1)y = x^3 - x^2 + (9370w - 208733)x\\
&\qquad\qquad + (2697263w - 61535794);\\
E_3 &: y^2 + (w+1)xy + (w+1)y = x^3 - wx^2 + (19636w + 434383)x\\
&\qquad\qquad + (5730650w + 125261893).
\end{aligned}$$

## Elkies-Donnelly search method: Example (cont'd)

By Freitas-Le Hung-Siksek, these curves are modular.

By a `Magma` computation, we check that there are exactly 6 Hilbert newforms of level (1) and weight 2 over $F$ with integer Hecke eigenvalues.

Therefore, these are the only elliptic curves with everywhere good reduction over $F$.

# Elliptic curves with everywhere good reduction

**Elkies-Donnelly search method: Example (cont'd)**

By Freitas-Le Hung-Siksek, these curves are modular.

By a `Magma` computation, we check that there are exactly 6 Hilbert newforms of level (1) and weight 2 over $F$ with integer Hecke eigenvalues.

Therefore, these are the only elliptic curves with everywhere good reduction over $F$.

**Exercise**

*Show that there are no elliptic curves with everywhere good reduction over* $F = \mathbf{Q}(\sqrt{2017})$*?*

# Abelian surfaces with everywhere good reduction

## Historical note

The only examples of abelian surfaces with everywhere good reduction in the literature before my work with A. Kumar were of the following kinds:

1. Surfaces with complex multiplication (D.-Donnelly);
2. $\mathbf{Q}$-surfaces (Casselman, Shimura);
3. Products of elliptic curves.

Except for (3), none of these examples is given by an explicit equation.

Possible explanations:

1. Not easy to embed such surfaces into projective spaces.
2. Additional complication: A curve can have bad reduction at a given prime while its Jacobian still has good reduction at the same prime.

# Abelian surfaces with everywhere good reduction

**Hilbert modular surfaces**

Let $K$ be a real quadratic field of discriminant $D'$.

The Hilbert modular surface $Y_-(D')$:

1. Is a (compactification of the) coarse moduli space.
2. Parametrizes principally polarized abelian surfaces with real multiplication by the ring of integers $\mathcal{O}_K$ of $K$, i.e. pairs $(A, \iota)$, where $\iota : \mathcal{O}_K \to \mathrm{End}_{\overline{\mathbf{Q}}}(A)$ is a homomorphism.

The surfaces $Y_-(D')$ have models over the integers, with good reduction outside $D'$.

# Abelian surfaces with everywhere good reduction

**Hilbert modular surfaces**

Elkies and Kumar compute explicit birational models over $\mathbf{Q}$ for these surfaces for all the fundamental discriminants $D'$ less than 100.

They describe $Y_{-}(D')$ as a double cover of $\mathbf{P}^2$, with equation $z^2 = f(r, s)$, where $r, s$ are parameters on $\mathbf{P}^2$.

They also give the map to $\mathcal{A}_2$, which is birational to $\mathcal{M}_2$, the moduli space of genus 2 curves.

It is given by expressing the Igusa-Clebsch invariants of the image point as rational functions of $r$ and $s$.

# Abelian surfaces with everywhere good reduction

## Our approach

Our strategy combines the Eichler-Shimura conjecture with the explicit equations of Elkies-Kumar.

To produce such a surface $A$, we proceed as follows:

1. Find a Hilbert newform $f$ of level $(1)$ and weight $2$ for a real quadratic field $F$ such that $\mathbf{Z}[a_{\mathfrak{m}}(f) : \mathfrak{m} \subseteq \mathcal{O}_F] = \mathcal{O}_{D'}$, the ring of integers for some real quadratic field $K_f$ of discriminant $D'$.
2. Find an $F$-rational point $x$ on the Hilbert modular surface $Y_{-}(D')$.
3. Compute the associated surface $A_x$, and check that $L_{\mathfrak{p}}(A', s)$ matches $L_{\mathfrak{p}}(A_f, s)$ for the first few primes, up to twist.
4. Reduce $A_x$ and compute the correct quadratic twist $A$.
5. Check that the abelian surface $A$ has good reduction everywhere.
6. Prove that the $L$-functions indeed match up, i.e. that $A$ is modular.

**Method 1: Point search on Hilbert modular surfaces**

We illustrate this with the following example.

The smallest discriminant for which we obtain a surface (of $\mathrm{GL}_2$-type) with everywhere good reduction is $D = 53$.

The abelian surface $A_f$ has real multiplication by $\mathbf{Z}[\sqrt{2}]$.

**Notations:**

$$w = \frac{1+\sqrt{D}}{2}.$$

# Abelian surfaces with everywhere good reduction

**Table:** The first few Hecke eigenvalues of a base change newform of level (1) and weight 2 over $\mathbf{Q}(\sqrt{53})$. Here $e = \sqrt{2}$.

| $\mathrm{N}\mathfrak{p}$ | $\mathfrak{p}$ | $a_\mathfrak{p}(f)$ | $s_\mathfrak{p}(f)$ | $t_\mathfrak{p}(f)$ |
|---|---|---|---|---|
| 4 | $2$ | $e+1$ | 2 | 7 |
| 7 | $-w-2$ | $-e-2$ | $-4$ | 16 |
| 7 | $-w+3$ | $-e-2$ | $-4$ | 16 |
| 9 | $3$ | $-3e+1$ | 2 | 1 |
| 11 | $w-2$ | $3e$ | 0 | 4 |
| 11 | $w+1$ | $3e$ | 0 | 4 |
| 13 | $w-1$ | $-2e+1$ | 2 | 19 |
| 13 | $-w$ | $-2e+1$ | 2 | 19 |
| 17 | $-w-5$ | $-3$ | $-6$ | 43 |
| 17 | $w-6$ | $-3$ | $-6$ | 43 |
| 25 | $5$ | $2e+4$ | 8 | 58 |
| 29 | $-w-6$ | $3e-3$ | $-6$ | 49 |
| 29 | $w-7$ | $3e-3$ | $-6$ | 49 |

# Abelian surfaces with everywhere good reduction

**Method 1: Point search on Hilbert modular surfaces**

An equation for the Hilbert modular surface $Y_{-}(8)$ is given in Elkies-Kumar's paper.

As a double-cover of $\mathbf{P}^2_{r,s}$, it is given by

$$z^2 = 2(16rs^2 + 32r^2s - 40rs - s + 16r^3 + 24r^2 + 12r + 2).$$

It is a rational surface (over $\mathbf{Q}$) and therefore the rational points are dense.

In particular, there is an abundance of rational points of small height.

# Abelian surfaces with everywhere good reduction

## Method 1: Point search on Hilbert modular surfaces

The Igusa-Clebsch invariants $(I_2 : I_4 : I_6 : I_{10}) \in \mathbf{P}^2_{(1:2:3:5)}$ are given by

$$\left(-\frac{24B_1}{A_1}, -12A, \frac{96AB_1 - 36A_1B}{A_1}, -4A_1B_2\right),$$

where

$$\begin{aligned} A_1 &= 2rs^2, \\ A &= -(9rs + 4r^2 + 4r + 1)/3, \\ B_1 &= (rs^2(3s + 8r - 2))/3, \\ B &= -(54r^2s + 81rs - 16r^3 - 24r^2 - 12r - 2)/27, \\ B_2 &= r^2. \end{aligned}$$

## Method 1: Point search on Hilbert modular surfaces

Expect to find a point of $Y_{-}(8)$ over $F = \mathbf{Q}(\sqrt{53})$, corresponding to the principally polarized surface $A$ which should match the Hilbert newform $f$.

The $L$-series of a surface $A$ arising from our search is obtained by counting points on the residue fields $\mathbb{F}_{\mathfrak{p}} = \mathcal{O}_F/\mathfrak{p}$ as $\mathfrak{p}$ runs over the set of primes.

On the other hand, the $L$-series of the conjectural surface $A_f$ attached to $f$ can be written as

$$L(A_f, s) = L(f, s)L(f^{\tau}, s) = \prod_{\mathfrak{p}} \frac{1}{Q_{\mathfrak{p}}(\mathrm{N}(\mathfrak{p})^{-s})},$$

where

$$\begin{aligned} Q_{\mathfrak{p}}(T) &:= (T^2 - a_{\mathfrak{p}}(f)T + \mathrm{N}(\mathfrak{p}))(T^2 - a_{\mathfrak{p}}(f)^{\tau}T + \mathrm{N}(\mathfrak{p})) \\ &= T^4 - s_{\mathfrak{p}}(f)T^3 + t_{\mathfrak{p}}(f)T^2 - \mathrm{N}(\mathfrak{p})s_{\mathfrak{p}}(f)T + \mathrm{N}(\mathfrak{p})^2. \end{aligned}$$

We would like the local factors of these two $L$-series to match.

# Abelian surfaces with everywhere good reduction

**Method 1: Point search on Hilbert modular surfaces**

A search of $Y_-(8)$ for all points of height $\leq 200$ using an algorithm of Doyle-Krumm (implemented in Sage) gives the parameters

$$r = -\frac{24 + 10w}{11^2}, \ s = \frac{136 - 24w}{11^2},$$

and the Igusa-Clebsch invariants

$$\begin{aligned} I_2 &= 208 + 88w, \\ I_4 &= -1660 - 588w, \\ I_6 &= -428792 - 135456w, \\ I_{10} &= 643072 + 204800w. \end{aligned}$$

**Method 1: Point search on Hilbert modular surfaces**

By using Mestre's algorithm, which is implemented in `Magma`, we obtain a curve with the above invariants.

We reduce this curve using the algorithm of Bouyer-Streng, implemented in `Sage`, to get the curve

$$\begin{aligned} C' : y^2 = (-6w+25)x^6 &+ (-60w+246)x^5 + (-242w+1017)x^4 \\ &+ (-534w+2160)x^3 + (-626w+2688)x^2 \\ &+ (-440w+1724)x - 127w + 567. \end{aligned}$$

# Abelian surfaces with everywhere good reduction

## Method 1: Point search on Hilbert modular surfaces

**Theorem**

*Let* $C: y^2 + Q(x)y = P(x)$ *be the curve over* $F = \mathbf{Q}(\sqrt{53})$, *where*

$$P := -4x^6 + (w-17)x^5 + (12w-27)x^4 + (5w-122)x^3 + (45w-25)x^2 + (-9w-137)x + 14w + 9,$$
$$Q := wx^3 + wx^2 + w + 1.$$

*Then*

**(a)** *The discriminant of this curve is* $\Delta_C = -\epsilon^7$. *Thus* $C$ *has everywhere good reduction.*

**(b)** *The surface* $A := \mathrm{Jac}(C)$ *has real multiplication by* $\mathbf{Z}[\sqrt{2}]$. *It is modular and corresponds to the unique Hecke constituent* $[f]$ *in* $S_2(1)$.

## Method 2: Splitting abelian varieties: Example

We can use this method when the Hilbert newform $f$ is a base change, i.e. when the Hecke eigenvalues of $f$ satisfy

$$a_{\mathfrak{p}}(f) = a_{\sigma(\mathfrak{p})}(f) \text{ for all } \mathfrak{p},$$

where $\mathrm{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$.

In this case, $f$ arises from a newform $g \in S_2(D, (\frac{D}{\cdot}))^{\mathrm{new}}$, whose coefficient field is a quartic $L_g$.

Let $B_g/\mathbf{Q}$ the fourfold associated to $g$.

## Method 2: Splitting abelian varieties: Example

Let $w_D$ be the Atkin-Lehner involution on $S_2(D, (\frac{D}{\cdot}))^{\mathrm{new}}$.

This induces an involution on $B_g$, which we still denote by $w_D$.

Shimura shows the followings:

1. $w_D$ is defined over $F$, and $w_D^\sigma = -w_D$;
2. We have

$$B_g \otimes_{\mathbf{Q}} F \sim (1 + w_D)B_g \times (1 - w_D)B_g \sim A_f \times A_f^\sigma .$$

BUT, this is an algebraic decomposition!

We want an analytic decomposition.

### Method 2: Splitting abelian varieties: Example

**Key facts:**

The Atkin-Lehner involution $w_D$ acts on:

1. $H_1(B_g, \mathbf{Z})$ (described by modular symbols);
2. $S_2(D, (\frac{D}{\cdot}))^{\mathrm{new}} \simeq H^0(B_g, \Omega^1_{B_g/\mathbf{Q}})$.

So we can write the analytic decompositions:

1. $H_1(B_g, \mathbf{Z}) = H_1(B_g, \mathbf{Z})^+ \oplus H_1(B_g, \mathbf{Z})^-$;
2. $H^0(B_g, \Omega^1_{B_g/\mathbf{Q}}) = H^0(B_g, \Omega^1_{B_g/\mathbf{Q}})^+ \oplus H^0(B_g, \Omega^1_{B_g/\mathbf{Q}})^-$.

Integrating (2) against (1) gives Period lattices $\Lambda^+$ and $\Lambda^-$ of $A_f$ and $A_f^\sigma$.

BUT, we also need $A_f$ and $A_f^\sigma$ to be principally polarized. (Can check this using intersection pairing.)

# Abelian surfaces with everywhere good reduction

## Method 2: Splitting abelian varieties: Example

We illustrate this with an example at the discriminant $D = 73$.

The abelian surface $A_f$ has real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$.

A symplectic basis for $H_1(B_g, \mathbf{Z})$ is given by the modular symbols:

$$\begin{aligned}
\gamma_1 &:= 2\{-1/57, 0\} - \{-1/62, 0\} - \{-1/52, 0\} + 2\{-1/29, 0\} + \{-1/18, 0\}, \\
\gamma_2 &= -\{-1/62, 0\} + 2\{-1/41, 0\} - \{-1/52, 0\} + 2\{-1/12, 0\} + 2\{-1/29, 0\} \\
&\quad + \{-1/18, 0\} - \{-1/36, 0\}, \\
\gamma_3 &:= \{-1/57, 0\} - \{-1/41, 0\} - \{-1/18, 0\} + \{-1/36, 0\}, \\
\gamma_4 &:= -\{-1/57, 0\} + \{-1/62, 0\} - \{-1/41, 0\} + \{-1/52, 0\} - \{-1/12, 0\} \\
&\quad - 2\{-1/29, 0\} - \{-1/18, 0\} + \{-1/24, 0\}, \\
\gamma_1' &:= \{-1/57, 0\} + \{-1/41, 0\} + \{-1/18, 0\} - \{-1/36, 0\}, \\
\gamma_2' &:= \{-1/57, 0\} + \{-1/62, 0\} + \{-1/41, 0\} - \{-1/52, 0\} - \{-1/12, 0\} \\
&\quad - \{-1/18, 0\} + \{-1/24, 0\}, \\
\gamma_3' &:= -\{-1/62, 0\} + \{-1/52, 0\} + \{-1/18, 0\}, \\
\gamma_4' &:= \{-1/62, 0\} - \{-1/52, 0\} - \{-1/18, 0\} + \{-1/36, 0\}.
\end{aligned}$$

## Method 2: Splitting abelian varieties: Example

We can also show that $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4\}$ are integral bases for $H_1(B_g, \mathbf{Z})^+$ and $H_1(B_g, \mathbf{Z})^-$.

Computing the intersection pairing in that basis, we see that:

1. $B_g$ is principally polarized.
2. $H_1(B_g, \mathbf{Z})^+$ and $H_1(B_g, \mathbf{Z})^-$ have the same polarization of type $(2, 2)$. Hence $A_f$ and $A_f^\sigma$ are principally polarized.

Integrating bases of differential forms of $H^0(B_g, \Omega^1_{B_g/\mathbf{Q}})^+$ and $H^0(B_g, \Omega^1_{B_g/\mathbf{Q}})^-$, respectively, against the Darboux bases $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\{\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4\}$, we obtain the Riemann period matrices $\Omega_{A_f}$ and $\Omega_{A_f^\sigma}$.

# Abelian surfaces with everywhere good reduction

## Method 2: Splitting abelian varieties: Example

$$\Omega_{B_g} = \Omega_{A_f} \times \Omega_{A_f^\sigma} = (\Omega_1 \,|\, \Omega_2) \times (\Omega_1^\sigma \,|\, \Omega_2^\sigma);$$

$$\Omega_1 := \begin{pmatrix} 101.34000... - 7.5977...i & -2.6423... - 2.6129...i \\ 23.92200... - 47.37900...i & 11.19300... - 4.6090...i \end{pmatrix}$$

$$\Omega_2 := \begin{pmatrix} 38.70800... - 12.29300...i & -6.9177... + 1.6149...i \\ -62.63000... + 19.89100...i & -4.275400... + 0.99804...i \end{pmatrix}$$

$$\Omega_1^\sigma := \begin{pmatrix} 0.53699... - 3.7425...i & 3.6304... - 3.4371...i \\ 0.86887... - 6.0555...i & -2.2437... + 2.1243...i \end{pmatrix}$$

$$\Omega_2^\sigma := \begin{pmatrix} -1.4059... + 2.3130...i & -1.3867... - 5.5613...i \\ -1.4059... - 2.3130...i & -1.3867... + 5.5613...i \end{pmatrix}$$

This yields the normalized period matrices

$$Z := \begin{pmatrix} -0.50106... + 0.29103...i & 0.43700... - 0.012594...i \\ 0.43700... - 0.012594...i & 0.41383... + 0.18028...i \end{pmatrix}$$

$$Z^\sigma := \begin{pmatrix} -0.22570... + 0.80024...i & 0.54639... - 0.32080...i \\ 0.54639... - 0.32080...i & -0.67931... + 0.47944...i \end{pmatrix}$$

# Abelian surfaces with everywhere good reduction

## Method 2: Splitting abelian varieties: Example

We compute the Igusa-Clebsch invariants $I_2, I_4, I_6$ and $I_{10}$ to 200 decimal digits of precision using $Z$ and $Z^\sigma$, and identify them as elements in $F$.

In the weighted projective space $\mathbf{P}^2_{(1:2:3:5)}$, this gives the point

$$(I_2 : I_4 : I_6 : I_{10}) = \Big(1, \frac{-3080592b + 36303121}{3750827536}, \frac{-72429788520b + 811909152327}{229715681614784}, \frac{680871365928b - 5817295179641}{6731436750404224780408}\Big),$$

where $b = \sqrt{73}$.

**Method 2: Splitting abelian varieties: Example**

By using Mestre's algorithm, which is implemented in `Magma`, we obtain a curve with the above invariants.

We reduce this curve using the algorithm of Bouyer-Streng, implemented in `Sage`, to get the curve

$$C' : y^2 = (4w - 19)x^6 + (12w - 56)x^5 + (12w - 74)x^4 + (16w - 10)x^3 \\ + (-12w - 63)x^2 + (12w + 46)x - 4w - 15.$$

# Abelian surfaces with everywhere good reduction

## Method 2: Splitting abelian varieties: Example

**Theorem**

*Let $C : y^2 + Q(x)y = P(x)$ be the curve over $F = \mathbf{Q}(\sqrt{73})$, where*

$$P := (w-5)x^6 + (3w-14)x^5 + (3w-19)x^4 + (4w-3)x^3 + (-3w-16)x^2 + (3w+11)x + (-w-4);$$

$$Q := x^3 + x + 1.$$

*Then*

**(a)** *The discriminant of this curve is $\Delta_C = -\epsilon^2$. Thus $C$ has everywhere good reduction.*

**(b)** *The surface $A := \mathrm{Jac}(C)$ has real multiplication by $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. It corresponds to the unique Hecke constituent $[f]$ in $S_2(1)$.*

## Method 2: Splitting abelian varieties: Example

**Proof.**

Only the proof of modularity is different from what we did in the previous example. Here the prime 3 is inert in $\mathcal{O}_f = \mathbf{Z}[\frac{1+\sqrt{5}}{2}]$. So we prove that the surface $A$ is modular by combining arguments of Ellenberg and Gee. $\square$