Sandpiles and Markov Partitions

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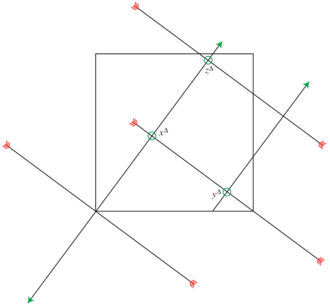
Consider the hyperbolic automorphism $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We can draw its expanding and contracting subspaces:



The intersections of these two subspaces are the homoclinic points of α.

We shall see that not all homoclinic points are created equal.

Maps Defined By Homoclinic Points

If x is one of these homoclinic points, (e.g., $x = y^{\Delta}$), then we can define a map $\xi_x \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathbb{T}^2$ by

$$\xi_x(v) = \sum\nolimits_{n \in \mathbb{Z}} v_n \alpha^{-n} x, \;\; v = (v_n) \in \ell^\infty(\mathbb{Z}, \mathbb{Z}).$$

The map ξ_x is equivariant:

$$\xi_x \circ \sigma = \alpha \circ \xi_x,$$

where σ is the shift $(\sigma v)_n = v_{n+1}$ on $\ell^\infty(\mathbb{Z}, \mathbb{Z})$.

Sofic Covers and Sofic Partitions

In 1992–94 Vershik showed that the restriction of ξ_x to the two-sided beta-shift $X_\beta \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ of the large eigenvalue $\beta = \frac{1+\sqrt{5}}{2}$ of α is surjective.

Since X_β is the Golden Mean shift (which is of finite type), this is a Markov cover of α.

Does this give a Markov partition (i.e., is ξ_x almost one-to-one on X_β)?

The answer depends on the homoclinic point chosen: it has to be fundamental in the sense that its orbit generates the group $\Delta(\alpha) \subset \mathbb{T}^2$ of all homoclinic points of α (Lind-S, 1999; Einsiedler-S, 1997). In our drawing the points x^Δ and y^Δ are fundamental, but z^Δ is not.

This connection between two-sided beta-shifts and hyperbolic toral automorphisms was extended to general quadratic Pisot numbers (Sidorov-Vershik, 1998), then to arbitrary Pisot numbers (S, 2000; Sidorov, 2001–02).

The General Recipe For This Construction

- Take a hyperbolic automorphism $\alpha \in \mathrm{GL}(n, \mathbb{Z})$;
- Find a fundamental homoclinic point $x^{\Delta} \in \mathbb{T}^n$ of α (for this α has to be conjugate — in $\mathrm{GL}(n, \mathbb{Z})$ — to a companion matrix; Lind-S, 1999);
- Define $\xi_{x^{\Delta}} \colon \ell^{\infty}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathbb{T}^n$ as before by $\xi_{x^{\Delta}}(v) = \sum_{n \in \mathbb{Z}} v_n \alpha^{-n} x^{\Delta}$;
- Find a full shift with sufficiently large alphabet $V \subset \ell^{\infty}(\mathbb{Z}, \mathbb{Z})$ such that the restriction of $\xi_{x^{\Delta}}$ to V is surjective.
- Use Marcus-Petersen-Williams (1984) to a sofic shift $V^* \subset V$ such that $\xi_{x^{\Delta}}$, restricted to V^* is surjective and finite-to-one (Kenyon-Vershik, 1998; S, 2000);
- Try to get this restriction to be almost one-to-one (S, 2000).

The Two-Dimensional Sandpile Model: Finite Volume

Let $\Lambda \subset \mathbb{Z}^2$ be a nonempty finite set ($\Lambda \Subset \mathbb{Z}^2$).

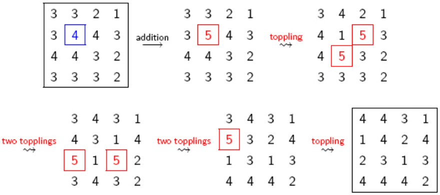
- A configuration on Λ is an element of $\mathbb{N}^\Lambda$, where $\mathbb{N} = \{1, 2, 3, \dots\}$;
- A configuration $y \in \mathbb{N}^\Lambda$ is stable if $y_{\mathbf{n}} \leq 4$ for every $\mathbf{n} \in \Lambda$;
- If $y \in \mathbb{N}^\Lambda$ is unstable at a site $\mathbf{n} \in \Lambda$ (i.e., if $y_{\mathbf{n}} > 4$), then the site $\mathbf{n}$ topples: $y \to y' = T_{\mathbf{n}} y$, where $y'_{\mathbf{n}} = y_{\mathbf{n}} - 4$, and $y'_{\mathbf{m}} = y_{\mathbf{m}} + 1$ for $\mathbf{m} \in \Lambda$ with $\|\mathbf{m} - \mathbf{n}\|_1 = 1$. If $y_{\mathbf{n}} \leq 4$ then $T_{\mathbf{n}} y = y$.

Lemma (Dhar): The toppling operators $T_{\mathbf{n}}$, $\mathbf{n} \in \Lambda$, commute.

We set $\mathcal{T}_\Lambda = \lim_{n \to \infty} \prod_{\mathbf{n} \in \Lambda} T_{\mathbf{n}}$. Then $\mathcal{T}_\Lambda \colon \mathbb{N}^\Lambda \longrightarrow \mathbb{N}^\Lambda$ is well-defined.

For $y \in \mathbb{N}^\Lambda$, $\mathcal{T}_\Lambda(y)$ is the stabilization of y.

Addition:



Under addition and stabilization the stable config's form a semigroup S_Λ.

The group property

Definition: The set $\mathcal{R}_\Lambda \subset \mathcal{S}_\Lambda$ of recurrent configurations is the unique maximal subgroup of $\mathcal{S}_\Lambda$.

Description of $\mathcal{R}_\Lambda$: For every $E \subset \Lambda$ and $\mathbf{n} \in E$ we denote by $\mathsf{N}_E(\mathbf{n})$ the number of neighbours of $\mathbf{n}$ in E. Put

$$\mathcal{P}_E = \{v \in \mathcal{S}_\Lambda : v_{\mathbf{n}} > \mathsf{N}_E(\mathbf{n}) \text{ for at least one } \mathbf{n} \in E\}.$$

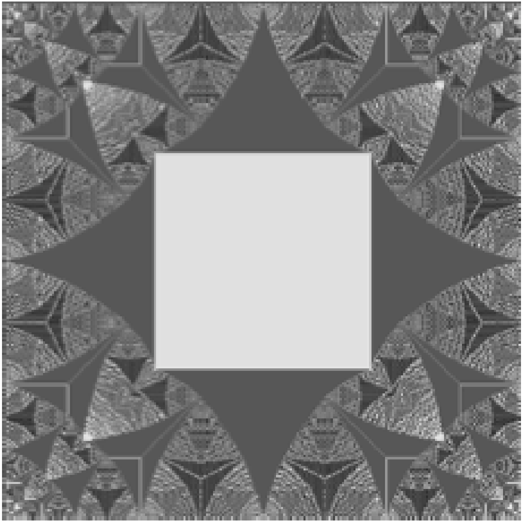
Then

$$\mathcal{R}_\Lambda = \bigcap\nolimits_{E \subset \Lambda} \mathcal{P}_E.$$

Note that $\pi_\Lambda(\mathcal{R}_{\Lambda'}) \subset \mathcal{R}_\Lambda$ whenever $\Lambda \subset \Lambda' \Subset \mathbb{Z}^2$.

Are the group operations in $\mathcal{R}_\Lambda$ and $\mathcal{R}_{\Lambda'}$ compatible? Unfortunately not!

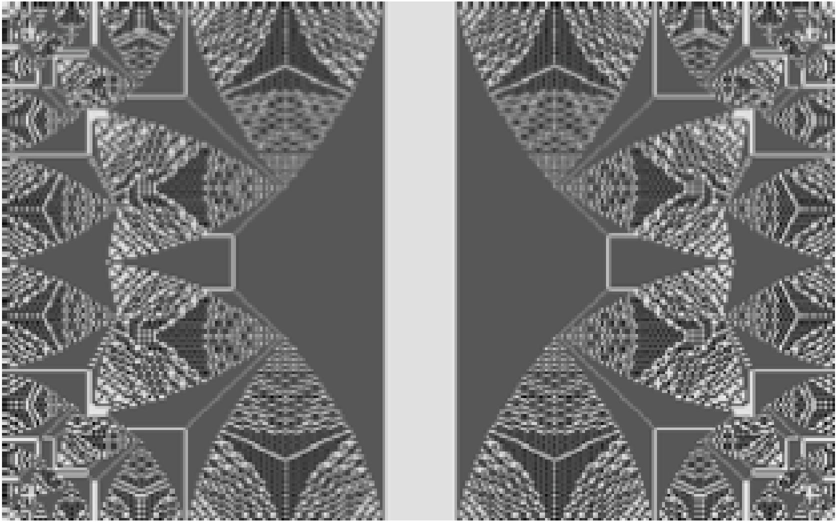
A Neutral Element Of $\mathcal{R}_\Lambda$



The identity for $\Lambda = 500 \times 500$. Dark grey = 4, light grey = 3, medium grey = 2, black = 1

(Le Borgne-Rossin, 2002)

Another Neutral Element



The identity for $\Lambda = 298 \times 198$. Dark grey = 4, light grey = 3, medium grey = 2, black = 1

(Le Borgne-Rossin, 2002)

From Finite to Infinite Volume

Since $\pi_\Lambda(\mathcal{R}_{\Lambda'}) \subset \mathcal{R}_\Lambda$ whenever $\Lambda \subset \Lambda' \Subset \mathbb{Z}^2$ we can define a closed shift-invariant subset

$$\mathcal{R}_\infty = \{v \in \{1, \dots, 2d\}^{\mathbb{Z}^2} : \pi_\Lambda(v) \in \mathcal{R}_\Lambda \text{ for every } \Lambda \Subset \mathbb{Z}^2\},$$

called the 2-dimensional critical sandpile model.

- *Is $\mathcal{R}_\infty$ a group?*
- *What are the dynamical properties of the shift-action of $\mathbb{Z}^2$ on $\mathcal{R}_\infty$?*
- The topological entropy of this-shift action is known: it is given by

$$h(\alpha) = \int_0^1 \int_0^1 \log\left(4 - cos(2\pi t_1) - cos(2\pi t_2)\right) dt_1 dt_2.$$

The Harmonic Model

Let α be the shift-action on the closed shift-invariant subgroup

$$X = \left\{ (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^2} : 4x_{\mathbf{n}} = x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}-(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}-(0,1)} \text{ for all } \mathbf{n} \right\}$$

of $\mathbb{T}^{\mathbb{Z}^2}$.

Theorem: α is nonexpansive and Bernoulli with entropy

$$h(\alpha) = \int_0^1 \int_0^1 \log\left(4 - cos(2\pi t_1) - cos(2\pi t_2)\right) dt_1 dt_2.$$

The Haar measure λ_X is the unique shift-invariant measure of maximal entropy on X.

Definition. The action α on the compact connected abelian group X (with Borel field $\mathcal{B}_X$ and Haar measure λ_X) is the *2-dimensional harmonic model*, determined by the Laurent polynomial $f = 4 - u_1 - u_2 - u_1^{-1} - u_2^{-1}$.

Symbolic Covers Of The Harmonic Model

Let $X \subset \mathbb{T}^{\mathbb{Z}^2}$ be the harmonic model with shift-action α. We call a point $y \in X$ homoclinic if

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} |y_{\mathbf{n}}| < \infty,$$

The set $\Delta_1(X) \subset X$ of all homoclinic points is a countable dense subgroup of X.

Fix a nonzero point $y \in \Delta_1(X)$ and define an equivariant map $\xi_y \colon L^\infty(\mathbb{Z}^2, \mathbb{Z}) \longrightarrow X$ by

$$\xi_y(v) = \sum_{\mathbf{n} \in \mathbb{Z}^2} v_{\mathbf{n}} \alpha^{-\mathbf{n}} y$$

for every $v \in L^\infty(\mathbb{Z}^2, \mathbb{Z})$. Although this map depends on the point y is can be shown that it is always surjective.

A closed, bounded, shift-invariant subset $V \subset L^\infty(\mathbb{Z}^2, \mathbb{Z})$ is an equal entropy symbolic cover of X if $\xi_y(V) = X$ and $h(\sigma_V) = h(\alpha)$, where σ_V is the shift-action of $\mathbb{Z}^2$ on V.

This definition does not depend on the homoclinic point $y \in \Delta_1(X) \smallsetminus \{0\}$.

Construction Of Homoclinic Points: A Green Function

We denote by ν the equidistributed probability measure on the set $\{\pm(1,0), \pm(0,1)\} \subset \mathbb{Z}^2$. For every $m \geq 1$ we set

$$\phi^{(m)} = \frac{1}{4} \cdot \sum_{k=0}^{m} \nu^k$$

and

$$\psi^{(m)} = \phi^{(m)} - \phi^{(m)}(0).$$

Then

$$w^{\Delta} := \lim_{m \to \infty} \psi^{(m)}$$

converges pointwise. The resulting map $w^{\Delta} \colon \mathbb{Z}^2 \longrightarrow \mathbb{R}$ has the following properties.

- w^{Δ} is unbounded;
- $w^{\Delta} = \frac{1}{4} \cdot G_{\nu}$, where G_{ν} is the Green function of the random walk with one-step transition law ν;
- $(w^{\Delta} \cdot f)_{\mathbf{n}} = (f \cdot w^{\Delta})_{\mathbf{n}} = \begin{cases} 1 & \text{if } \mathbf{n} = \mathbf{0}, \\ 0 & \text{otherwise,} \end{cases}$ i.e., $w^{\Delta} = f^{-1}$.

L^1-Multipliers Of The Green Function

We set $\mathcal{I} = \{h \in R_2 = \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}] : h \cdot w^\Delta \in L^1(\mathbb{Z}^2)\}$.

Theorem. $\mathcal{I} = f \cdot R_2 + (u_1 - 1)^3 \cdot R_2 + (u_2 - 1)^3 \cdot R_2$.

Define $x^\Delta \in X$ by setting

$$x^\Delta_{\mathbf{n}} = w^\Delta_{\mathbf{n}} \pmod 1, \;\; \mathbf{n} \in \mathbb{Z}^2.$$

Theorem. The point x^Δ is not homoclinic, but $\Delta_1(\alpha) = \{h(\alpha)(x^\Delta) : h \in \mathcal{I}\}$, where

$$h(\alpha) = \sum\nolimits_{\mathbf{n} \in \mathbb{Z}^2} h_{\mathbf{n}} \alpha^{\mathbf{n}} \;\text{ for every }\; h = \sum\nolimits_{\mathbf{n} \in \mathbb{Z}^2} h_{\mathbf{n}} u^{\mathbf{n}} \in R_2.$$

Sandpiles Cover The Harmonic Model

Theorem. For every nonzero $y \in \Delta_1(\alpha)$, $\xi_y(\mathcal{R}_\infty) = X$ and $h(\sigma_{\mathcal{R}_\infty}) = h(\alpha)$. In other words, $\mathcal{R}_\infty$ is an equal entropy symbolic cover of α with covering map ξ_y (Verbitskiy-S, 2008).

Problem. When is the restriction of ξ_y to $\mathcal{R}_\infty$ almost injective?

The conjecture is that by taking all such maps simultaneously one obtains an almost injective covering map from $\mathcal{R}_\infty$ to a certain (explicitly given) quotient of X by an α-invariant subgroup $Y \subset X$.

The Dissipative Sandpile Model

For the dissipative sandpile model, where the toppling operator $T_{\mathbf{n}}$ removes, e.g., 5 grains of sand at the site $\mathbf{n}$ and deposits one grain at each of the four neighbours, one can show that this sandpile model is an almost one-to-one symbolic cover of the group

$$X' = \left\{ (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^2} : 5x_{\mathbf{n}} = x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}-(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}-(0,1)} \ \text{ for all } \mathbf{n} \right\}.$$

The shift-action α' of $\mathbb{Z}^2$ on X' is expansive and has a genuine fundamental homoclinic point, which makes life (and proofs) *much* easier than in the critical case.