Z^d Group Shifts

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1. Vocabulary

These are definitions for this talk.

An algebraic action is an action of Z^d on a compact metrizable group by continuous automorphisms. It has completely positive entropy (is cpe) if it and every nontrivial factor has positive entropy. (The topological condition in this case is equivalent to the measurable condition with respect to Haar measure.)

A group shift is an expansive algebraic action on a zero-dimensional group.

A conjugacy or factor map unless indicated is topological. An algebraic conjugacy or factor map is one with the additional property of being a group homomorphism.

A Bernoulli group shift is the Z^d shift on G^{Z^d}, for some finite group G (the alphabet group). Up to algebraic conjugacy, every group shift is the shift on a closed invariant subgroup of a Bernoulli group shift. Usually we take this presentation without comment. Every Z^d group shift is a Z^d shift of finite type (Kitchens-Schmidt).

A Z^d action on X may be denoted by α, and $v \in Z^d$ acts by a map denoted α^v. We often suppress α, e.g. writing its Z^d entropy as $h(X)$.

A group shift is abelian if its domain is abelian.

p always denotes a prime number.

2. Main results

THEOREM 1 Every group shift has an equal entropy Bernoulli group as a topological factor.

THEOREM 2 Every abelian group shift has an equal entropy Bernoulli group shift as an algebraic factor. This factor x is unique up to isomorphism of the alphabet group.

Remark: although this Bernoulli is canonical, we do not have a canonical factor map to it.

Remark: For $d > 1$ there are Z^d shifts of finite type of arbitrarily large entropy which do not have any positive entropy Bernoulli shift as a factor.

COROLLARY Every finite entropy algebraic Z^d action has a canonical maximum entropy Bernoulli group shift as an algebraic factor.

THEOREM 3 Every cpe group shift is weakly algebraically equivalent to a Bernoulli group shift (i.e., each is an algebraic factor of the other).

"COROLLARY" An easier proof in dimension zero of the Rudolph-Schmidt result that a cpe algebraic Z^d system is measurably (w.r.t. Haar measure) Bernoulli:

By theorem 3, a group shift is a measurable (since cont.) factor of a Bernoulli. A cpe algebraic system is an inverse limit of cpe group shifts and thus an inverse limit of Bernoullis, hence Bernoulli.

We have other technical results and counterexamples regarding the issue of when the Pinsker factor splits.

3. Our methods

We use little of the sophisticated Laurent module/duality theory for algebraic Z^d actions. Why?

– The possible value of another viewpoint.

– In dimension zero, we can see some things more simply.

We use coding arguments and homoclinic points (developed in the Lind-Schmidt JAMS paper and others).

4. Homoclinic points

DEFN A point x is a homoclinic point for a Z^d action α if $\alpha^v(x) \to \text{id}$ as $||v|| \to \infty$.

For x in a group shift, with e the identity in the alphabet group: x is a homoclinic point iff $x(v) = v$ for all but finitely many $v \in Z^d$.

Let Δ_X denote the group of homoclinic points of X. Lind and Schmidt used Fourier analysis to prove for abelian algebraic Z^d actions the following:

1. $h(X) > 0$ iff Δ_X is nontrivial

2. $h(X) = h(\overline{\Delta_X})$

3. The Pinsker factor map is the algebraic factor map with kernel $\overline{\Delta_X}$.

4. X is cpe iff Δ_X is dense.

For group shifts: these Lind-Schmidt results have elementary proofs which do not use Fourier analysis. The proofs work for nonabelian group shifts.

Remark: also in the possibly nonabelian case: an algebraic factor of a group shift is a group shift.

5. Entropy

Addition formula (Yuzvinskii; Lind-Schmidt-Ward). For an algebraic factor map $X \to Y$ with kernel K, we have $h(X) = h(Y) + h(K)$.

(Mostly Einsiedler-Schmidt) Suppose γ is an algebraic factor map between group shifts. TFAE:

1. $h(\ker(\gamma)) = 0$.
2. $h(X) = h(Y)$
3. For every closed shift-invariant subgroup W of X, $h(\gamma W) = h(W)$.
4. The only homoclinic point in $\ker(\gamma)$ is the identity.

5. Proof of Theorem 1

DEFN Let X be a group shift with alphabet group G and e the identity in G. For x in X, define
$F(x) = \{y(0) : y(v) = x(v) \text{ for all } v \prec 0\}$,
where $\prec$ is lexicographic order on Z^d.

Well known:
– $F(\text{id})$ is a normal subgroup of G.
– Every $F(x)$ is a coset of $F(\text{id})$.
– $h(X) = \log |F(\text{id})| = h(B)$ (redone below)
(so, if $pX = 0$, then $h(X) \in \{0, p\}$).
– By general meas.th. entropy theory:
$h_\mu(X) \leq \log |F(\text{id})|$
for any invariant Borel probability μ.

For each $F(\text{id})$ coset $[g]$ in G, choose a bijection $\beta : [g] \to F(\text{id})$. Define a one-block code ϕ from X to the Bernoulli shift B with alphabet $F(\text{id})$ by the rule $\phi(g) = \beta([g])$. It is not hard to check that ϕ is surjective. So $h(X) = \log |F(\text{id})| = h(B)$. QED

There are group shifts X such that no map ϕ constructed in this way can be a group homomorphism.

7. Kitchens Theorem

Theorem (Kitchens 1987) If X is a Z group shift, then X is topologically conjugate to a product of a Bernoulli shift B and a shift P on a finite set. (Here P is the Pinsker factor.)

That is a pretty decisive classification theorem.

If $h(X) = \log n$ with n square free, then the conjugacy $X \to B$ above can be made algebraic. For n not square free this fails badly (Kitchens; Fagnani; Schmidt).

What about Z^d, $d > 1$?

All those Z group shift top-but-not-alg conjugate examples give rise to Z^d examples via a construction of Hochman and Meyerovitch.

Kitchens Example: a Z^2 cpe group shift X with entropy log 2, with $2X = 0$ and with four fixed points. So, X is not top. conjugate to the full Z^2 two shift S. (Contrast Z.)

To show this X is cpe, Kitchens gave an algebraic argument (torsion free dual module). Alternately (B-S) one can check that the homoclinic points are dense.

There are easier examples ("USAir"). Here is an infinite family of pairwise not conjugate (by periodic point counts) cpe Z^2 group shifts of entropy log 2.

For a positive integer k define maps $S \to S$ by
$\phi_k(x) = x - \sigma^{2^{k-1}(1,0)}(x)$,
$\psi_k(x) = x - \sigma^{2^{k-1}(0,1)}(x)$
and set
$\gamma_k = \phi_k \times \psi_k, \quad X_k =$ image of γ_k.

The kernel of γ_k is finite, so X_k has entropy log 2. Can check: γ_k maps Fix($2^k Z^2$) onto Fix($2^{k-1}Z^2$) x Fix($2^{k-1}Z^2$) := F_k, and F_j is not contained in X_k if $j > k$.

For example, X_1 has $2 \times 2 = 4$ fixed points, while the 2-shift has two.

Proof sketch for Theorem 2

For n a positive integer, $B(n)$ denotes a Bernoulli group shift with alphabet Z/n, with Z/n given as the subgroup of R/Z generated by $1/n$.

Reduce to case X is a p-group, a subgroup of $B := B_1 \times \cdots \times B_r$ where $B_i = B(p^{n_i})$.

If X is not B: choose $f = (f_1, ..., f_r)$ with each $f_i \in Z[u_1^{\pm 1}, \ldots, u^{\pm d}]$ and associated factor map π_f from X into $(R/Z)^{Z^d}$ which annihilates X but not B. Let $f_i = 0$ if f_i annihilates B_i. Set $f = p^r g = p^r(g_1, ..., g_r)$ with r nonnegative and say g_1 not identically zero mod p. Define factor map ϕ on X by
$x = (x_1, ..., x_r) \mapsto (px_1 - \pi_g(x), x_2, \ldots, x_r)$. Now $\phi X < pB_1 \times B_2 \times \cdots \times B_r$ and $h(\phi X) = h(X)$. Continue until image is Bernoulli. QED

Uniqueness of the Bernoulli equal entropy factor follows from consideration of the entropies of the subgroups $pB, p^2B, p^3B, ...$

Proof of Theorem 3

Let X be a cpe group shift. We have already built an algebraic factor map $\phi : X \to B$, with $h(X) = (B)$.

We need to show a cpe group shift X is the image of an equal entropy Bernoulli group shift B by an algebraic factor map.

PROOF 1

A part of Einsiedler-Schmidt: given an algebraic action Y, let Y^* be the algebraic action which is dual to the discrete Laurent module which is the homoclinic group of Y. We know that ϕ is a module monomorphism from Δ_X into Δ_B). By duality we get B^* mapping onto X^*. E-S show generally X^* maps algebraically onto X. So B^* maps algebraically onto X. But B^* is alg. conjugate to B.

PROOF 2

"Just" take the natural generators of the homoclinic group of B, and define a module monomorphism ψ into the homoclinic group of X. Such a ψ extends to a factor map $B \to X$.

This works but the proof becomes a little messy, at least in our hands.

Problems

1. Classify cpe abelian group shifts up to topological conjugacy. (Reduce to the case the group shift X is a p shift and $pX = 0$.)

2. Can one understand the periodic point possibilities for cpe Z^2 group shifts of entropy log 2? If cpe abelian group shifts have the same periodic point counts, must they be top. conjugate? If one is a full shift on 2 symbols?

3. The only result we know in this area for nonabelian group shifts is the existence of the equal entropy Bernoulli factor.

So, for example: must a nonabelian group shift have an equal entropy algebraic Bernoulli factor?