Enriching Bézout's Theorem

Stephen McKean (Georgia Tech)

June 12th, 2019

PIMS Workshop on Arithmetic Topology

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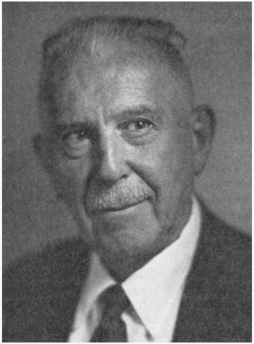

"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry."
– Lefschetz, 1924.

Bézout's Theorem

Theorem

Let k be an algebraically closed field. If $f, g \subset \mathbb{P}^2_k$ are generic algebraic curves of degree c, d, respectively, then

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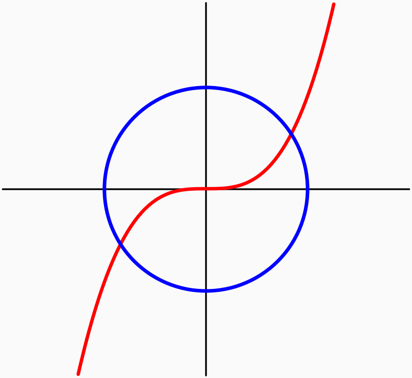

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$\mathbb{A}^1$-enumerative geometry: extra information has geometric meaning.

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- Over $\mathbb{C}$: counts intersection points.
- Over $\mathbb{R}$: equal number of positive/negative crossings.
- Over $\mathbb{F}_q$: counts crossing types mod 2.

Example

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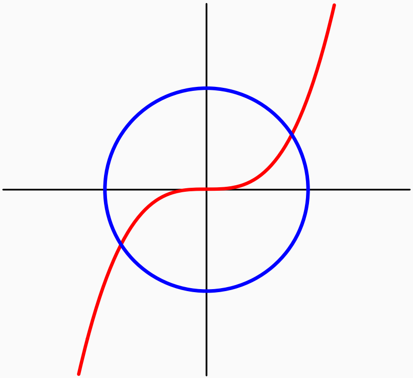

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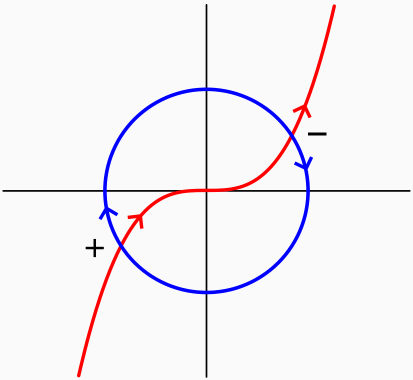

Relative Orientability

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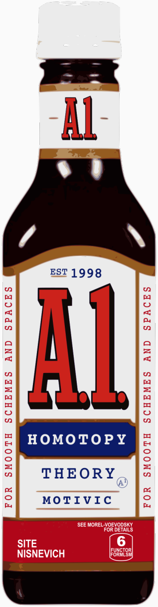
What's left to do?

- Explicit calculation of a_p when $i_p > 1$.
- Address c, d odd case.

Thanks!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6 FUNCTOR FORMLSM

Hurwitz Space Statistics and Dihedral Nichols Algebras

Gregory Michel

PIMS: Workshop in Arithmetic Topology

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Number Theory

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Theorem (Bhargava-Shankar-Tsimerman)

When $n = 3$, this number is given by

$$\frac{1}{12\zeta(3)}X + \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + \textit{(smaller order terms).}$$

Quantum Shuffle Algebras

Theorem (Ellenberg-Tran-Westerland (2017))

$$H_j(Hur^c_{G,m}, k) \cong Ext^{m-j,m}_{\mathfrak{A}(V)}(k, k),$$

where $\mathfrak{A}(V)$ denotes a quantum shuffle algebra.

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There is a subalgebra $\mathfrak{B}(V) \subseteq \mathfrak{A}(V)$ called the *Nichols Algebra*

Idea: Replace $\text{Ext}_{\mathfrak{A}(V)}(k, k)$ with $\text{Ext}_{\mathfrak{B}(V)}(k, k)$.

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Idea: Replace $\mathrm{Ext}_{\mathfrak{A}(V)}(k,k)$ with $\mathrm{Ext}_{\mathfrak{B}(V)}(k,k)$. *At the moment, this is completely unjustified.*

The Third Fomin-Kirillov Algebra

Definition (Fomin-Kirillov Algebras)

For $n \geq 2$, the n^{th} Fomin-Kirillov algebra FK_n over k is the quadratic algebra with generators x_{ij} for $1 \leq i < j \leq n$ subject to the relations

- $x_{ij}^2 = 0$,
- $x_{ij}x_{kl} = x_{kl}x_{ij}$ when i, j, k, l are all distinct,
- $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$ when i, j, k are distinct.

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When $G = S_3$, the corresponding Nichols Algebra $\mathfrak{B}$ is isomorphic to the third Fomin-Kirillov Algebra FK_3.

Theorem (Ștefan-Vay (2016))

$$Ext_{\mathfrak{B}}(k, k) \cong \mathfrak{B}^![Z],$$

where $\mathfrak{B}^!$ *is generated by three classes* A, B, C *of degree* $(1, 1)$ *and* Z *has degree* $(4, 6)$.

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Use Deligne's bounds to approximate the trace of "Frob"

Resulting point count:

Hurwitz Space Statistics

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Theorem (In Progress, M.)

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Naively applying G-L in this situation yields

$$CX + DX^{\frac{p+2}{2p}}$$

Thank you!

Spaces of Noncollinear Points

Ben O'Connor
joint with Ronno Das

University of Chicago

PIMS Workshop on Arithmetic Topology
June 12, 2019

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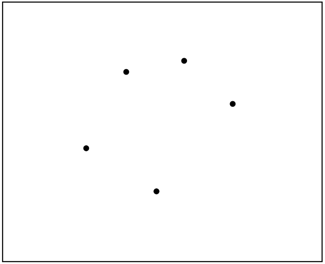

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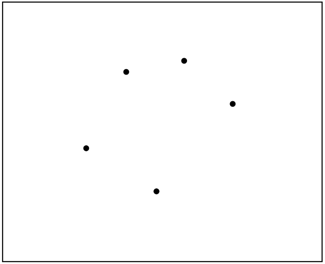 $\in B_5$

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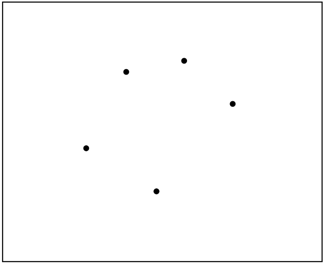

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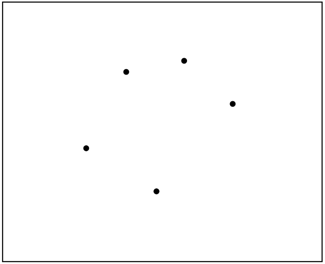

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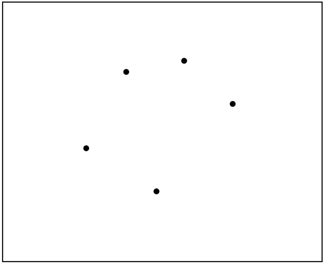

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Goal

Compute $H^*(B_n; \mathbb{Q})$

Ordered Version

Ordered cover F_n:

$$\begin{array}{c} F_n \\ \downarrow \\ B_n = F_n/S_n \end{array}$$

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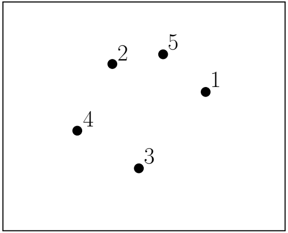

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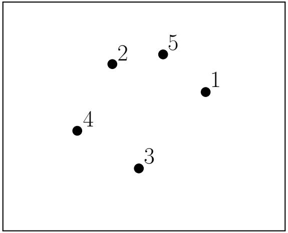

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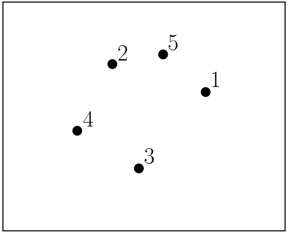

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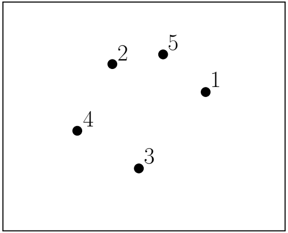

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Compute $H^*(F_n; \mathbb{Q})$ as an S_n-representation

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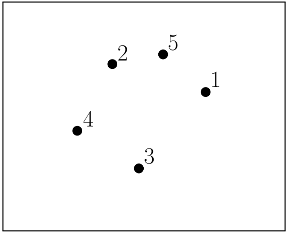

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Refined Goal

Compute $H^*(F_n; \mathbb{Q})$ as an S_n-representation

- By transfer, $H^*(B_n; \mathbb{Q}) \cong H^*(F_n; \mathbb{Q})^{S_n}$

Forgetting Points

- Ordering gives maps $F_n \to F_{n-1}$ by “forget the last point”

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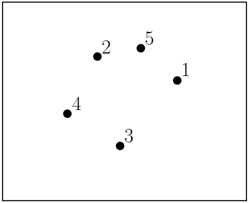

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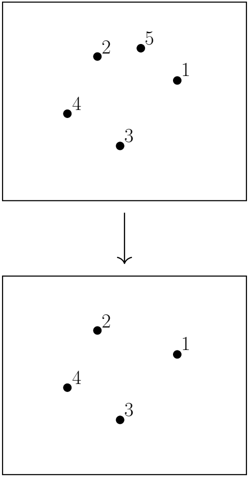

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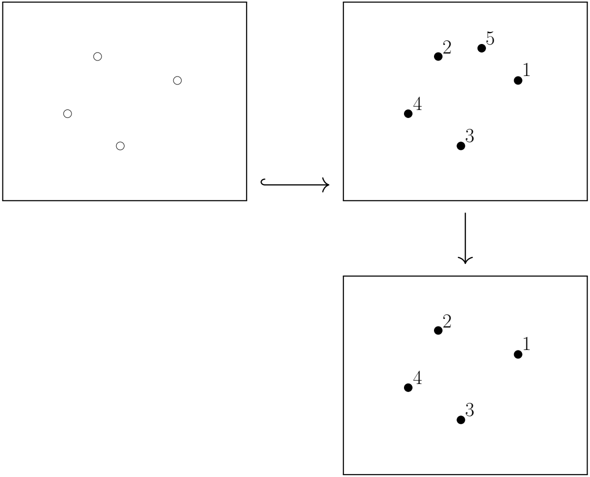

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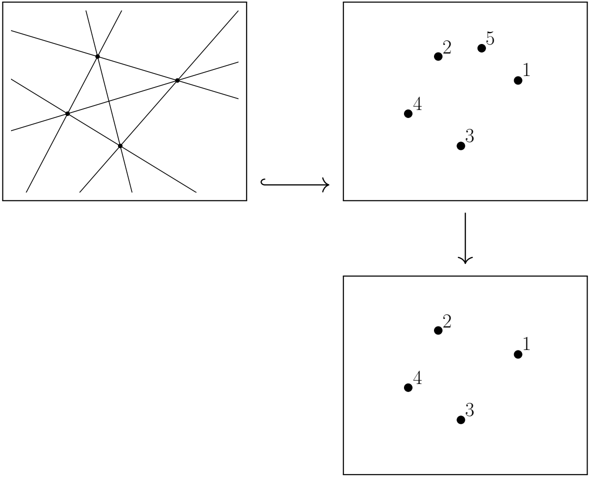

State of Knowledge

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- $H^*(\mathbb{F}_n; \mathbb{Q})$ known for $n = 2, 3$
- $F_4 \cong \mathrm{PGL}_3(\mathbb{C})$
- Finitely presented group surjecting onto $\pi_1(F_n)$ (Moulton)

Theorem (Das-O.)

For $X_5 = F_5 / \mathrm{PGL}_3(\mathbb{C})$, *there are isomorphisms of* S_5*-representations*

$$H^*(X_5; \mathbb{Q}) \cong \begin{cases} U & \text{if } * = 0, \\ S_{3,2} & \text{if } * = 1, \\ \wedge^2 V & \text{if } * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (Das-O.)

For $X_6 = F_6/\operatorname{PGL}_3(\mathbb{C})$, there are isomorphisms of S_6-representations

$$H^*(X_6;\mathbb{Q}) \cong$$

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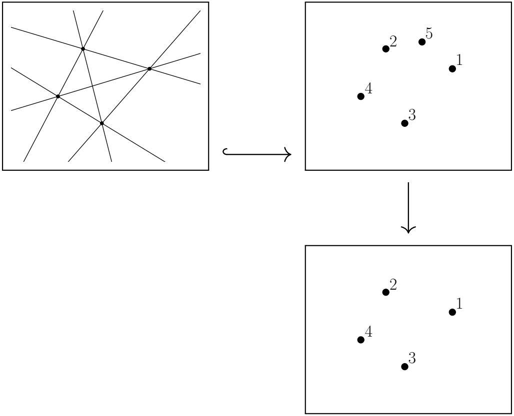
Proof(?)



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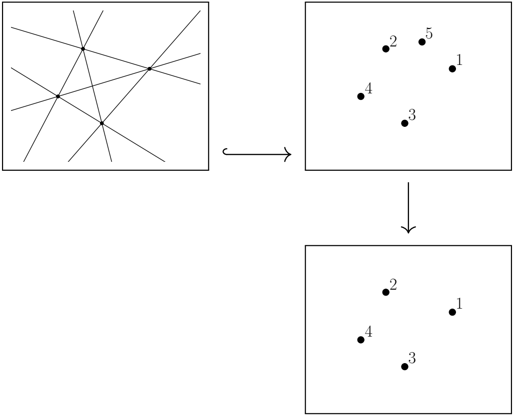
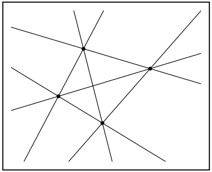


Fiber bundle $\longrightarrow$ Serre spectral sequence

Proof(?)



$$\hookrightarrow F_5 \circlearrowleft S_5$$

$$\downarrow$$

$$F_4 \circlearrowleft S_4$$

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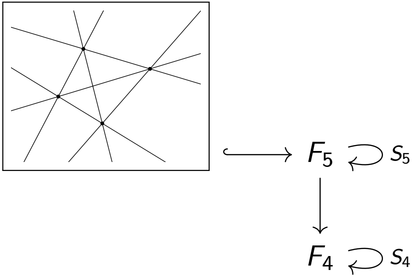

Topology comes up short - what do we do?

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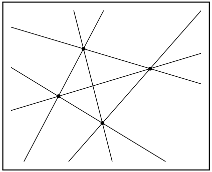

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F_n (smooth) variety defined over $\mathbb{Z}$

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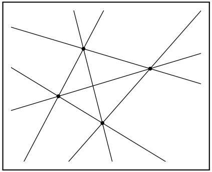

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Topology comes up short - what do we do?

F_n (smooth) variety defined over $\mathbb{Z}$

Use point counts and Grothendieck-Lefschetz trace formula

Refined Point Counting

$$B_n(\mathbb{F}_q) \ni p = \{p_1, \ldots, p_n\}$$

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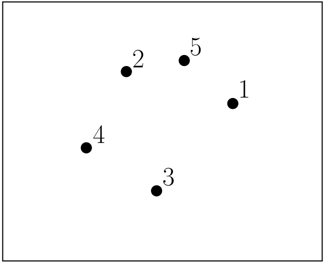


$$= p \circlearrowleft \mathrm{Frob}_q \to \sigma_p \in S_5$$

$$p_{n,C}(q) = |\{p \in B_n(\mathbb{F}_q) \mid \sigma_p \in C\}|$$

Example: $n = 6$, $C = (123)(45)$

- Choices of a

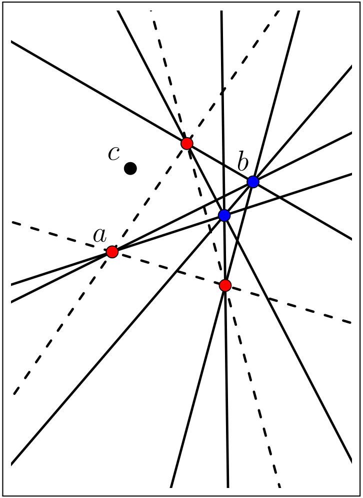
$$(q-1)^2 q^3 (q+1)$$

- Choices of b

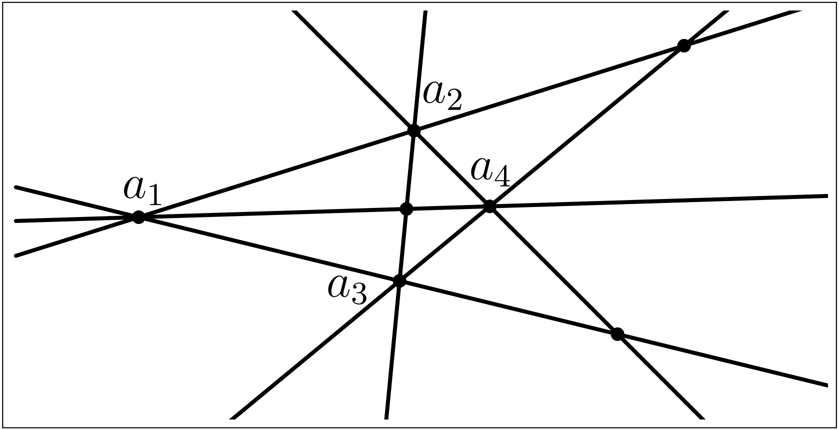
$$(q-1)q(q^2+q+1)$$

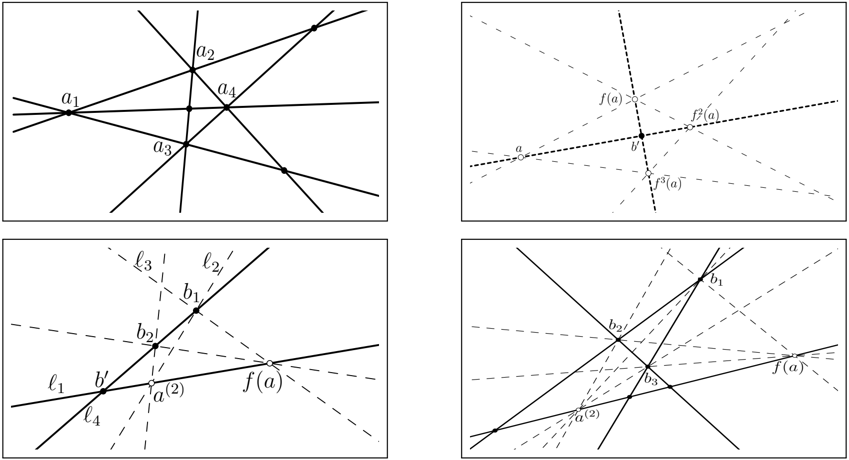
- Choices of c

$$q^2$$



$$p_{6,(123)(45)}(q) = \frac{1}{6}(q-1)^3 q^6 (q+1)(q^2+q+1)$$


a_2
a_1
a_4
a_3


a_1
a_2
a_3
a_4
$f(a)$
$f^2(a)$
a
b'
$f^3(a)$
ℓ_3
ℓ_2
b_1
b_2
ℓ_1
b'
$a^{(2)}$
$f(a)$
ℓ_4
b_1
b_2
b_3
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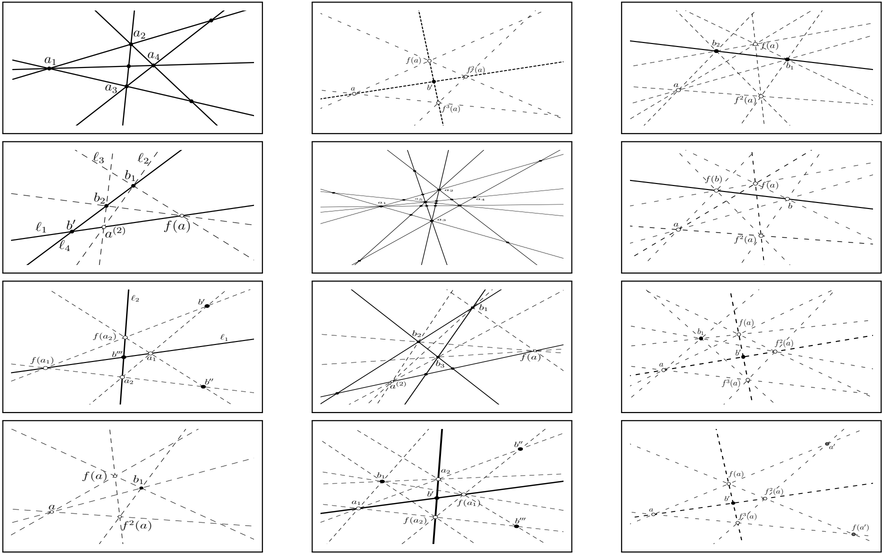

a_1 a_2 a_3 a_4
a $f(a)$ $f^2(a)$ $f^3(a)$ b'
b_1 b_2 a $f(a)$ $f^2(a)$
ℓ_1 ℓ_2 ℓ_3 ℓ_4 b_1 b_2 b' $a^{(2)}$ $f(a)$
a_1 a_2 a_3 a_4 a_5
$f(b)$ $f(a)$ a b $f^2(a)$
ℓ_1 ℓ_2 b' b'' b''' $f(a_1)$ $f(a_2)$ a_1 a_2
b_1 b_2 b_3 $a^{(2)}$ $f(a)$
a b_1 b' $f(a)$ $f^2(a)$ $f^3(a)$
a $f(a)$ b_1 $f^2(a)$
a_1 a_2 b_1 b' b'' b''' $f(a_1)$ $f(a_2)$
a a' b' $f(a)$ $f^2(a)$ $f^3(a)$ $f(a')$

Tables of Point Counts

Class (C)	$p_{5,C}(q)$
e	$\frac{1}{120}(q-3)(q-2)(q-1)^2q^3(q+1)(q^2+q+1)$
(12)	$\frac{1}{12}(q-1)^3q^4(q+1)(q^2+q+1)$
(12)(34)	$\frac{1}{8}(q-2)(q-1)^2q^3(q+1)^2(q^2+q+1)$
(123)	$\frac{1}{6}(q-1)^2q^4(q+1)^2(q^2+q+1)$
(123)(45)	$\frac{1}{6}(q-1)^3q^4(q+1)(q^2+q+1)$
(1234)	$\frac{1}{4}(q-1)^2q^4(q+1)^2(q^2+q+1)$
(12345)	$\frac{1}{5}(q-1)^2q^3(q+1)(q^2+1)(q^2+q+1)$

Table: Point counts for $B_5(\mathbb{F}_q)$ twisted by conjugacy classes of S_5.

Class (C)	$p_{6,C}(q)$
e	$\frac{1}{720}(q-3)(q-2)(q-1)^2q^3(q+1)(q^2+q+1)(q^2-9q+21)$
(12)	$\frac{1}{48}(q-1)^3q^4(q+1)(q^2+q+1)(q^2-3q+3)$
(12)(34)	$\frac{1}{6}(q-2)(q-1)^2q^3(q+1)^2(q^2+q+1)(q^2-q-3)$
(12)(34)(56)	$\frac{1}{48}(q-1)^2q^3(q+1)(q^2+q+1)(q^4-6q^2+q+8)$
(123)	$\frac{1}{18}(q-1)^2q^6(q+1)^2(q^2+q+1)$
(123)(45)	$\frac{1}{6}(q-1)^3q^6(q+1)(q^2+q+1)$
(123)(456)	$\frac{1}{18}(q-1)^2q^3(q+1)(q^2+q+1)(q^4-2q^3-3q+9)$
(1234)	$\frac{1}{8}(q-1)^2q^4(q+1)^2(q^2+q+1)(q^2+q-1)$
(1234)(56)	$\frac{1}{8}(q-1)^2q^3(q+1)(q^2+q+1)(q^4-2q^2-q-2)$
(12345)	$\frac{1}{5}(q-1)^2q^3(q+1)(q^2+1)(q^2+q+1)^2$
(123456)	$\frac{1}{6}(q-1)^2q^3(q+1)(q^2+q+1)(q^4+q-1)$

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Table: Point counts for $B_6(\mathbb{F}_q)$ twisted by conjugacy classes of S_6

- Now we cross the bridge back to topology(!)

Grothendieck-Lefschetz Trace Formula

$$\sum_{p \in X(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q \mid \mathcal{V}_p) = \sum_i (-1)^i \,\mathrm{tr}(\mathrm{Frob}_q : H^{2n-i}_{\text{ét},c}(X;\mathcal{V}))$$

$$\downarrow$$

$$\sum_C \chi_V(C) p_{n,C}(q) = q^n \sum_{i,w} q^{-w} (-1)^i \langle \chi_V, \chi^i_w(F_n) \rangle_{S_n}$$

Theorem (Das-O.)

For $X_n = F_n / \mathrm{PGL}_3(\mathbb{C})$, there are isomorphisms of S_n-representations

$$H^*(X_5; \mathbb{Q}) \cong \begin{cases} U & \text{if } * = 0, \\ S_{3,2} & \text{if } * = 1, \\ \wedge^2 V & \text{if } * = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks for listening!

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Types of Lines on Quintic Threefolds and Beyond

Sabrina Pauli

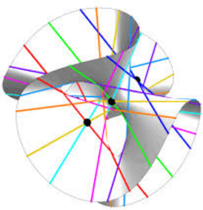
University of Oslo

June 12, 2019

Lines on a Cubic Surface

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface.

- $k = \mathbb{C}$: #complex lines on $X = 27$ (Cayley, Salmon 19th century)



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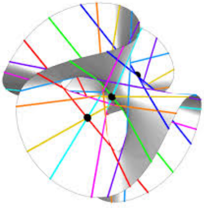
- $k = \mathbb{R}$: There are two types of real lines, called hyperbolic and elliptic (Segre).
- # real hyperbolic lines on X − # real elliptic lines on $X = 3$ (Finashin–Kharlamov, Okonek–Teleman, Horev-Solomon, Benedetti-Silhol)



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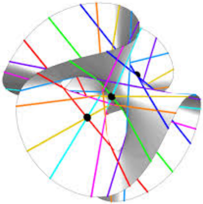
- # real hyperbolic lines on X − # real elliptic lines on $X = 3$ (Finashin–Kharlamov, Okonek–Teleman, Horev-Solomon, Benedetti-Silhol)
- k arbitrary ($\text{char}(k) \neq 2$): can assign an arithmetic type in $k^*/(k^*)^2$ (Kass-Wickelgren) $\rightsquigarrow$ can count lines in $\text{GW}(k)$: $15 < 1 > +12 < -1 >$



The Type of a Line on a Cubic Surface.

Let $L \subset X$ be a line. To each point $p \in L$, there is exactly one other point q such that $T_pX = T_qX$.

Definition

The morphism $i : L \to L$ that swaps p and q is called *Segre involution*. Its fixed points are called *Segre fixed points*.

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The Segre fixed points are defined over the field $k(\sqrt{\alpha})$ for some $\alpha \in k^*/(k^*)^2$.

Definition

The *type of a line on a cubic surface* is $< \alpha > \in \mathrm{GW}(k)$.

Local degree

Let $\text{Gr}(2,4)$ be the Grassmannian of lines in $\mathbb{P}^3$. A homogeneous degree 3 polynomial f defines a section σ_f of the vector bundle $\mathcal{E} := \text{Sym}^3 \mathcal{S}^\vee \to \text{Gr}(2,4)$ where $\mathcal{S}$ is the tautological subbundle of $\text{Gr}(4,2)$.

$$\{\text{zeros of } \sigma_f\} \leftrightarrow \{\text{lines on } X = \{f = 0\} \subset \mathbb{P}^3\}$$

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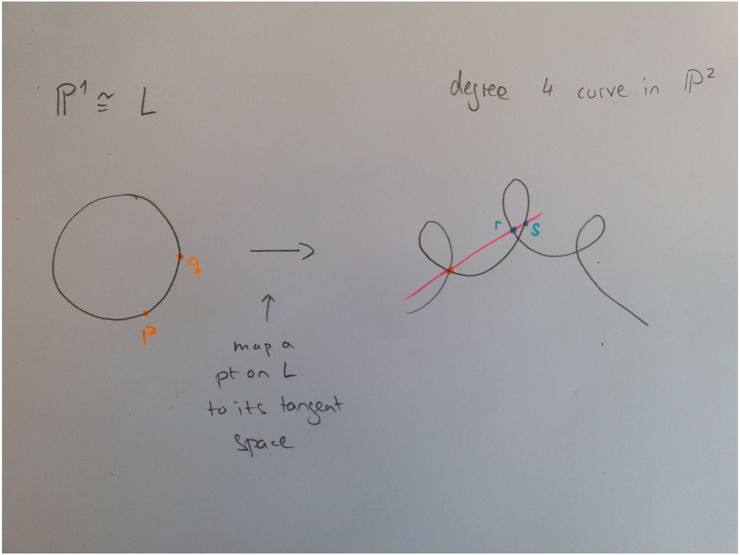
Locally σ_f is a morphsim $\mathbb{A}^4 \to \mathbb{A}^4$. The local degree of σ_f at a zero is $< J > \in \mathrm{GW}(k)$ where J is the determinant of the Jacobian at the zero. We define the Euler number $e(\mathcal{E}) := \sum$ local degrees.

Theorem (Kass-Wickelgren)

The local degree of a zero of σ_f is equal to the type of the corresponding line on $X = \{f=0\} \subset \mathbb{P}^3$ in $\mathrm{GW}(k)$.

The Type of a Line on a Quintic Threefold

lines quintic threefold.jpg



The Type of a Line on a Quintic Threefold

Let $L \subset X \subset \mathbb{P}^4$ be a line on a quintic threefold X.

- There are 3 pairs of points on L with the same tangent space in X (might only be defined over a field extension F/k).

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- Let $p, q \in L \otimes F$ be such a pair, i.e., $T := T_p(X \otimes F) = T_q(X \otimes F)$. For $r \in L \otimes F$ there is exactly one other point $s \in L \otimes F$ such that

  $$T \cap T_r(X \otimes F) = T \cap T_s(X \otimes F).$$

  $\rightsquigarrow$ 3 Segre involutions $i_j : L \otimes F_j \to L \otimes F_j$ with fixed points defined over $F_j(\sqrt{\alpha_j})$, $j = 1, 2, 3$.

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Definition

The *type of a line on a quintic threefold* is $< \prod N_{F_j/k}(\alpha_j) > \in \mathrm{GW}(k)$ where the product runs over the Galois orbits of the pairs of points with the same tangent space.

This has been defined for $k = \mathbb{R}$ by Finashin and Kharlamov.

My Theorem

Let f be a homogeneous degree 5 polynomial in 5 variables and σ_f the corresponding section of $\mathrm{Sym}^5\,\mathcal{S}^\vee \to \mathrm{Gr}(2,5)$.

$$\{\text{zeros of } \sigma_f\} \leftrightarrow \{\text{lines on } X = \{f=0\} \subset \mathbb{P}^4\}$$

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$$\{\text{zeros of } \sigma_f\} \leftrightarrow \{\text{lines on } X = \{f = 0\} \subset \mathbb{P}^4\}$$

Theorem (P.)

The local degree of a zero of σ_f is equal to the type of the corresponding line on a $X = \{f = 0\} \subset \mathbb{P}^4$ in $\mathrm{GW}(k)$.

My Theorem

Let f be a homogeneous degree 5 polynomial in 5 variables and σ_f the corresponding section of $\mathrm{Sym}^5 \mathcal{S}^\vee \to \mathrm{Gr}(2,5)$.

$$\{\text{zeros of } \sigma_f\} \leftrightarrow \{\text{lines on } X = \{f = 0\} \subset \mathbb{P}^4\}$$

Theorem (P.)

The local degree of a zero of σ_f is equal to the type of the corresponding line on a $X = \{f = 0\} \subset \mathbb{P}^4$ in $\mathrm{GW}(k)$.

The definition of the type of a line can be generalized to lines on degree $2n-1$ hypersurfaces in $\mathbb{P}^{n+1}$.

Twin Prime Polynomials
Joint with Sawin

Mark Shusterman

UW Madison

6/10/2019

Main Result

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- Theorem (Sawin, S): There exists a prime power q such that for every $h \in \mathbb{F}_q[T]$ there exist infinitely many monic irreducible $f \in \mathbb{F}_q[T]$ such that $f + h$ is irreducible as well.

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- Actually, we have a quantitative version where the number of such f (having a certain degree) is obtained (with a power saving error term).

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- Level of Distribution for Primes: Counting primes up to X in a certain residue class, with modulus larger than $\sqrt{X}$.
- Parity Problem: How often do both f and $f + h$ have an odd number of prime factors?
- We focus on the second problem.

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- Theorem (Sawin, S): For distinict $h_1, \ldots, h_k \in \mathbb{F}_q[T]$ we have

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- Idea: Split the sum into subsums over those f having the same derivative, and show that (on these subsums) the Möbius function can be mimicked by a multiplicative Dirichlet character.
- We are then able to reduce the problem to a short character sum.

Improving the Burgess Bound

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- Theorem (Sawin, S): Let $g \in \mathbb{F}_q[T]$ be squarefree, and χ a nonprincipal Dirichlet character mod g. Then

$$\left| \sum_{\substack{h \in \mathbb{F}_q[T] \\ d(h) < t}} \chi(f+h) \right| \leq (q^{1/2}+1)\binom{\deg(g)}{t} q^{\frac{t}{2}}$$

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- We write down a variety whose $\mathbb{F}_q$-point count controls the above character sum.
- Study the geometry (e.g. singularities) of our variety in order to estimate the dimensions of the associated cohomology groups.
- Using Deligne's RH and the Grothendieck-Lefschetz trace formula, we are then able to estimate the number of $\mathbb{F}_q$-ponits on our variety.

Euler characteristics for spaces of string links and the modular envelope of $\mathcal{L}_\infty$

Paul Arnaud Songhafouo Tsopméné

University of Regina

(Joint with Victor Turchin)

June 12, 2019

Definition of the space of string links

Fix an integer $d \geq 1$, which represents the dimension of the ambient space, and let $r \geq 1, m_1, \cdots, m_r \geq 1$.

Definition

Define $Emb_c(\coprod_{i=1}^{r} \mathbb{R}^{m_i}, \mathbb{R}^d)$ *to be the space of smooth embeddings* $f\colon \coprod_{i=1}^{r} \mathbb{R}^{m_i} \hookrightarrow \mathbb{R}^d$ *that coincide outside a compact set with a fixed affine embedding* ι*. Such embeddings are called string links of* r *strands.*

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For convenience, we consider a variation of that space, denoted $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$. To be more precise, $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ is the homotopy fiber over ι of the obvious inclusion $\mathrm{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d) \hookrightarrow \mathrm{Imm}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, where $\mathrm{Imm}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ is the space of smooth immersions $\coprod_{i=1}^r \mathbb{R}^{m_i} \looparrowright \mathbb{R}^d$ that coincide outside a compact set with ι.

Examples

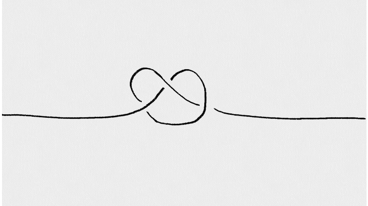

Figure: A string link of one strand ($r = 1, m_1 = 1$), also called a long knot

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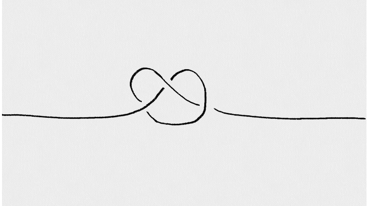

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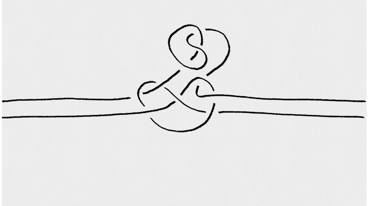

Figure: A string link of two strands ($r = 2, m_1 = m_2 = 1$)

Literature Review

Many people studied $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^{r} \mathbb{R}^{m_i}, \mathbb{R}^d)$ for various r and m_i:

- For $r = 1, m_1 = 1$, we have the space $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^d)$ which has been studied by: V. Turchin (2004, 2013), D. Sinha (2006), P. Salvatore (2006), R. Budney (2007, 2012), K. Sakai (2008), Lambrechts-Volić-Turchin (2010), Dwyer-Hess (2012), P. Songhafouo Tsopméné (2013), S. Moriya (2013), T. Willwacher (2015).

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- For $r \geq 1, m_1, \cdots, m_r \geq 1$, we have the space $\overline{\mathrm{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ sutied by J. Ducoulombier (2018), Songhafouo Tsopméné - Turchin (two papers, 2018).

Right Γ-modules and Right Ω-modules

- Define Γ to be the category whose objects are finite pointed sets $n+ = \{0, 1, \cdots, n\}$, with 0 as the basepoint, and whose morphisms are pointed maps.
- Let Ω denote the category of finite unpointed sets $\{1, \cdots, n\}, n \geq 0$, and surjections. (Some authors denote that category by FI).
- For $X = \Gamma$ or $X = \Omega$, define a right X-module as a contravariant functor from X to chain complexes.
- For $X = \Gamma$ or $X = \Omega$, the category of right X-modules is denoted Rmod_X. We endow this category with the projective model structure.
- Given two objects $A, B \in \mathrm{Rmod}_\Omega$, we write $\mathrm{hRmod}_\Omega(A, B)$ for the space of derived morphisms from A to B.

Right Γ-modules and Right Ω-modules (continued)

- For $k \geq 0$, define $C(k, \mathbb{R}^d)$ denotes the configuration space of k labeled points in $\mathbb{R}^d$.
- One can show that the sequence $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d), d \geq 3$, has a natural structure of a right Γ-module.
- Let $\mathrm{cr}\colon \mathrm{Rmod}_\Gamma \longrightarrow \mathrm{Rmod}_\Omega$ be the cross-effect functor constructed by Pirashvili. And let $\mathbb{Q} \otimes \widehat{\pi}_* C(\bullet, \mathbb{R}^d)$ denote the cross effect of $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d)$.
- A sequence of r integers $s_1, \cdots, s_r$ is written as $\vec{s}$. Also we write $|\vec{s}|$ for $s_1 + \cdots + s_r$, and $\Sigma_{\vec{s}}$ for $\Sigma_{s_1} \times \cdots \times \Sigma_{s_r}$. If $x_1, \cdots, x_r$ is another sequence, we write $\vec{s} \cdot \vec{x}$ for $s_1 x_1 + \cdots + s_r x_r$, and $\vec{x}^{\vec{s}}$ for $\prod_i x_i^{s_i}$.
- Let $Q_{\vec{s}}^{\vec{m}}$ be the right Ω-module defined by

$$Q_{\vec{s}}^{\vec{m}}(k) = \begin{cases} 0 & \text{if} \quad k \neq |\vec{s}|; \\ \mathrm{Ind}_{\Sigma_{\vec{s}}}^{\Sigma_k} \widetilde{H}_*(S^{\vec{s}\cdot\vec{m}}; \mathbb{Q}) & \text{if} \quad k = |\vec{s}|. \end{cases}$$

 

Theorem (S.T.-Turchin, 2018)

For $d > 2max\{m_i : 1 \leq i \leq r\} + 1$, there is an isomorphism

$$\mathbb{Q} \otimes \pi_*(\overline{Emb}_c(\coprod_{i=1}^{r} \mathbb{R}^{m_i}, \mathbb{R}^d)) \cong \bigoplus_{\vec{s},t} hRmod_{\Omega}\left(Q_{\vec{s}}^{\vec{m}}, \mathbb{Q} \otimes \widehat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d)\right)$$

We also have the homology version of this.

The functions $\mu(-)$, $E_l(-)$, $S_j(-)$, and $F_l(-)$

- Let $\mu(-)$ denote the standard Möbius function.
- Given a variable x and an integer $l \geq 1$, let $E_l(x)$ denote the sum $E_l(x) = \frac{1}{l}\sum_{p|l}\mu(p)x^{\frac{l}{p}}$.
- Let B_p denote the pth Bernoulli number, so that $\sum_{p\geq 0}\frac{B_p x^p}{p!} = \frac{x}{e^x - 1}$. Recall that $B_{2n+1} = 0$, $n \geq 1$. Bernoulli's summation formula equates $1^j + 2^j + \cdots + n^j$ with $S_j(n)$ where $S_j(x) = \frac{1}{j+1}\sum_{p=0}^{j}(-1)^p\binom{j+1}{p}B_p x^{j+1-p}$, $j \geq 1$.
- Define $F_l(u)$ by $F_l(u) = lu^l E_l(\frac{1}{u})$.

Euler characteristics for $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$

For $\vec{s} \geq 0$ and $t \geq 0$, let $\mathcal{X}_{\vec{s},t}$ be the Euler characteristic of the summand of the previous theorem indexed by $\vec{s}, t$. The associated generating function is $F^{\pi}_{\vec{m},d}(x_1, \cdots, x_r, u) = \sum_{\vec{s},t \geq 0} \mathcal{X}_{\vec{s},t} \cdot u^t \vec{x}^{\vec{s}}$.

Theorem (S.T.-Turchin, 2018)

The generating function $F^{\pi}_{\vec{m},d}(x_1, \cdots, x_r, u)$ is given by the formula

$$F^{\pi}_{\vec{m},d}(x_1, \cdots, x_r, u) = \sum_{k,l,j \geq 1} \frac{\mu(k)}{kj} S_j \left(\sum_{i=1}^{r} (-1)^{m_i - 1} E_l(x_i^k) \right) \left(\frac{(-1)^{d-1} l u^{kl}}{F_l(u^k)} \right)^j - \sum_{k,l \geq 1} \sum_{i=1}^{r} \frac{\mu(k)}{k} (-1)^{m_i - 1} E_l(x_i^k) \ln(F_l(u^k)),$$

We also have the homology version of this.

- Very roughly, an operad is an algebraic structure consisting of an object of n-ary operations for all n. The compositions of operations are encoded by a certain category of trees.

Modular operads and Modular envelope of $\mathcal{L}_\infty$

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- When dealing with moduli spaces $\overline{M}_{g,n}$ of stable marked complex curves, one encounters general graphs (of certain genus), the case of trees corresponding to curves of genus 0. So one can consider a "higher genus" analogue of the theory of operads, in which graphs replace trees. The resulting object, introduced by E. Getzler and M. Kapranov, is called modular operad.

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- A cyclic operad is a usual symmetric operad for which the output of its elements has the same role as the inputs.
- One has an adjunction $\mathbf{Mod}\colon \mathrm{CycOp} \rightleftarrows \mathrm{ModOp}\colon \mathbf{Cyc}$ between the categories of cyclic and modular operads.
- Let $\mathcal{L}_\infty$ be the operad for homotopy Lie algebras. We consider the modular operad $\mathbf{Mod}(\mathcal{L}_\infty) = \{\mathbf{Mod}(\mathcal{L}_\infty)((g,n))\}_{g,n}$.

The notion of supercharacter

Let $M = (\oplus_i M_i, \partial)$ be a finite dimensional chain complex of Σ_k-modules over a ground field $\mathbb{K}$ of characteristic 0.

- By the *supercharacter* we understand the character of the Σ_k action on the virtual representation $\mathcal{X}M$ defined as $\mathcal{X}M := \sum_i (-1)^i M_i$. The latter virtual representation is similar to the Euler characteristic in the sense that $\mathcal{X}M \simeq \mathcal{X}(H_* M)$, that's why we use this notation.
- Let Z_{M_i} denote the cycle index sum of M_i. The cycle index sum encoding the supercharacter of the Σ_k action on M can be defined as $Z_{\mathcal{X}M} = \sum_i (-1)^i Z_{M_i}$,

For a symmetric sequence of chain complexes $M = \{M(k)\}_{k\geq 0}$, we similarly define $Z_{\mathcal{X}M} := \sum_{k\geq 0} Z_{\mathcal{X}M(k)}$.

The supercharacter of the symmetric group action on $\mathbf{Mod}(\mathcal{L}_\infty)$

For any stable collection $\{M((g,n))\}$ define a symmetric sequence $M((\bullet)) = \{\oplus_g M(g,n),\ n \geq 0\}$.

Theorem (S.T. - Turchin, 2018)

The supercharacter of the symmetric group action on the modular envelope of $\{\mathbf{Mod}(\mathcal{L}_\infty)((k))\}_{k\geq 0}$ of $\mathcal{L}_\infty$ is described by the cycle index sum

$$Z_{\chi_{\mathbf{Mod}(\mathcal{L}_\infty)((\bullet))}}(w; p_1, p_2, p_3, \cdots) =$$
$$w \sum_{k,l,j\geq 1} \frac{\mu(k)}{kj} S_j \left(\frac{1}{l} \sum_{a|l} \mu\left(\frac{l}{a}\right) \frac{p_{ak}}{w^{ak}} \right) \left(\frac{l w^{kl}}{F_l(w^k)} \right)^j -$$
$$w \sum_{k,l\geq 1} \frac{\mu(k)}{kl} \left(\sum_{a|l} \mu\left(\frac{l}{a}\right) \frac{p_{ak}}{w^{ak}} \right) \ln(F_l(w^k))$$

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- the formula we obtained for the generating function $F^{\pi}_{\vec{m},d}(x_1, \cdots, x_r, u)$, and
- certain graph complexes introduced by M. Kontsevich.

Thanks for listening!

Incidence strata of affine varieties with complex multiplicities

Hunter Spink, joint with Dennis Tseng

Incidence Strata in A^1

Incidence Strata in $\mathbb{A}^1$

Consider 4 unordered points in $\mathbb{A}^1$

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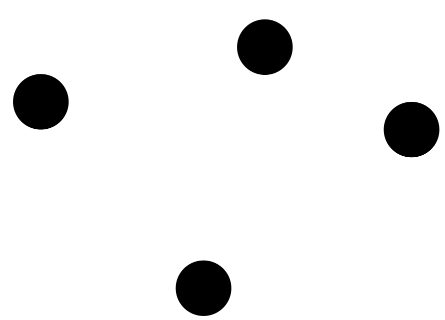
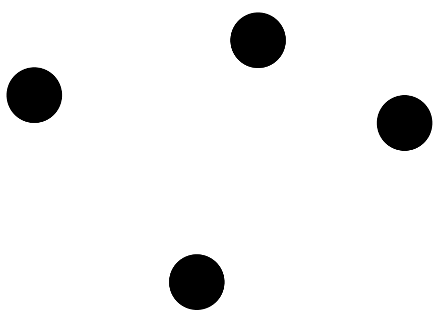


Incidence Strata in A^1

Consider 4 unordered points in $\mathbb{A}^1$

$\mathrm{Sym}^4\mathbb{A}^1 = \mathbb{A}^4$ freely parametrizes coefficients a, b, c, d of

$$(z - x_1)(z - x_2)(z - x_3)(z - x_4) = z^4 + az^3 + bz^2 + cz + d$$



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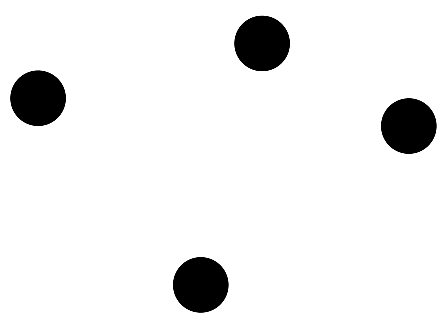

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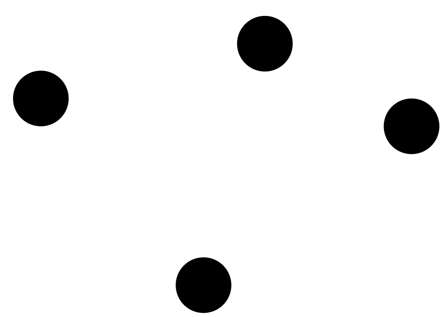

Partitions of [4] are in bijection with <u>closed incidence strata</u>, the closure of the set of configurations where the multiplicities are exactly given by the partition.

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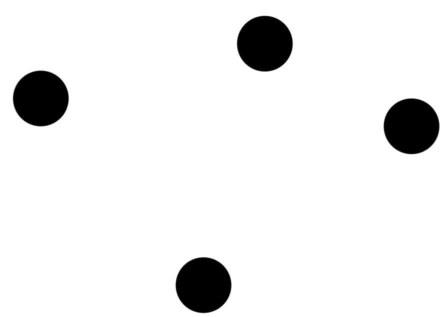
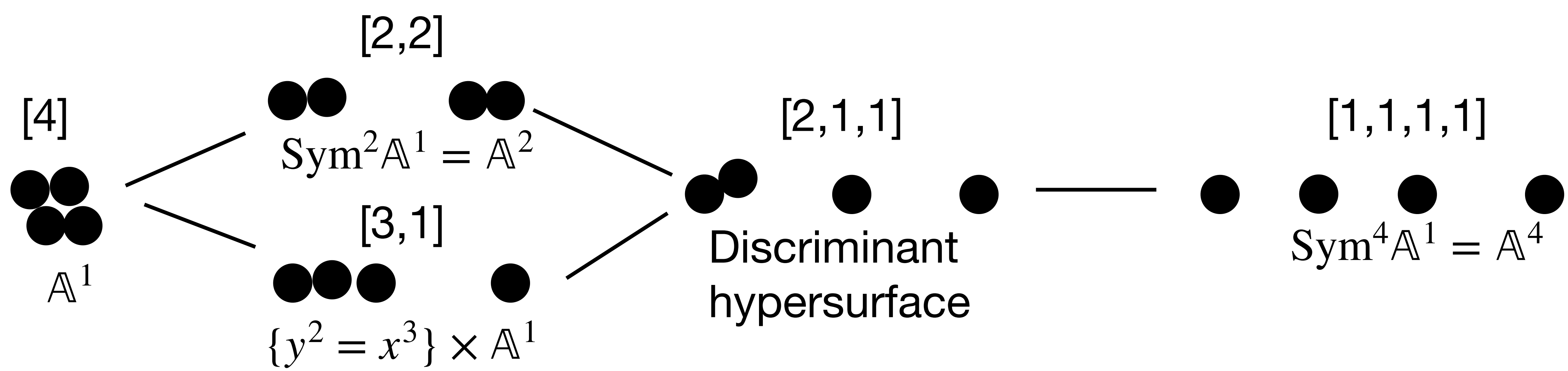
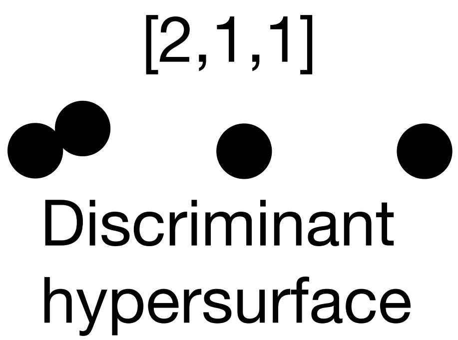


Partitions of [4] are in bijection with <u>closed incidence strata</u>, the closure of the set of configurations where the multiplicities are exactly given by the partition.

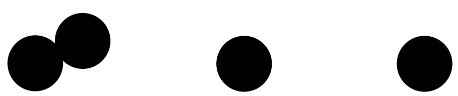


Incidence Strata in A^1



Incidence Strata in A^1

[2,1,1]

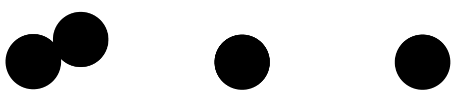


$$= \{(z - x_1)^2(z - x_2)(z - x_3)\} \subset \{z^4 + ax^3 + bx^2 + cx + d\} = \mathrm{Sym}^4\mathbb{A}^1 = \mathbb{A}^4$$

Discriminant
hypersurface

Incidence Strata in A^1

[2,1,1]



Discriminant hypersurface

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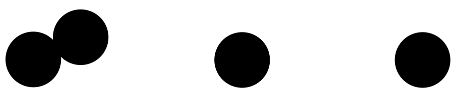
$$= \mathrm{Spec}\,\mathbb{C}[e_1(x_1,x_1,x_2,x_3), e_2(x_1,x_1,x_2,x_3), e_3(x_1,x_1,x_2,x_3), e_4(x_1,x_1,x_2,x_3)]$$

Incidence Strata in A¹

[2,1,1]



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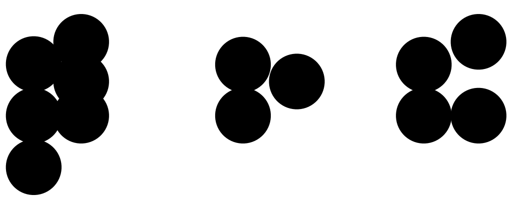
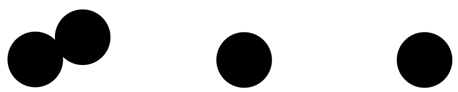
Discriminant hypersurface

[r,s,t]



Incidence Strata in A^1

[2,1,1]



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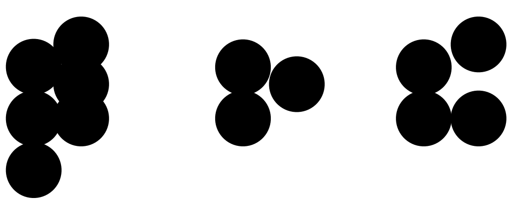
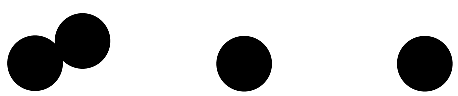
[r,s,t]



$$= \{(z-x_1)^r(z-x_2)^s(z-x_3)^t\} \subset \mathrm{Sym}^{r+s+t}\mathbb{A}^1 = \mathbb{A}^{r+s+t}$$

Incidence Strata in A^1

[2,1,1]



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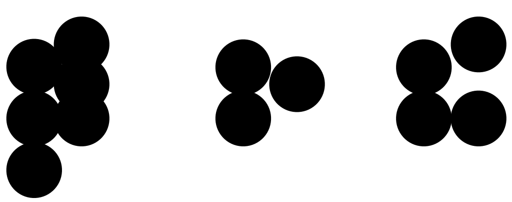
[r,s,t]



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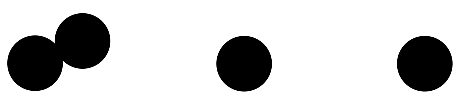
Incidence Strata in A¹

[2,1,1]



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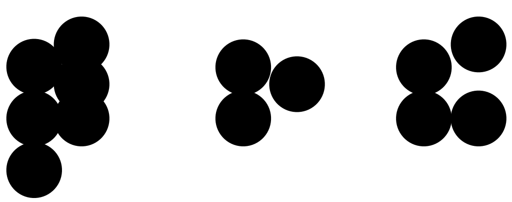

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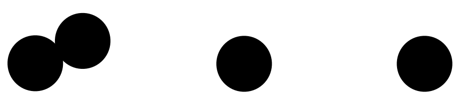
Incidence Strata in A¹

[2,1,1]



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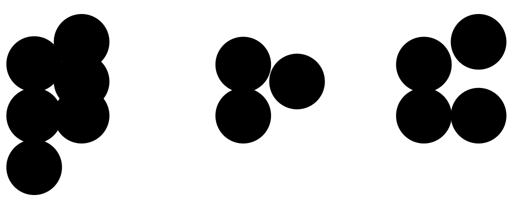
[r,s,t]



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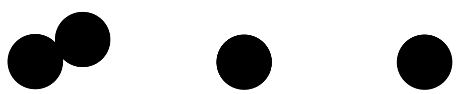

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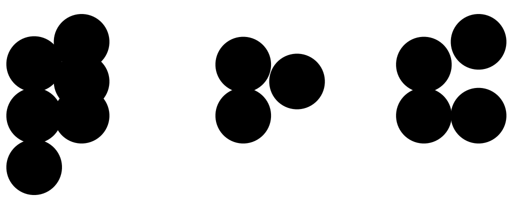

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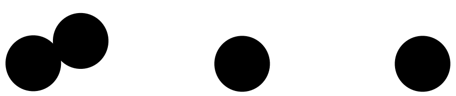
Idea: Let r,s,t be arbitrary complex numbers, obtain continuous family of ``ℂ-incidence strata”

Incidence Strata in A¹

[2,1,1]



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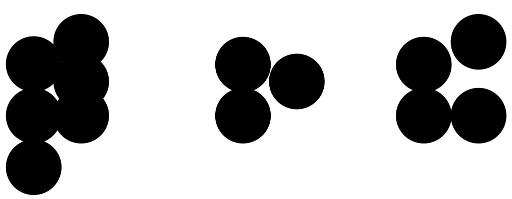

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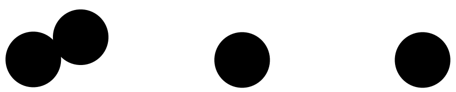
Problem: $\mathbb{C}[r,s,t,\{e_i(\underbrace{x_1,\ldots,x_1}_{r},\underbrace{x_2,\ldots,x_2}_{s},\underbrace{x_3,\ldots,x_3}_{t})\}_{i\in\mathbb{N}}]$ isn't finitely generated.

Incidence Strata in A¹

[2,1,1]



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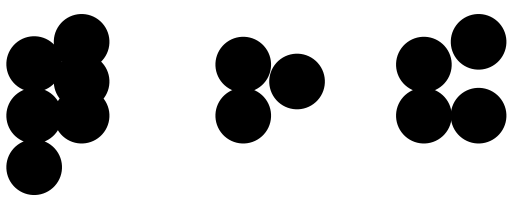
[r,s,t]



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Idea: Let r,s,t be arbitrary complex numbers, obtain continuous family of "$\mathbb{C}$-incidence strata"

Problem: $\mathbb{C}[r,s,t,\{e_i(\underbrace{x_1,\ldots,x_1}_{r},\underbrace{x_2,\ldots,x_2}_{s},\underbrace{x_3,\ldots,x_3}_{t})\}_{i\in\mathbb{N}}]$ isn't finitely generated.

Other possible obstruction: If we have a sequence of varieties $X_1, X_2, \ldots$ then a necessary condition for them to be fibers of a finite-type family is that their "affine embedding dimensions" $\min\{n \mid X \hookrightarrow \mathbb{A}^n\}$ are bounded.

Incidence Strata in $\mathbb{A}^1$

Solution: (Etingof, Rains, Sam) It's finite-type after inverting

$$r^{-1}, s^{-1}, t^{-1}, (r+s)^{-1}, (r+t)^{-1}, (s+t)^{-1}, (r+s+t)^{-1}$$

(i.e. we avoid all collisions which would cause a point of multiplicity 0).

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Concretely: There exists a threshold N such that for all $i > N$ we may find a polynomial expression for $e_i(\underbrace{x_1, \ldots, x_1}_{r}, \underbrace{x_2, \ldots, x_2}_{s}, \underbrace{x_3, \ldots, x_3}_{t})$ in terms of $e_j(\underbrace{x_1, \ldots, x_1}_{r}, \underbrace{x_2, \ldots, x_2}_{s}, \underbrace{x_3, \ldots, x_3}_{t})$ with $j \leq N$ and coefficients in $\mathbb{C}[r, s, t][r^{-1}, s^{-1}, t^{-1}, (r+s)^{-1}, (r+t)^{-1}, (s+t)^{-1}, (r+s+t)^{-1}]$

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Question: Does something similar work for other affine varieties?

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Theorem(S, Tseng): There is a functor Δ^k from affine varieties X to affine varieties over $\mathbb{C}[m_1, \ldots, m_k][\{(\sum_{i \in A} m_i)^{-1}\}_{A \subset \{1, \ldots, k\}}]$ such that the fiber of $\Delta^r(X)$ over any $(m_1, \ldots, m_k) \in \mathbb{N}^k$ is precisely the k part incidence strata of $\mathrm{Sym}^{m_1 + \cdots + m_k} X$ associated to $(m_1, \ldots, m_k)$.

Incidence Strata in A¹

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$r^{-1}, s^{-1}, t^{-1}, (r+s)^{-1}, (r+t)^{-1}, (s+t)^{-1}, (r+s+t)^{-1}$

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Theorem(S, Tseng): By explicit elimination, if we use **power sum polynomials** instead of elementary symmetric sums, this works over any ring (e.g. ring of integers).

Why power sums?

$$p_i(\underbrace{x_1, \ldots, x_1}_{r}, \underbrace{x_2, \ldots, x_2}_{s}, \underbrace{x_3, \ldots, x_3}_{t}) = rx_1^i + sx_2^i + tx_3^i$$

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$(m_1, \ldots, m_k)$ -incidence strata in $\mathrm{Spec}(R)$

$$= \mathrm{Spec}(\texttt{subalgebra of } R^{\otimes k} \texttt{ generated by } \{m_1 r_1 + \ldots + m_k r_k\}_{r \in R})$$

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For $X = \mathbb{A}^1$, difference between $(m_1, \ldots, m_k)$-incidence strata in $\mathrm{Sym}^{m_1+\ldots+m_k}\mathbb{A}^1$ and

$$\{\frac{m_1}{z - x_1} + \ldots + \frac{m_k}{z - x_k} \mid x_i \in \mathbb{A}^1\}$$

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THANK YOU

Arithmetic groups and characteristic classes of manifold bundles

Bena Tshishiku

Workshop on arithmetic topology

June 2019

Main Theorem

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$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

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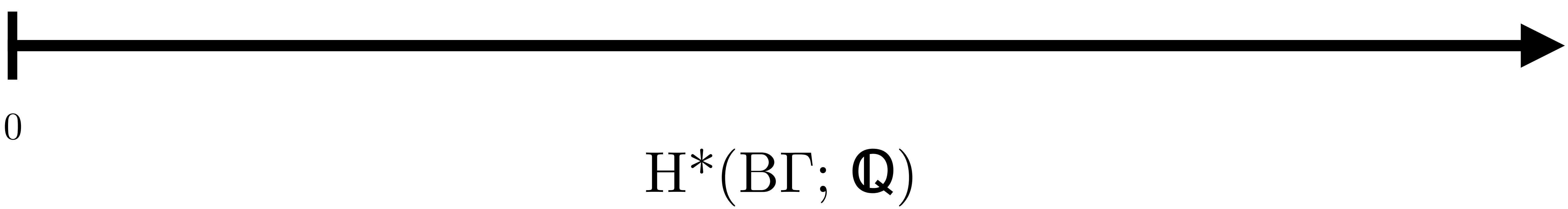

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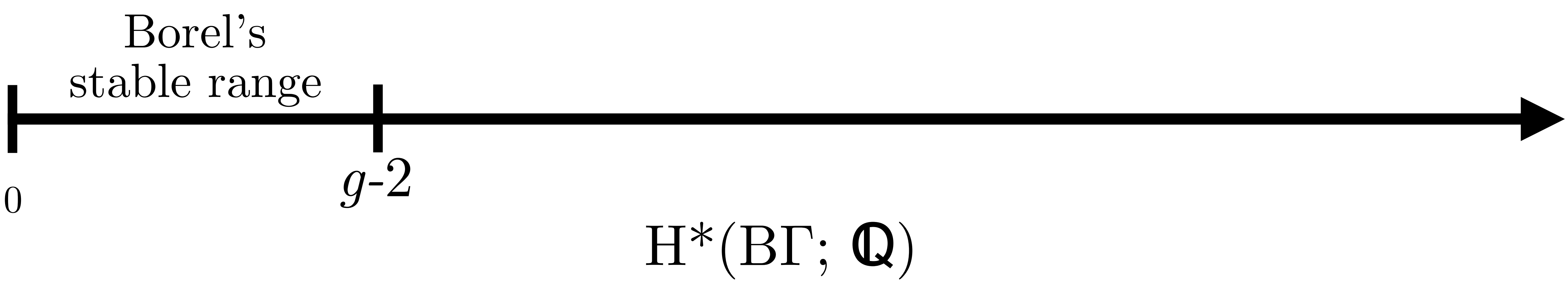

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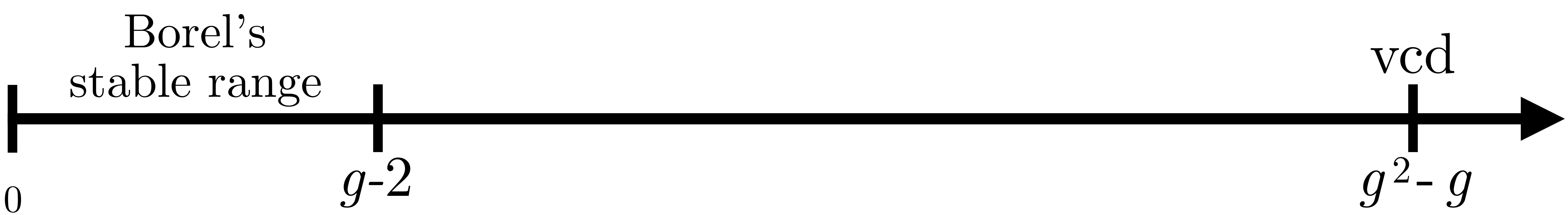

$\mathrm{H}^*(B\Gamma; \mathbb{Q})$

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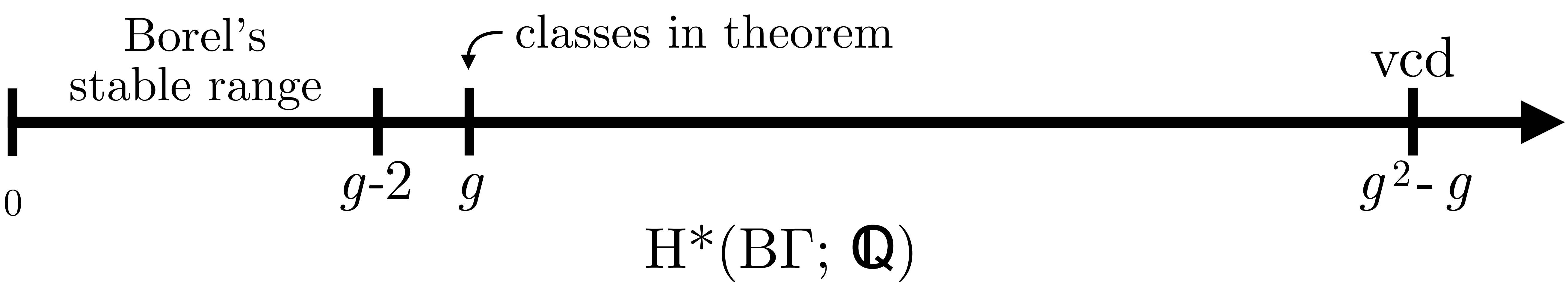

Characteristic class construction

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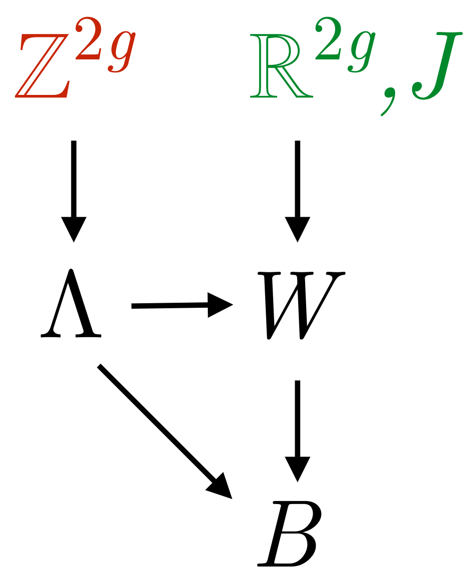

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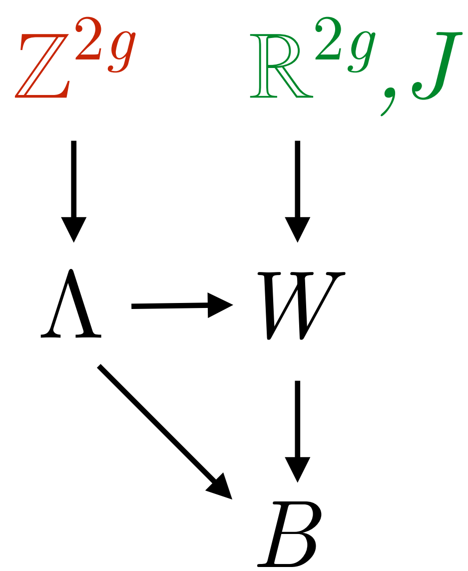

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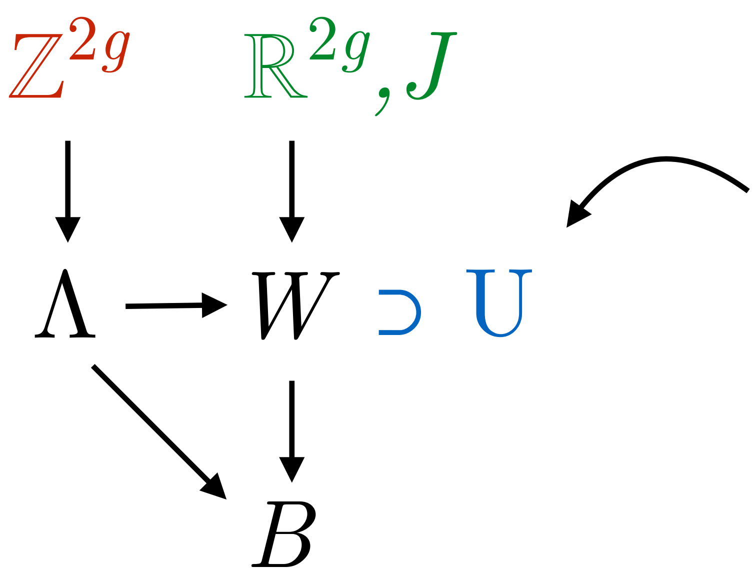

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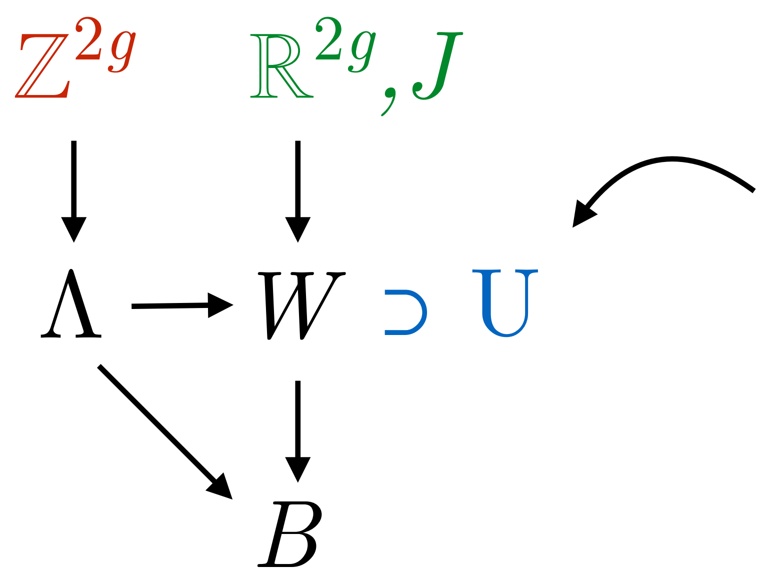

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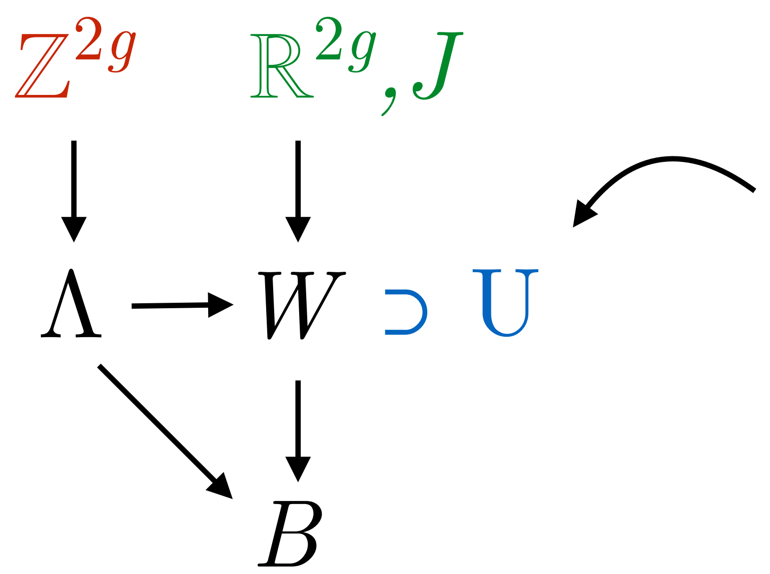

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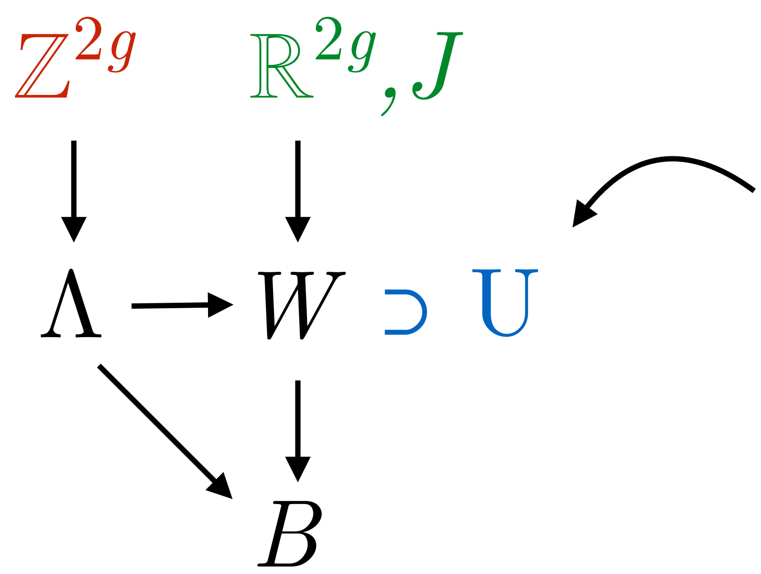

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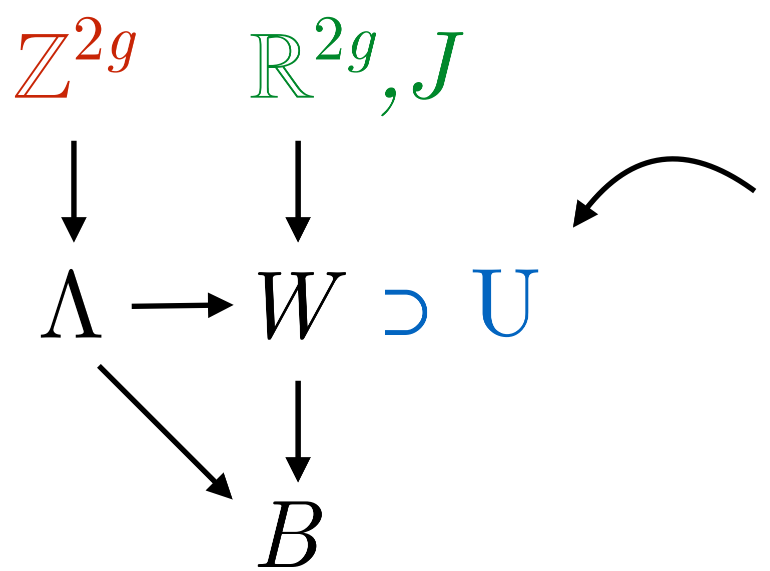
Theorem. There are $\mathrm{SO}_{g,g}(\mathbb{Z})$ bundles $E \to B^g$ where these characteristic classes are nonzero.

Characteristic class construction



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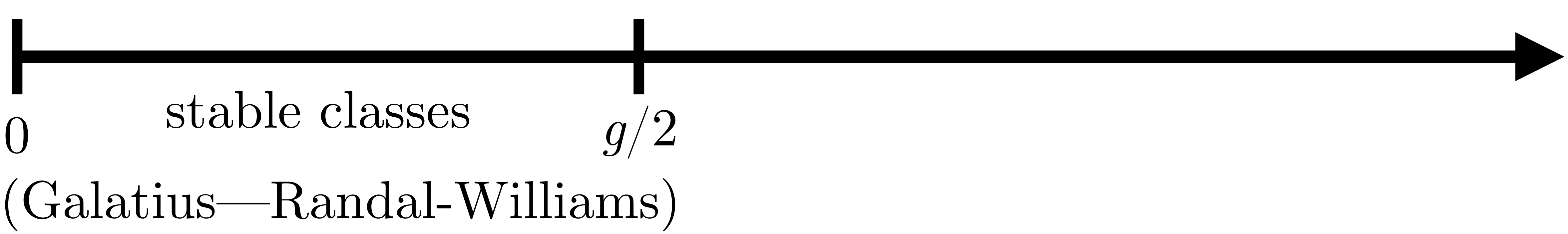

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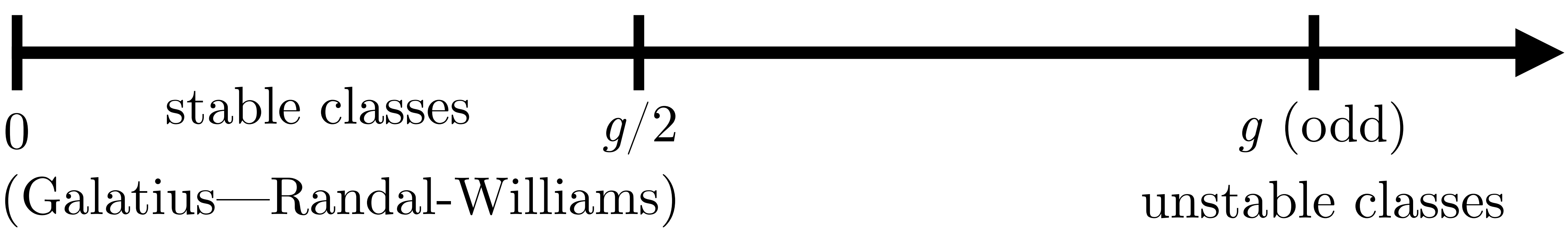

Application 2

Cohomology in the mapping class group of a K3 surface.

$$M \text{ K3 surface}, \; M \simeq \{ \; x^4 + y^4 + z^4 + w^4 = 0 \; \} \subset \mathbb{C}\mathrm{P}^3$$

$$\mathrm{Diff}(M) \to \mathrm{SO}_{3,19}(\mathbb{Z})$$

Input: Global Torelli theorem for Einstein metrics.

Further direction

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Thank you.

An enriched count of bitangents to a smooth plane quartic

(based on joint work with Hannah Larson)

Isabel Vogt

Stanford University

June 12, 2019

Hannah Larson ⟶

(demonstrating types of lines)

Thanks to Kirsten Wickelgren, Jesse Kass, and AWS!

⟵ cubic surface

or: How I learned to stop worrying and "love" the lack of orientations

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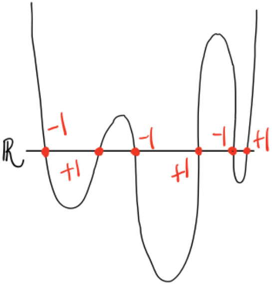
Warmup:

Signed count of real zeros of a real polynomial

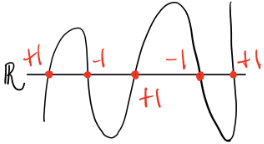
even degree:



signed count $= 0$

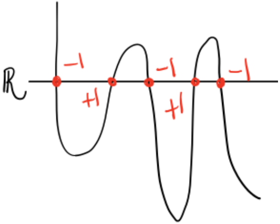
odd degree:

leading coefficient positive



signed count $= +1$

leading coefficient negative



signed count $=$ -1

The $\mathbb{A}^1$-enumerative package for bitangents (after Kass-Wickelgren)

- $X = \{(L, Z) : Z \subset L \subset \mathbb{P}^2$, degree 2 subscheme of a line$\}$
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Hope

$$\sum_{(L,Z) \text{ zero of } \sigma_f} \operatorname{ind}_{(L,Z)} \sigma_f = \text{fixed count in } \mathrm{GW}(k)$$

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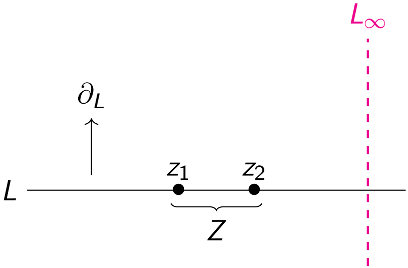
A new hope

Fix any L_∞ in $\mathbb{P}^2_k$, then if σ_f has no zeros in D_∞, can we understand

$$\sum_{(L,Z) \text{ zero of } \sigma_f} \operatorname{ind}^{L_\infty}_{(L,Z)} \sigma_f \in \mathrm{GW}(k)?$$

Geometric information in $\mathrm{ind}_{(L,Z)}^{L_\infty}\sigma_f$:

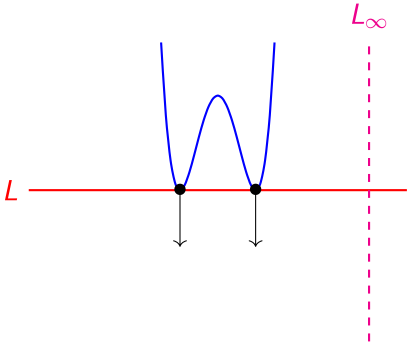
- ∂_L is a derivation determined by L
- f some affine equation for the quartic in $\mathbb{P}^2 \setminus L_\infty = \mathbb{A}^2$



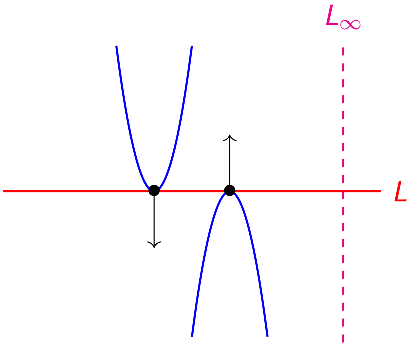
Define the type of L:

$$\mathrm{Qtype}_{L_\infty}(L) := \mathrm{ind}_{(L,Z)}^{L_\infty}\sigma_f = \langle \partial_L f(z_1) \cdot \partial_L f(z_2) \rangle$$

Over $\mathbb{R}$:



$\mathrm{Qtype}_{L_\infty}(L) = \langle 1 \rangle$



$\mathrm{Qtype}_{L_\infty}(L) = \langle -1 \rangle$

A new hope

Fix any L_∞ in $\mathbb{P}^2_k$, then if σ_f has no zeros in D_∞, can we understand

$$\sum_{(L,Z) \text{ zero of } \sigma_f} \operatorname{ind}^{L_\infty}_{(L,Z)} \sigma_f \in \mathrm{GW}(k)?$$

Theorem (Hannah Larson-V.)

Let L_∞ be a **bitangent** *of the quartic Q. Relative to this,*

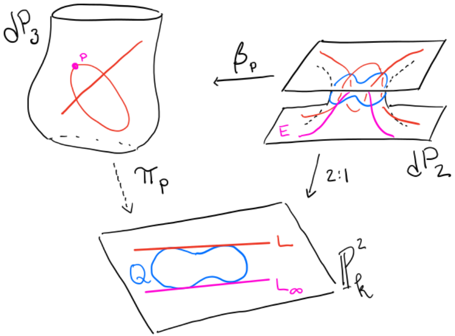
$$\sum_{\substack{\text{lines } L \text{ bitangent to } Q \\ L \neq L_\infty}} \operatorname{Tr}_{k(L)/k} \operatorname{Qtype}_{L_\infty}(L) = 15\langle 1\rangle + 12\langle -1\rangle \in \operatorname{GW}(k).$$

Proof Sketch:

27 lines

signed count:

$15\langle 1\rangle + 12\langle -1\rangle$



56 lines

27 bitangents $+ L_\infty$

signed count rel to L_∞:

$15\langle 1\rangle + 12\langle -1\rangle$

What about other choices of L_∞?

- When $k = \mathbb{R}$, compute

$$\sum_{\text{lines } L \text{ bitan to } Q} \mathrm{Tr}_{\mathbb{R}(L)/\mathbb{R}}\, \mathrm{Qtype}_{L_\infty}(L)$$

  for *all* possible choices of L_∞

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- It seems to always be one of:

$$\begin{aligned} &18\langle 1\rangle + 10\langle -1\rangle, \\ &17\langle 1\rangle + 11\langle -1\rangle, \\ &16\langle 1\rangle + 12\langle -1\rangle, \\ &15\langle 1\rangle + 13\langle -1\rangle, \\ &14\langle 1\rangle + 14\langle -1\rangle \end{aligned}$$

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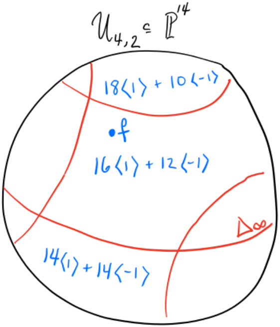
$$14\langle 1\rangle + 14\langle -1\rangle$$



$\Delta_\infty =$ {quartics with bitangent along L_∞}

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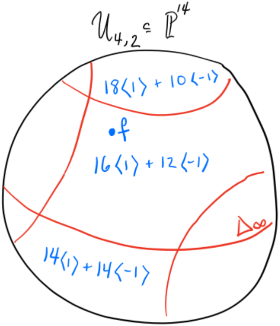

$\Delta_\infty =$ {quartics with bitangent along L_∞}

WHY??