π in the Sky

300 years of Euler

Mathematician and Scholar

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Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

Contributions Welcome

Pi in the Sky accepts materials on any subject related to mathematics and its applications, including articles, problems, cartoons, statements, and jokes. Copyright of material submitted to the publisher and accepted for publication remains with the author, with the understanding that the publisher may reproduce it without royalty in print and electronic forms. Submissions are subject to editorial review and revision. We also welcome Letters to the Editor from teachers, students, parents, and anybody interested in math education (be sure to include your full name, phone number and e-mail address).

On the Cover

The Hermitage museum in St Petersburg, Russia, with portrait of Leonhard Euler.

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Editorial

Mathematical Research: Invention and Discovery

by Florin Diacu, Guest Editor
University of Victoria

This issue of *Pi in the Sky* celebrates the work of Leonhard Euler—the most prolific mathematician ever. At his death in 1783, he left so many unpublished manuscripts behind that it took more than half a century to print them. The ongoing publication of his collected works, which started in 1910, will fill an estimated 80 large quarto volumes. His research touched most branches of pure mathematics, from number theory to calculus, and dealt with applications in various fields, some as far apart from each other as music theory and celestial mechanics.

But what is mathematical research, and how is it done? Many people are surprised to learn that such an enterprise exists. One of them asked me once: "Are you reinventing the numbers, or what?" In his view, our only job was to teach students, and was appalled that we lectured only a handful of hours per week. He didn't know how fully committed we are to finding new theorems, often sparing no nights or weekends in search of results.

In a way, our work resembles what students do. Like them, we use our knowledge and skills to answer mathematical questions. We follow logical reasoning, employ methods, and apply computational techniques to achieve our goals. But here the analogy ends. Unlike students, we cannot check the answer key to our unsolved problems. Often we have no clue how to approach a question or get stuck in every direction we take. Sometimes we have to invent novel methods because the old ones don't help.

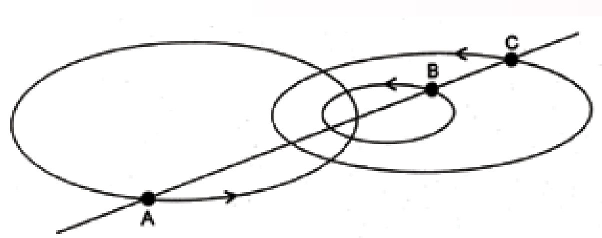
Most problems we attempt are hard. We may need weeks or months to make progress with them. If the results are significant enough, we publish them, thus giving others a chance to take our work further. Some questions, however, are so difficult that only collective efforts of years or decades lead to answers. And there are problems that have eluded our understanding for centuries.

Leonhard Euler in 1753–as portrayed by the Swiss painter Emanuel Handmann.

One of the most celebrated unsolved mathematical questions is the 3-body problem of celestial mechanics, which Isaac Newton posed in 1686. Given three point masses that move in space under the influence of gravity—the problem states—determine their trajectories for any choice of initial positions and velocities. Many famous mathematicians contributed to this question, Euler among them (see the pink box), but after more than three centuries and thousands of published papers, this problem continues to elude our understanding. Still, what we achieved so far allows us to compute the orbits of planets, asteroids, and comets, and to design missions in space. From the theoretical point of view, this problem led to the creation of new branches of mathematics, such as algebraic topology and dynamical systems, and to the discovery of chaos, an intriguing mathematical property encountered in many time-evolving phenomena, the weather and the stock market among them.

Like the 3-body problem, other questions inspire researchers and lead to the development of mathematics, a world open to everyone who has the desire and strength to enter it. No doubt, the young readers of this magazine are eager to explore realms of thought nobody has conquered before. I trust that, someday, many of them will.

The Eulerian solution of the 3-body problem



In 1767, Euler proved that if one assigns suitable initial positions and velocities to 3 point-masses, A, B, C, among which gravitation acts, they move forever along ellipses without leaving a rotating straight line.

Euler and the Gamma Function

by Tom Archibald
Simon Fraser University

The *factorial* function is defined for the positive integers as follows:

$$n! = n \times (n \times 1) \times (n \times 2) \times \ldots\ 2 \times 1.$$

Aside from being useful in counting, this function has interesting characteristics. For one thing it grows very fast. Leonhard Euler (1707-1783) thought that it might be useful to try to extend it to values other than positive integers. There are various ways that you might be able to think of doing this, but Euler wanted a *continuous* function—one whose graph is an unbroken curve that shared the correct values at the positive integers. He also wanted it to have the "factorial" property everywhere, not just at the integers: For *every* x,

$$F(x+1) = (x+1)F(x).$$

Eventually, Euler came up with a definition that is still used now, for a function called the gamma function: the notation is $\Gamma(x)$.

The gamma function is usually defined today using an integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

In what follows we will see how this relates to Euler's way of defining it.

Euler was Swiss, born in Basel. His father was a successful pastor, and encouraged Leonhard to study theology. On entering university at the age of 14, Euler gravitated toward mathematics instead, and got private tutoring from Johann Bernoulli, one of the best mathematicians then alive. He began to publish papers in 1726, with one early paper being on the best placement for masts on a ship. By this time his great talent was well-known, as was his amazing memory: Euler, for example, knew the first six powers of the first hundred prime numbers. Euler moved to St. Petersburg in 1727, joining his mentor's son Daniel Bernoulli at the newly founded Imperial Academy of Sciences, as an adjunct in physiology. He worked only in mathematics, mechanics and physics, and soon had a regular position.

In 1729, Euler (then 23 years old) was working in St. Petersburg at the Academy of Sciences. He wrote a letter to Christian Goldbach, in Moscow, in which his ideas about the Gamma function were first expressed. Goldbach had published several papers on general terms of series, and it was natural that Euler would communicate his results to Goldbach. He gave his initial definition of the function that coincides with the factorials in a letter of Oct. 13:

> For the sequence 1, 2, 6, 24, 120 etc., which I know has been extensively treated by you, I have found the general term
>
> $$\frac{1 \cdot 2^m}{1+m}\,\frac{2^{1-m} \cdot 3^m}{2+m}\,\frac{3^{1-m} \cdot 4^m}{3+m}\,\frac{4^{1-m} \cdot 5^m}{4+m} \text{ etc.}$$
>
> consisting of an infinite number of factors, which expresses the term of order *m*. (Euler in Juskevic and Winter 1965, p. 19. My translation.)

Before figuring out what this has to do with the factorials, though perhaps you can see it already, let us notice that this expression is defined for any *m*–value except at the negative whole numbers.

To see the relation to the factorials, let us rearrange it a little. Write the same expression in the form

$$F(m) = \left(\frac{2}{1}\right)^m \cdot \frac{1}{m+1} \cdot \left(\frac{3}{2}\right)^m \cdot \frac{2}{m+2} \cdot \left(\frac{4}{3}\right)^m \cdot \frac{3}{m+3} \ldots.$$

In this form, you see that the numerator of one factor of the form $\left(\frac{k+1}{k}\right)^m$ cancels with the denominator of the next factor, of the form $\left(\frac{k+2}{k+1}\right)^m$, and we have a 'telescoping product'. Euler would have thought of this as cancelling all the way to infinity. Since, after that cancellation, the denominator is $(m+1)(m+2)(m+3)\ldots$, we see that all that is left in the numerator after cancellation is $m!$. Write this out for a few small values to get the idea.

We notice that Euler is freely rearranging this infinite product. Nowadays we have strict rules for when infinite sums and products can be rearranged. Euler did not formulate these rules explicitly, though he had a very good understanding of when such procedures would work and when they would not, even if he did not always explain it the way we would do now.

Euler pointed out that you can also look at the *partial* product—what you get after the first n steps—as

$$\frac{n!}{(1+m)(2+m)\ldots(n+m)}(n+1)^m,$$

and again you might want to write this out for some small values to convince yourself that the regrouping works. It means that the function you really want is the limit of this value as n gets as big as you like.

Now the key thing is to see that in fact we have the factorial property, that is, $F(m+1)=(m+1)F(m)$, if you accept the idea that rearrangement is allowed (as Euler did). ***Exercise***: Write the expressions on the left and the right of this equation, using the form given for $F(m)$ above, and convince yourself that if you allow rearrangements the two sides are equal.

Having defined the function, Euler noted in his letter that evaluating it at non-integer values can be tricky. For example, what would be the term of order 1/2? Euler's letter to Goldbach would have been hard for Goldbach to understand, and I will quote it first before explaining it a little:

> Moreover, the term of order 1/2 is found to be equal to this:
>
> $$\frac{1}{2}\sqrt{\sqrt{-1}\log(-1)}$$
>
> [where log is the complex logarithm], or what is equal to it, the side of a square equal to a circle with diameter =1 [thus, √π/2]. From this it is obvious that the nature of the thing does not allow the number to be expressed [exactly]. But from the approximate ration of the radius to the circumference, the term with exponent 1/2 is found to be 0.886269; which, it is multiplied by 3/2, gives the term of exponent 3/2, namely 1.3293403 (p. 20).

Poor Goldbach. There was no generally agreed upon idea at this time of the logarithm of a negative number, nor were complex numbers well-known. In fact, there was a famous controversy between Leibniz and Johann Bernoulli about the logs of negative numbers, so that even the best mathematicians of the day found these subjects difficult. To see how Euler thought about this, let us ask two questions. First, how did he figure out the value? Second, how did he write it in that particular form?

First, let us see how the value involving the logarithm and the 'geometric' value are related. One thing we should note is that this is not exactly what we now call the Γ function. The value $F(1/2)=\Gamma(3/2)$ the way we would calculate today, and in general $F(x)=\Gamma(x+1)$.

Euler had done some work on the logarithms of negative numbers, and the *polar form* of a complex number is also his idea. If

$$z = re^{i\theta}, r>0$$

is a complex number in polar form, then using the usual properties of logarithms we should have that

$$\log z = \log r + i\theta$$

(though this is actually just one value). To find the polar form of a complex number, we plot it in the complex plane. For the number –1, the value of r (the distance from the origin) is 1 and the value of θ, the angle of counterclockwise rotation from the real axis, is π in radian measure. Thus $\log(-1)=\log(1)+i\pi=i\pi$.

If we substitute this in Euler's answer, we get

$$\frac{1}{2}\sqrt{-\pi} = i\frac{\sqrt{\pi}}{2},$$

the magnitude of which is $\sqrt{\pi}/2$, as Euler said.

How would he know this value? First, let us look at the expression for $F(1/2)$, substituting in the definition. Looking at the first few terms we get

$$F\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{1}\frac{1}{3/2}\frac{\sqrt{3}}{\sqrt{2}}\frac{2}{5/2}\frac{\sqrt{4}}{\sqrt{3}}\frac{3}{7/2}\cdots = \frac{2\times 4\times 6\times 8\times \ldots}{3\times 5\times 7\times 9\times \ldots}$$

Now, surprising though it may seem, Euler would have recognized this product because it is closely related to a product formula for the area of the circle. This had been devised around 75 years earlier by the Oxford mathematician John Wallis. In particular, the Wallis product formula for the area of a quadrant of the circle we would now write as

$$\frac{\pi}{4} = \frac{3\times 3\times 5\times 5\times 7\times 7\ldots}{2\times 4\times 4\times 6\times 6\times 8\ldots}.$$

By cancelling in the correct way, Euler was able to convince himself that his series had the value we said above.

This leaves us with the question of the relation to logarithms. Fortunately, Euler himself decided to help Goldback out with understanding where this had come from. In his next letter, dated Jan. 8, 1730, he was a bit more charitable in his explanations, and the relationship between his product and integrals was discussed. Various integrals were known at Euler's time to yield factorials. The one Euler selected was

$$\int_0^1 (-\log x)^n\,dx.$$

If you have studied integration by parts, you will know how to do this for $n=1$, letting $u=\log x$ and $dv=dx$ and applying the formula

$$\int udv = uv - \int vdu.$$

It is a little surprising that this integral gives you $n!$ generally, when evaluated on the interval [0,1], but it is easy to verify up to $n=4$ and the pattern is reasonably clear. ***Exercise***: Try this.

What Euler then did was, in essence, to change the n to a continuous variable z. Here I will depart a little from a strictly historical account, and explain the relation to the product expansion in modern terms.

Consider the integral

$$\int_0^1 (-\log x)^z \, dx.$$

The logarithm is not easy to handle, and our goal is to get this in a form that Euler could have treated. If we let $x=e^{-t}$, so that $-t=\log(x)$ and $dx=-e^{-t}dt$, we get

$$\int_0^\infty e^{-t} t^z \, dt.$$

This, by the way, is the integral we mentioned at the beginning, and its value is $\Gamma(z+1)$ using the modern definition. Now this integral may be rewritten as a limit:

$$\int_0^\infty e^{-t} t^z \, dt = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^z \, dt.$$

But if we now let $y=t/n$, the integral on the right in this expression becomes

$$n^{z+1} \int_0^1 (1-y)^n y^z \, dy.$$

This one was well-known already in Euler's day, based on calculating the first few values directly. To evaluate it, we will use a trick involving integration by parts.([1], 459.)

Let $I_s = \int_0^1 (1-y)^{n-s} y^{z+s} dy$. We want to find I_0. Then using integration by parts once we see that

$$I_s = \frac{n-s}{s+z+1} I_{s+1}.$$

Successively applying this gives

$$I_0 = I_n \frac{n!}{(1+z)(2+z)\dots(n+z)}.$$

But

$$I_n = \frac{1}{n+1+z}$$

and therefore

$$F(z+1) = \lim_{n\to\infty} \frac{n^{z+1} n!}{(1+z)(2+z)\dots(n+z+1)},$$

which has the same limit as the Euler partial product expression.

Euler's ingenuity in manipulating relatively elementary techniques to get deep and interesting formulas became a hallmark of his career. It shows the value of 'just fooling around' with whatever mathematics you do know, learning to ask yourself questions and seeking patterns.

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[1] Bromwich, T. J. I'A. *Introduction to the Theory of Infinite Series*, London: Macmillan, 1908.

[2] Juskevic, A. P. and E. Winter. "Leonhard Euler un Christian Goldbach: Briefwechsel 1729-1764", *Abh. Deutsch. Akad. Wiss. Berlin* 1965, 1.

Pi in the Sky Readers:
University of the Philippines, Cebu College

Pi in the Sky has readers in at least forty countries around the world. One of our readers, Lorna Almocera, Associate Professor of Mathematics at University of the Phillipines, Cebu College, contacted us for a subscription.

Upon receiving some earlier issues of Pi in the Sky, Lorna kindly sent photos. The one on the left is her Set Theory class. The photo on the right is the Cebu College Math Faculty plus (far right in photo) an aspiring young mathematician.

Building Bridges to Discrete Mathematics (But Crossing Them Only Once!)

by Peter Dukes
University of Victoria

Imagine it's 1707 and you're a resident of Königsberg[1], a Prussian city on the banks of the river Pregel. Nearby, war is brewing in the Baltics. But Europe is finally beginning to emerge from the bloody Reformation. Across the channel, Isaac Newton's famous *Arithmetica Universalis* is published, spanning the realms of number theory, geometry and algebra. To the south, in Switzerland, Leonhard Euler is born. You, Newton, and the newborn Euler are each separated by a week's journey, yet you have a close connection in the world of mathematics.

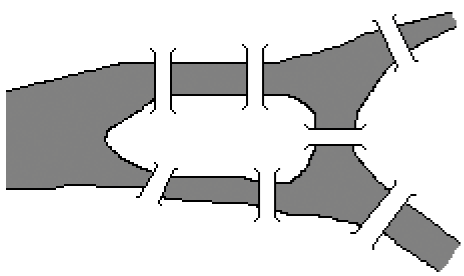

Figure 1: The city of Königsberg in 1735.

One of your pastimes is trying to walk in a continuous route across the seven bridges of your city, crossing each bridge *once and only once*. The bridges join four regions, defined by a branching of the river with a small island at the branch (see Figure 1). Over the years, your many attempts to walk the bridges in this way have led to some good exercise, but no success. Your hobby catches on and eventually people become suspicious that this 'bridges of Königsberg problem' has no solution. Each time, it seems that some bridge is left out or a bridge needs to be crossed more than once. But you persist, still believing there's a way to do it. In fact, one time you swim home in frustration

It's August 26, 1735, and Euler, now in St. Petersburg, shows in a presentation at the Petersburg Academy that all your walking was in vain. *There is no possible tour of the seven bridges.* (Well, none without getting wet!) Not only has Euler forced you to adopt a new pastime, but he has built a bridge of another sort —a bridge to the new land of discrete mathematics.

Euler understood that the seven bridges problem can be phrased as one of 'geometry of position,' which "does not involve distances, nor calculations made with them." In fact, for some time Euler himself doubted whether the problem was even mathematical in nature! Fundamentally, there are only two objects in the problem: regions and bridges between them. In modern day, this is best modelled in the language of *graph theory.*

A *graph* is simply a pair (V,E), where V is a set of *vertices* (represented with points) and E is a collection of *edges* (represented with line segments or curves), such that each edge joins two different vertices. For example, one possible graph with $V=\{A,B,C,D\}$ has the seven edges[2] AB,AB,BC,BC,AD,BD,CD, as shown in Figure 2. Let's call this graph K.

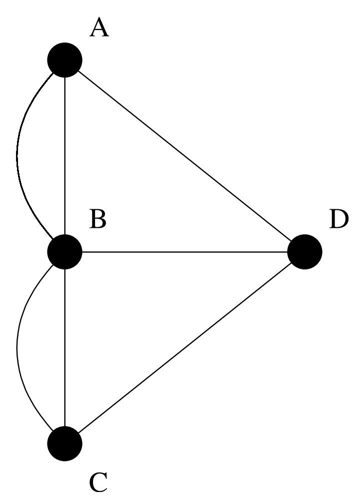

Figure 2: A graph for the seven bridges problem.

Returning to Königsberg for a moment, we are free to walk anywhere *within* one of the four regions. So we can identify each region of land with one of the vertices A,B,C,D. The seven bridges shrink precisely to the seven edges in K. Now, the bridge question can be rephrased. We must sequence the vertices so that: (1)

[1] In present day, this city is Kaliningrad, located in a small piece of Russia between Poland and Lithuania.

[2] In what we consider, edges are 'undirected' to model the fact that Königsberg's bridges were two-way streets! So for instance an edge joining A and B also joins B and A, but we will abbreviate it simply as AB.

consecutive vertices are joined by an edge; (2) no edge is used more than once; and (3) the start and end vertex are the same. In graph theory, such a sequencing is called a *circuit*. But the Königsberg bridge problem requires the additional condition that: (4) every edge is used. A circuit with this property is called *Eulerian*, named after our friend Euler. The question now becomes whether graph K above has an Eulerian circuit. Roughly speaking, we must trace the diagram in Figure 2 without backtracking or lifting our pen from the paper.

Not only did Euler show that this is not possible for our graph K, but he answered the same question for *any graph*! Actually, this generality is perhaps what makes Euler's observation mathematical in nature. With modern terminology, here's what it says.

Theorem 1. *A graph G has an Eulerian circuit if and only if it is connected*[3] *and every vertex has even degree.*

A graph is *connected* if, between any two vertices x and y, there exists a sequencing of vertices beginning with x and ending with y, such that consecutive vertices are joined by an edge of the graph. Certainly our graph K in Figure 2 is connected. In fact, almost all pairs of vertices are joined directly by an edge. The exception is pair $\{A,C\}$, for which there is the sequencing A,B,C (using two edges) or its reverse C,B,A.

The *degree* of a vertex is the number of edges with which it is incident. In K, vertices A,C and D have degree 3, while the degree of B is 5. So by Theorem 1, K has no Eulerian circuit! This was essentially Euler's argument to settle the bridges of Königsberg problem.

Proof of Theorem 1. Connectedness of G is clearly necessary for an Eulerian circuit. Let E be an Eulerian circuit beginning and ending at vertex u. Each time some vertex v occurs as an internal vertex of E, two edges through v are used. Since E uses each edge of G, the degree of each vertex besides u is even. Similarly, since E begins and ends at u, the degree of u is also even. Thus each vertex in K has even degree. This proves the 'only if' part of Theorem 1.

To see that the conditions are sufficient (the other implication), we outline how to construct an Eulerian circuit in a connected graph G with all vertices of even degree. Pick any random circuit in G to start. If it uses all edges already, then stop because it is already an Eulerian circuit. Otherwise erase the edges in the circuit and find another random circuit in the remaining graph which has at least one vertex in common with the existing circuit. Repeat this until all edges are used and join the circuits together at the common vertices. Note that erasing edges in a circuit preserves the property that the degree of every vertex is even. This is because the deleted circuit has even degree at every vertex. So the construction will end with a circuit through all edges exactly once. □

[3] To be precise, a graph with 'isolated vertices', not incident with any edge, might be disconnected but still have an Eulerian circuit.

An *Eulerian trail* is defined similarly as an Eulerian circuit, but without condition (3) that the start and end vertex be the same.

Corollary 2. *A graph G has an Eulerian trail if and only if G is connected and the number of vertices of odd degree is* 0 *or* 2.

Actually, if there are two vertices x,y of odd degree, we can get an Eulerian trail between (only) *those* vertices as follows: join them with a new edge xy. In this modified graph, every vertex has even degree, since the degree of x and y have increased by 1. So by Theorem 1, there is an Eulerian circuit E in the new graph. Deleting edge xy from E 'breaks it' into a trail.

Corollary 2 begs another question. What happens if a graph has exactly 1 vertex of odd degree? Is this even possible?

Theorem 3. *The sum of all degrees of a graph G equals twice the number of edges in G.*

Returning to the Königsberg graph K, the sum of degrees is $3+5+3+3=14$, and there are 7 edges. Theorem 3, another one of the starting points of graph theory, is usually known as the 'Handshaking Lemma' due to the following interpretation. If certain pairs of people shake hands at a party, then the sum, over all people, of the number of handshakes made by that person equals twice the number of handshakes at the party.

Proof of Theorem 3. In summing the degrees of vertices of G, each edge, say between x and y is counted twice, once in the degree of x and once in the degree of y. □

Corollary 4. *Any graph has an even number of vertices with odd degree.*

These observations weren't the only foundations of graph theory initiated by Euler. In considering a completely different question, Euler began the investigation of a problem which still taunts us today. What if we would like to walk along some (not necessarily all) edges of a graph so as to visit each *vertex* exactly once? Today, this is called a *Hamiltonian path*, named after William Rowan Hamilton, who lived about a century after Euler.

Euler's investigation of Hamiltonian paths in graphs began with a mathematical analysis of 'knight's tours' on a chessboard. The problem is to repeatedly move a chess knight (legally, in the usual L-shape) so as to visit every square, exactly once each. Persian chess players had long known that a knight's tour is possible on an 8×8 board. But yet again, Euler's generalized analysis gave birth to the mathematics behind the puzzle. The graph in question has as its vertices the set of squares of the chessboard. Two vertices are joined by an edge if and only if their corresponding squares are a single knight's move apart. Some vertices (in the corner) have degree 2, while others have degree 8 (near the centre). In any case, a knight's tour is equivalent to a Hamiltonian path in this graph. One particularly nice knight's tour on a 5×5 board is shown in Figure 3. This was

1	16	21	8	3
12	7	2	15	20
17	22	13	4	9
6	11	24	19	14
23	18	5	10	25

Figure 3: A 5×5 knight's tour, beginning in the upper-left and in sequence with the given numbers.

found by Euler in 1759, and exhibits symmetry and arithmetic structure in addition to the graph theoretic condition.

We're now in 2007 and, unlike Eulerian trails and circuits, there is presently no known efficient characterization of which graphs have Hamiltonian paths. Perhaps all your practice walking along bridges will one day pay off after all!

References

[1] G. Alexanderson, Euler and Königsberg's bridges: A historical view, *Bulletin (New Series) of the American Mathematical Society*, ***43*** (2006), 567—573.

[2] G. Chartrand, "The knight's tour," in *Introductory Graph Theory*, New York, Dover, 1985.

From Our Readers

Pi in the Sky ran this joke in Issue 6 (March, 2003)

Denis Diderot was a French philosopher in the 18th century. He traveled Europe extensively, and on his travels also stopped at the Russian court in St. Petersburg. His wit and suave charm soon drew a large following among the younger nobles at the court - and so did his atheist philosophy. That worried Empress Catherine the Great very much... Swiss mathematician Leonhard Euler was working at the Russian court at that time, and unlike Diderot, he was a devout Christian. So, the empress asked him for help in dealing with the threat posed by Diderot. Euler had himself introduced to Diderot as a man who had found a mathematical proof for the existence of God. With a stern face the mathematician confronted the philosopher:

> *"Monsieur, (a+bn)/n=x holds! Hence, God exists. What is your answer to that?"*

Quick-witted Diderot was speechless, was laughed at by his followers, and soon returned to France.

Pi in the Sky has received this correspondence:

Dear Sir,

Enjoyable as your collection of mathematical humor is, I ask that you please remove or at least correct the anecdote referring to the almost certainly fictional encounter between Leonhard Euler and Denis Diderot in the court of Catherine the Great in St. Petersburg. Not only is this anecdote almost certainly not true, it also implies, at the very least, that:

1) Diderot was mathematically illiterate (he was not)

and/or

2) The nonsensical expression somehow proves (or supports) the existence of a god or gods (it does not).

For a treatment of the falsehood of this anecdote, I ask you to refer to: Brown, B.H. (May 1942). "The Euler-Diderot Anecdote". The American Mathematical Monthly 49 (5): 302-303.

Regardless of your personal beliefs about the factuality of religious beliefs, I would hope you agree that false historical information is contrary to the goals of an educator, and to the strength of ethical, democratic societies.

Thank you for your time,

Samuel Cornwell, Merced, California

Guest-Editor's response: We published the scenario of the discussion between Euler and Diderot as a joke, not as an anecdote. Therefore we agree with Mr. Cornwell that it is unlikely to have historical value.

Dürer's "Melancholia I": A 16th Century Tribute to Mathematics

by Michael McNeely
Victoria, BC

The winged, disheveled woman sits, slumping forward with her chin resting on the heel of her left hand. She is dejected, despairing and thoughtful. Around her feet we see a disordered array of builder's tools and measurement instruments. Beside her, an emaciated dog, an angry cupid, and an irregular stone polyhedron. Above, a curious magic square shares a wall with a bell, an hourglass, and a hanging pan-balance. In the distance, a bat with 'Melancholia I' imprinted on his wings, flies over the sea across the tail of a comet. The meaning of this precise, beautiful, and mysterious 1514 engraving by Albrecht Dürer has puzzled viewers for five centuries and has particular significance for mathematicians.

Although there are many interpretations, the figure is generally considered to be the personification of the Renaissance artistic genius who has the tools of craftsmanship but is cast into a state of depression upon realizing that perfection will require greater theoretical (mathematical and geometric) knowledge than may ever be attainable [1].

Let us consider how Dürer blends the symbols of craftsmanship, artistic genius, geometry, and melancholy.

In Dürer's time, it was believed there were four personalities (sanguine, choleric, melancholic, and phlegmatic) created by the balance of the four humours (blood, yellow bile, black bile, and phlegm) which were also linked to aspects of the earth and to celestial bodies. Melancholia (associated with Saturn and the earth) was well recognized and also feared for it was associated with insanity. Our figure is a symbolic embodiment of the condition. She is not a real person for she has wings and she is not an angel for she has no halo. In addition, the scattered collection of tools and devices immediately identify the figure (to a

Melancholia I

Melancholia I by Albrecht Dürer (1471-1528) is an engraving which measures 18.5 x 24 cm. The extraordinary detail is achieved by cutting the design into a copper plate using a sharpened steel tool known as a burin. The plate is then inked and paper reproductions made. Dürer is the greatest master of this art form.

16^{th} century viewer) as the personification of ‘Art in General’. Dürer has thus created a new image to reflect a popular notion of the day that persons of genius, including Dürer himself, were particularly susceptible to the melancholic state (*Saturnine Genius*).

The collection of architectural and carpentry tools are symbolic of the practical skills essential to an artisan and these particular professions were subject to the governance of Saturn. In contrast, the scales, pentahedron, sphere, and magic square are devices of measurement and geometry and symbolize the scientific principles which underlie all craft. It was a Renaissance idea, to which Dürer was known to subscribe, that consummate mastery resulted from coordination of technique and theoretical insight. As god of the earth, Saturn is associated with stone and wood and agriculture (particularly the partitioning of lands). To deal with such matters, geometry was ascribed to Saturn. Thus, with Saturnine allusions, Dürer weaves associations amongst melancholy, craftsmanship, and geometry.

16	3	2	13
5	10	11	18
9	6	7	12
4	15	14	1

Magic Square;
In recreational mathematics, a magic square of order n is an arrangement of n^2 numbers, usually distinct integers, in a square, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant. A normal magic square contains the integers from 1 to n^2. The term “magic square” is also sometimes used to refer to any of various types of word square.

Normal magic squares exist for all orders $n \geq 1$ except $n = 2$, although the case $n = 1$ is trivial–it consists of a single cell containing the number 1.
-Wikipedia

The mathematical allusions are quite fascinating. The stone object is obviously a three-dimensional solid and is bounded, as far as we can see, by pentagons. Such figures were of great interest in the 16^{th} century and scholars of the day recognized that there were only five ‘regular’ convex solids (tetra-, hexa-, octa-, dodeca-, and icosahedron) which had been described in Plato’s *Timaeus* (360 B.C.E) and were therefore known as the Platonic Solids. But why did Dürer depict such a strangely distorted polyhedron whose sides are clearly not equal? Certainly not out of ignorance, for he had published a treatise on the subject [2]. Perhaps it is a further commentary on the imperfect knowledge of geometry?

Above the figure’s head we see a magic square. In the 16^{th} century, the magic square was a very popular form of recreational puzzle which intellectuals would create for one another, rather like the Sudoku craze of today. These squares produce the same sum when the numbers in each row, column, or major diagonal are added together. Dürer’s example uses the sequential numbers 1 to 16 whose orthogonals each add to 34. This magic square is a special example known as ‘symmetrical’ since each number, added to the one symmetrically opposite the center, adds to 17. Making it even more unique is the equal summation (to 34) of the four numbers in each quadrant. Such 4^{th} order magic squares were linked to Jupiter by Renaissance astrologers and were believed to combat Saturn’s melancholy. A particular feature of this matrix is that the bottom, central squares contain the numbers ‘15’ and ‘14’, the year in which the engraving was produced.

Dürer’s mysterious engraving calls for greater mathematical skill in order to elevate art. But if Dürer had known that viewers with vastly superior mathematical sophistication would continue to enjoy the beauty of his art and debate the meaning of his composition some five centuries later, he would most certainly have been lifted from the melancholy which plagued his soul.

References:

[1] Panofsky, Erwin, 1955: *The Life and Art of Albrecht Dürer*. Princeton University Press, Princeton NJ.
[2] Dürer, Albrecht, 1525: *Underweysung der Messung*.

Why Proof?

by Volker Runde
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Once upon a time, there was a prince who was educated by private tutors. One day, the math tutor set out to explain the Pythagorean theorem to his royal student. The prince wouldn't believe it. So, the teacher proved the theorem, but the prince was not convinced. The teacher presented another proof of the theorem, and then yet another, but the prince would still shake his head in disbelief. Desperate, the teacher exclaimed: "Your royal highness, I give you my word of honour that this theorem is true!" The prince's face lit up: "Why didn't you say so right away?!"

Wouldn't that be wonderful? A simple word of honour from the teacher, and the student accepts the theorem as true...

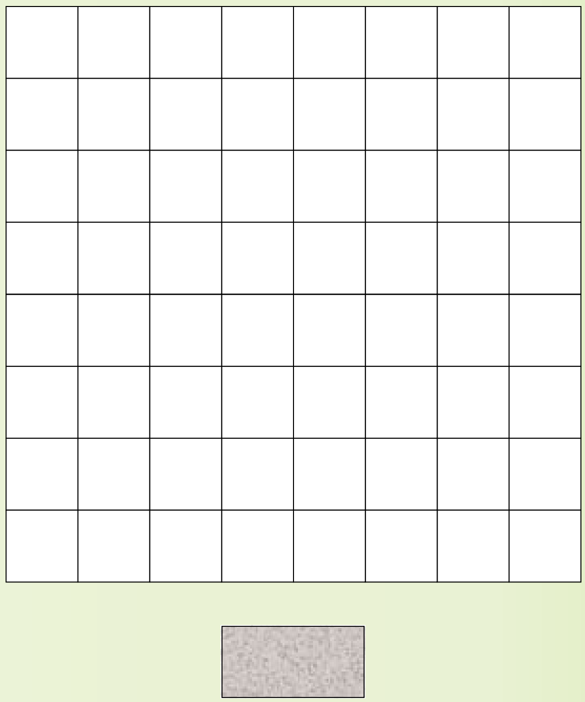
Of course, it would be awful. Who makes sure that the teacher can be trusted? Where did he get his knowledge from? Did he rely on another person's word of honour? Was the person from whom the teacher learned the theorem trustworthy? Where did that person get his/her knowledge from? Did that person, too, trust someone else's word of honour? The longer the chain of words of honour gets, the shakier the theorem starts to look. It can't go on indefinitely: someone must have established the truth of the theorem some other way. My guess is: that someone *proved* it.

Why are mathematicians so obsessed with proofs? The simple answer is: because they are obsessed with the truth. A proof is a procedure which, by applying certain rules, establishes an assertion as true. Proofs do not only occur in mathematics. In a criminal trial, for instance, the prosecution tries to *prove* that the defendant is guilty. Of course, the rules according to which a proof is carried out depend very much on the context: it is one thing to prove in court that Joe Smith stole his neighbour's hubcaps and another one to give a proof that there are infinitely many prime numbers. But in the end all proofs serve one purpose: to get to the truth.

Here is a joke: *A mathematician and a physicist are asked to check whether all odd numbers greater than one are prime. The mathematician says: "Three's a prime, five's a prime, seven's a prime, but nine isn't. Therefore it's false." The physicist says: "Three's a prime, five's a prime, seven's a prime, nine isn't—but eleven and thirteen are prime again. So, five out of six experiments support the hypothesis, and it's true!"* We laugh at the physicist. How can he simply dismiss a counterexample? It's not as silly as it seems. Experimental data rarely fits theoretical predictions perfectly, and scientists are used to a certain amount of data that is somewhat out of line. What makes the physicist in the joke look foolish is that he treats a mathematical problem like an experimental one: he applies rules of proof that are valid in one area to another area where they don't work.

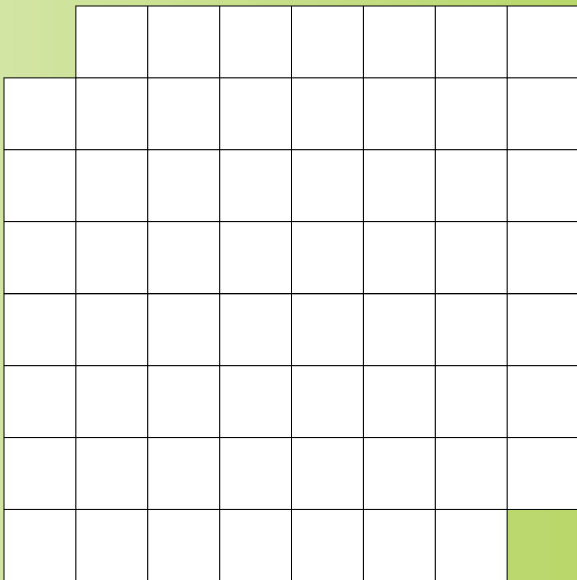
Instead of musing further on the nature of mathematical proof, let's try and do one.

Consider a chessboard consisting of 64 squares:



Then take rectangular tiles as shown below the board: each of them covers precisely two adjacent squares on the chessboard. It's obvious that you can cover the entire board with such tiles without any two of them overlapping. That's straightforward, so why do we need proof here? Not yet...

To make things slightly more complicated, take a pair of scissors to the chessboard and cut away the squares in the upper left and in the lower right corner. This is what it will look like:

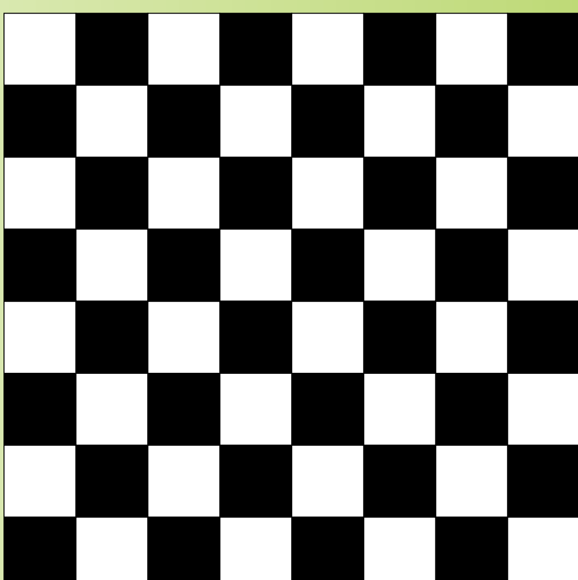


Now, try to cover this altered chessboard with the tiles without any two of them overlapping...

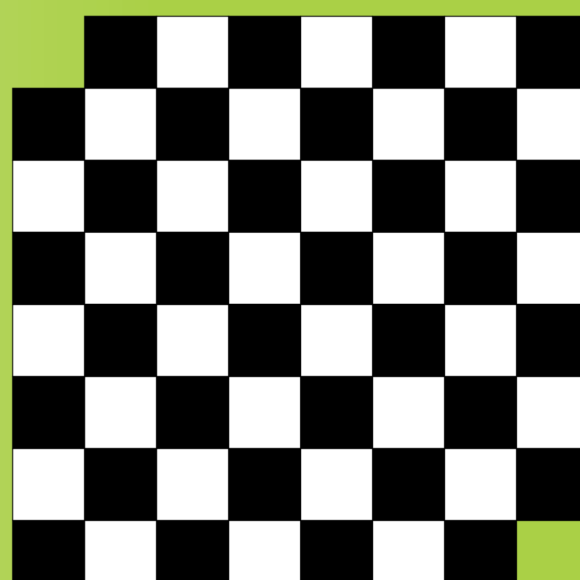
If you really try this (preferably with a chessboard drawn on a piece of paper...), you'll soon find out that—to say the least—it's not easy, and maybe the nagging suspicion will set in that it's not even possible—but why?

There is, of course, the method of brute force to find out. There are only finitely many ways to place the tiles on the chessboard, and if we try them all and see that in no case the area covered by them is precisely the altered chessboard, then we are done. There are two problems with this approach: firstly, we need to determine *every* possible way to arrange the tiles on the chessboard, and secondly, even if we do, the number of possible tile arrangements may be far too large for us to check them all. So, goodbye to brute force...

So, if brute force fails us, what can we do? Remember, we are dealing with a chessboard, and a chessboard not only consists of 64 squares in an eight by eight pattern—the squares alter in color; 32 are white, and 32 are black:



Each tile covers precisely two adjacent squares on the board, and two adjacent squares on a chessboard are always different in color; so each tile covers one white square and one black square. Consequently, any arrangement of tiles on the chessboard must cover the same number of white and of black squares. But now, check the altered chessboard:



We removed two white squares, so the altered board has 30 white squares, but 32 black ones. Therefore, it is impossible to cover it with tiles—we *proved* it.

Does this smell a bit like black magic? Maybe, at the bottom of your heart, you prefer the brute force approach: it's the harder one, but it's still doable, and maybe you just don't want to believe that the tiling problem is unsolvable unless you've tried *every* possibility.

There are situations, however, where a brute force approach to truth is not only inconvenient, but impossible. Have a look at the following mathematical theorem:

Theorem 1 *Every integer greater than one is a product of prime numbers.*

Is it true? And if so, how do we *prove* it?

Let's start with checking a few numbers: 2 is prime (and thus a product of prime numbers), so is 3, $4=2\cdot2$, 5 is prime again, $6=2\cdot3$, 7 is prime, $8=2\cdot2\cdot2$, $9=3\cdot3$, and $10=2\cdot5$. So, the theorem is true for all integers greater than 2 and less than or equal to 10. That's comforting to know, but what about integers greater than 10? Well, 11 is prime, $12=2\cdot2\cdot3$, 13 is prime, $14=2\cdot7$, $15= 3\cdot5$, $16=2\cdot2\cdot2\cdot2$, ... I stop here because it's useless to continue like this. There are infinitely many positive integers, and no matter how many of them we can write as a product of prime numbers, there will always remain infinitely many left for which we haven't shown it yet. Is $10^{10^{10^{10}}}+1$ a product of prime numbers? That number is awfully large. Even with the help of powerful computers, it might literally take an eternity to find the prime numbers whose product it is (if they exist...). And if we have shown that the theorem holds true for every integer up to $10^{10^{10^{10}}}+1$, we still don't know about $10^{10^{10^{10^{10}}}} + 1$.

Brute force leads nowhere here. Checking the theorem for certain examples might give you a feeling for it—but it doesn't help to establish its truth for *all* integers greater than one.

Is the theorem possibly wrong? What would that mean? If not every integer greater than one is a product of prime numbers, then there must be at least one integer a_0 which is *not* a product of prime numbers. Maybe, there is another integer a_1 with $1 < a_1 < a_0$ which is also not a product of prime numbers; if so replace a_0 by a_1. If there is an integer a_2 with $1 < a_2 < a_1$ which is not a product of prime numbers, replace a_1 by a_2. And so on... There are only finitely many numbers between 2 and a_0, and so, after a finite number of steps, we hit rock bottom and wind up with an integer $a > 1$ with the following properties: (a) a is not a product of prime numbers, and (b) it is the smallest integer with that property, i.e., every integer greater than one and less than a is a product of prime numbers.

Let's think about this (hypothetical) number a. It exists if the theorem is false. What can we say about it? It can't be prime because then it would be a product (with just one factor) of prime numbers. So, a isn't prime, i.e., $a = bc$ with neither b nor c being a or 1. This, in turn, means that $1 < b, c < a$. By property (b) of a, the numbers b and c are thus products of prime numbers, i.e., there are prime numbers $p_1, \dots, p_n, q_1, \dots, q_m$ such that $b = p_1 \cdots p_n$ and $c = q_1 \cdots q_m$. But then

$$a = bc = p_1 \cdots p_n q_1 \cdots q_m$$

holds, and a is product of prime numbers, which contradicts (a).

Math Joke

The numbers π and i are having an argument.

"I cannot have a logical discussion with you," says i. "You are irrational."

At this, π looks at i with contempt and says: "Get real!"

We assumed that the theorem was wrong, and—based on that assumption—obtained an integer a that is not a product of prime numbers only to see later that this was not possible. The only way out of this dilemma is that our assumption was wrong: the theorem is true! (And we now know that $10^{10^{10^{10}}}+1$ is a product of prime numbers without having to find them...)

The strategy we used to prove Theorem 1 is called *indirect proof*. We can't show something directly, so we assume it's wrong and (hopefully) arrive at a contradiction.

Let's try another (indirect) proof:

Theorem 2 *There are infinitely many prime numbers.*

Is this believable? There is no easy formula to calculate the n^{th} prime number, and after putting down the first few prime numbers, it gets harder and harder to come up with the next prime. So, is the theorem wrong and do we simply run out of prime numbers after a while?

Assume this is so: there are only finitely many prime numbers, say $p_1, \dots, p_n$. Set $a := p_1 \cdots p_n + 1$. By Theorem 1, a is a product of prime numbers. In particular, there is a prime number q and a non-negative integer b with $a = qb$. Since $p_1, \dots, p_n$ are all the prime numbers there are, q must be one of them. Let c be the product of all those p_j that aren't q, so that $a = qc + 1$. We then obtain

$$0 = a - a = qc + 1 - qb = q(c - b) + 1,$$

and thus $q(c - b) = -1$. This, however, is impossible because $c - b$ is a non-zero integer and $q \geq 2$.

We have thus again reached a contradiction, and Theorem 2 is proven.

The proof of Theorem 2 isn't as straightforward as that of Theorem 1. Why did we define a the way we did? The answer is simply that it works this way, and it's been working for over two thousand years: Theorem 2 was first stated (and proven) in Euclid's *Elements*, which appeared around 300 B.C. As the American mathematician Saunders Mac Lane once said: "Mathematics rests on proof—and proof is eternal."

Math Joke

A math major studied hard in the university library all evening.

On his way home, he felt very hungry so he stopped at a nearby pizza place and ordered a large pizza. When it was ready, the pizza guy asked him if he wanted it cut into six or eight pieces.

The Math major replied, "Cut it into six—I don't think I could eat eight pieces."

Euler, infinite sums, and the zeta function

by Michael P. Lamoureux
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As you can read from the many articles in this edition of *Pi in the Sky*, Euler did a wide variety of mathematics. One thing he liked to do was to play around with adding up numbers. For instance, he asked what would happen if you added the reciprocals of all the integers $1, 2, 3, 4, \ldots$, their squares, their cubes, and so forth. That is, how do you compute the sums

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = ?$$
$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = ??$$
$$\frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = ???$$

What we will see in this article is that Euler found a way to relate these sums involving integers with products involving primes. Now primes are nice, and there are a lot fewer primes than integers, so this is a useful trick. For instance, the first sum above can be rewritten as a product

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \cdots.$$

This equation just says if we sum the reciprocals of all the integers $1, 2, 3, 4, \ldots$, we get the same result as taking the product of all the prime numbers $2, 3, 5, 7, 11, \ldots$ divided by one less than each prime.

Well, you might wonder what it means to add up infinitely many numbers, since in most arithmetic classes we only add a few numbers at a time. In fact, you might expect that adding up infinitely many numbers should give you infinity. After all, if someone promises to give you one dollar a day, FOREVER, you will expect to get an infinite amount of money.[1] Of course, it's not always true that adding infinitely many numbers gives infinity, since we can add up a bunch of zeros to find

$$0 + 0 + 0 + 0 + \cdots = 0$$

which is another way of saying "A whole lot of nothing is still nothing!"

Can you get something in between zero and infinity? Well, yes you can. Think of adding a bunch of fractions with powers of 10 in the denominator. This gives you a decimal expansion of a number, so

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = 0.1 + 0.01 + 0.001 + 0.0001 + \cdots = 0.1111111\ldots$$

which we recognize as the repeating decimal expansion of 1/9. So we have a nice "infinite sum" formula that says

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = \frac{1}{9}.$$

There is nothing special about 1/9. We can get other neat numbers using infinite sums, for instance

$$3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \cdots = 3.1415\ldots = \pi$$

gives the decimal expansion for pi, which is a tricky sum since we don't really know offhand what the pattern of digits 31415... should be.

[1] If you don't ever die!

But back to Euler. What about the first sum we saw, where

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\cdots.$$

What does this equal? Well, it turns out that this equals infinity! So maybe it is not that exciting, since after all, we know that the left hand side has infinitely many terms in it, as there are infinitely many integers. But wait: this also tell us something about the right hand side. Since that product equals infinity as well, there must be infinitely many factors in the product. That means there must be infinitely many primes $2,3,5,7,\ldots$. Now, most of us reading here know this already, but it really is not obvious that there are infinitely many primes. For instance, they are hard to find. If I ask you what is the first prime bigger than 20, well you can quickly figure out it is the prime number 23. But, if I ask you what is the first prime bigger than 20 million, then you would have to work! Or look it up on the net! There is no simple formula for finding primes; in fact some mathematicians spend a good chunk of their career looking for really big primes. The biggest prime currently known is $2^{32582657}-1$, which, in binary notation, is simply a string of ones, with 32482657 binary digits in it! In our usual decimal notation, it is a number with almost 10 million digits in it!

The point is, Euler's equation with the infinite sum in it tells us there are infinitely many primes, which is pretty neat.[2]

But, Euler did more than this. He also showed a similar formula holds when we square the integers, and square the primes. He found that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots=\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\cdot\frac{121}{120}\cdot\cdots.$$

If you're not sure what the pattern is here, on the left side we have the reciprocals of the squares of the numbers $1,2,3,4,5,\ldots$ and on the right side the squares of the primes $2,3,5,7,11,\ldots$ divided by one less than these squares.

In this case, though, the sum is not infinity, it is $\pi^2/6$. Amazing.

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots=\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\cdot\frac{121}{120}\cdot\cdots=\frac{\pi^2}{6}.$$

What's more, you can use the fourth power of the integers and primes and find out, for instance, that

$$1+\frac{1}{64}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\cdots=\frac{16}{15}\cdot\frac{81}{80}\cdot\frac{625}{624}\cdot\frac{2401}{2400}\cdot\frac{14641}{14640}\cdot\cdots=\frac{\pi^4}{90}.$$

In fact, Euler showed more than just these formulas for integers, or their squares, or their fourth power; he showed it for any power. If s is any number bigger than one, then raising to the power s gives a formula connecting the integers and the primes, that says:

$$1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\cdots=\frac{2^s}{2^s-1}\frac{3^s}{3^s-1}\frac{5^s}{5^s-1}\frac{7^s}{7^s-1}\frac{11^s}{11^s-1}\cdots.$$

This sum, and the corresponding product, is so important that it is given a special name, $\zeta(s)$, pronounced 'zeta of s', which is called the Riemann zeta function.

What Euler showed, then, is there are at least two ways to compute this zeta function, one using integers and sums, another using primes and products. We also saw above that $\zeta(1)=\infty$, $\zeta(2)=\pi^2/6$, and $\zeta(4)=\pi^4/90$.

How did Euler know that these two ways of computing give the same result? Well, it's a pretty neat argument, where we just divide up the sum into integers that have certain primes in it, and those that don't. So, take the sum and split into those with even denominators, and those with odd, and then pull out a factor of two from the first half of the sum:

$$\begin{aligned}\zeta(1)&=\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\right)+\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots\right)\\&=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\right)+\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots\right)\\&=\frac{1}{2}\zeta(1)+\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots\right)\end{aligned}$$

[2] Euclid came up with a completely different proof in ancient times.

There is a $\zeta(1)$ on both sides of the equation, so we pull them both to the left and get

$$\frac{1}{2}\zeta(1) = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots$$

which is just the sum of the odd terms. Now split these odd terms into those divisible by three, and those not, so

$$\begin{aligned}\frac{1}{2}\zeta(1) &= \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \cdots\right) + \left(\frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \cdots\right)\\ &= \frac{1}{3}\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots\right) + \left(\frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \cdots\right)\end{aligned}$$

where now we see in the first half of the sum, a sum of odd terms, which we know from before is $1/2\,\zeta(1)$. So we can write

$$\frac{1}{2}\zeta(1) = \frac{1}{3} \cdot \frac{1}{2}\zeta(1) + \left(\frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \cdots\right)$$

and pulling the zetas over to one side, we have

$$\frac{2}{3} \cdot \frac{1}{2}\zeta(1) = \frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \cdots$$

Now keep doing this. Split the remaining sum into those terms divisible by five, and not; then divisible by 7 and not, and so on, all the way down the primes. On the left hand side, we get an infinite product involving the primes, and on right hand side all the fractions disappear, except for 1/1. So we have

$$\cdots \frac{10}{11} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{2}\zeta(1) = \frac{1}{1}$$

Or, pulling the fractions onto the other side, we have

$$\zeta(1) = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \cdots$$

which is another way of stating Euler's formula,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \cdots.$$

The same explanation shows why this works for other powers s.

Now to be completely honest, there is something fishy about this explanation. Namely, the last equation really tells us $\infty = \infty$, and everywhere in the equations above we are tossing around infinities like they are regular numbers, which is a dangerous thing to do. Not to worry, though: there are some easy fixes to make this explanation completely valid.

By the way, the Riemann zeta function can also be defined on imaginary and complex numbers.[3] Strictly speaking, the defining sums only make sense when the power s has real part bigger than one, but good functions like this have precise extensions to the whole complex plane.

There is a very famous question, called the Riemann hypothesis, that asks for a proof to show that $\zeta(s)$ equals zero ONLY for some values of s in the form of a complex number $s=1/2+x\sqrt{-1}$, plus a few easy ones.[4] Believe it or not, mathematicians have calculated the first billion or so places where the zeta function is zero, and they are all of that special form. But that's not a proof, and the hunt goes on.

A real proof of the Riemann hypothesis will win you fame and fortune; there is even a one million dollar prize for its solution. However, it's a tough nut to crack—our chances with the lottery might be better!

[3] Calvin and Hobbes describe imaginary numbers as things like eleventeen and thirty-twelve, but here we mean things like the square root of -1.

[4] The 'trivial' values $s=-2,-4,-6,\ldots$

Who Invented Euler's Circle?

by Klaus Hoechsmann
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Common sense would suggest that the answer should be "Euler", but sometimes things are not that simple: Julius Caesar had more to do with the Tsar and the Kaiser than with the Caesar Salad. Most mathematicians would say that the answer is "Nobody", since they cannot imagine that the circle in question did not exist before it was *discovered*.

Actually, we are not concerned with a *particular* circle (like the Arctic Circle), but one circle for every conceivable triangle. As shown in Figure 1, any triangle ABC spawns (at least) three other triangles: A*B*C*, A'B'C', and A"B"C", the first of these being formed by the midpoints of the sides, the second by the feet of the altitudes, and the third by the points midway between the vertices and the so-called *orthocentre* H of ABC. As every triangle has a unique circle containing its three vertices (its *circumcircle*), we would have three derived circumcircles K*, K', and K". The subject of the present article is the amazing fact that these all turn out to be one and the same!

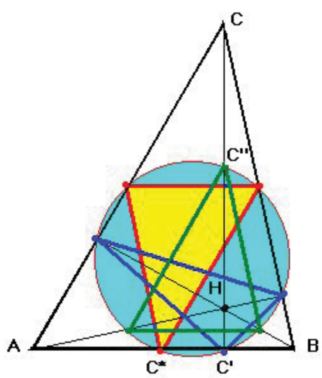


Figure 1

Since "orthocentre" is not exactly a household word, an explanation is in order. The diagram hints that H is the intersection of the *altitudes* (perpendiculars dropped from each vertex to the opposite side), but why should these three lines, unlike the sides of ABC itself, intersect in a single point?

Figure 2 shows why. If you focus on the thin red line sprouting from midpoint C* of AB, you will readily agree that any point on this *perpendicular bisector* of AB must be equidistant from A and B (why?). By the same token, all points of the perpendicular bisector of BC are equidistant from B and C. The point of intersection of these two bisectors, traditionally labelled O, is therefore equidistant from of A, B, and C. In other words, the circle of radius OA (=OB=OC) is the circumcircle K of ABC.

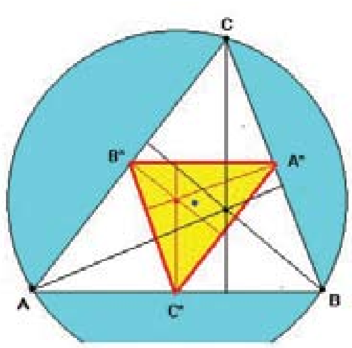


Figure 2

Now comes a surprise: the three perpendicular bisectors of ABC are exactly the altitudes of A*B*C* (because AB is parallel to A*B*, etc.—see?). Since A*B*C* is *similar* to ABC (why?) the altitudes of ABC must behave like those of A*B*C* and intersect in a single point. Thus the orthocentre H is born—or discovered.

Since the triangle A'B'C' formed by the *feet* of the altitudes of ABC (blue outline in Figure 1) will play a very important role, it deserves a special name. To bring together the notions of *orthogonal* (Greek for "right-angled") and feet, it would seem reasonable to call it "orthopaedic"—but tradition obliges us to shorten this to *orthic*.

The blue dot which shows up between O and H in Figure 2 is called the *centroid* of ABC and usually denoted by G for "gravity". It is where you would stick a pin through a cardboard triangle to make it spin freely. We shall not call on Archimedes to explain this (though *you* may do so), but instead define G without reference to it as the common meeting point of the three *medians* AA*, BB*, CC*. A careful study of Figure 3 will convince you that they do indeed meet in a single point—which cuts each median in the ratio 2:1, and is also the centroid of the smaller triangle A*B*C*. We shall presently see that G lies *on* OH, which—though only a segment—is known as the *Euler line* of ABC. In fact, the collinearity of OGH is one of the by-products of an article written by Euler in 1765.

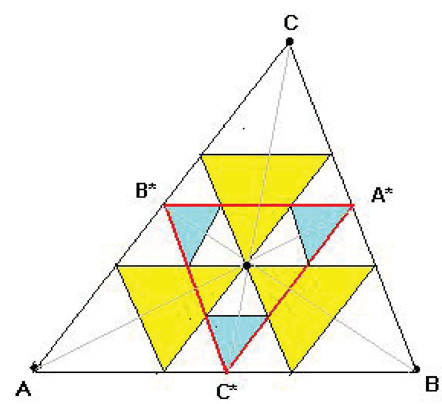


Figure 3

With that study, Euler launched what might be called the "Modern geometry" of the triangle, as opposed to its "Ancient geometry". The latter was an old hat, amply described in Euclid's *Elements*, further extended by Hipparchus with his trigonometry, visited by the likes of Heron of Alexandria, and refined by numerous Islamic scientists. It was by no means dead-wood, but of daily beneficial use in navigation, construction, artillery, etc.—and even in the exploration of fancier geometric and kinematic phenomena by mathematicians and physicists. But to worry about the humble triangle itself must have seemed like a waste of time: all that could be known about it appeared to be out in the open.

Euler's paper was published by the "Petropolitan" (from St. Petersburg) Academy of Science, under the title, "An easy solution to some very difficult geometrical problems" (in Latin). It set out to explore the relations, in terms of distances and angles, between the following four special points associated to any non-equilateral triangle:

(1) the orthocentre, (2) the centroid, (3) the circumcentre, (4) the incentre.

The arguments to be given here are, however, quite unlike Euler's. He was, after all, first and foremost a master of algebraic computation, perhaps the greatest virtuoso of all times on this difficult instrument, and he derived those relations with consummate artistry by means of cleverly concocted formulas—whereas this column of *Pi in the Sky* has consistently aimed at side-stepping algebra to bring up the geometric rear.

To see the collinearity OGH, forget triangles for a moment, and just think of a single point G stranded in the middle of the plane. For any other point P in this plane, one could define the "Euler image through G" (a term *ad hoc* just invented a minute ago) of P to be the point P* which lies on the line PG *beyond* G but only half as far away from G as P. Putting it differently, P* is obtained from P by turning the segment GP around G by 180 degrees, and then shrinking it by 1/2 to get P*G. If you keep this firmly in mind, you will see that every segment PP* is cut by G in the ratio 2:1. It will also be clear that A* is the Euler image (through G) of A, and so on; in fact all, of A*B*C* can be thought of as the Euler image of ABC.

Since the two stages of creating P* from P—namely the 180 degree turn as well as the shrinkage by 1/2—both preserve angles (why?), the whole transition from some planar structure to its Euler image is *conformal* (i.e., angle-preserving). In particular, the altitudes of A*B*C* are the Euler images of the altitudes of ABC, and their intersection O must be H*. Thus O and H lie on either side of G on a single line, the latter twice as far from G as the former. Since the wondrous circle K* is the Euler image of the circumcircle K, its centre O* is the Euler image of O. Just like O itself, O* also gets a special label, namely N. Now we know four points of the Euler line: O, G, N, H in that order. As the segment OG, which is 1/3 of OH, must be 2/3 of OO*=ON, the latter must be 1/2 of OH. In other words:

OGNH are collinear, with OG one third and ON one half of the segment OH.

Everything said so far would apply *verbatim* to triangles with an obtuse angle. Indeed, if you can look past the confusing lines and letters of Figure 4, you will see a triangle ABC with an obtuse angle at B. Now the feet of the altitudes from A and C lie on the *extensions* of the sides BC and CA, respectively, and the orthocentre H lies outside the bluish halo of the circumcircle. This makes for a nice and long Euler line OH, on which G and N are clearly visible. For a first time through, you should probably skip Figure 4 and the text that goes with it. All it does is justify what is stated under "Upshot" a couple of paragraphs further down, namely that obtuse angles can be circumvented. It looks messy but it is only a question of sorting things out—no tricky thinking—you can can come back to it later.

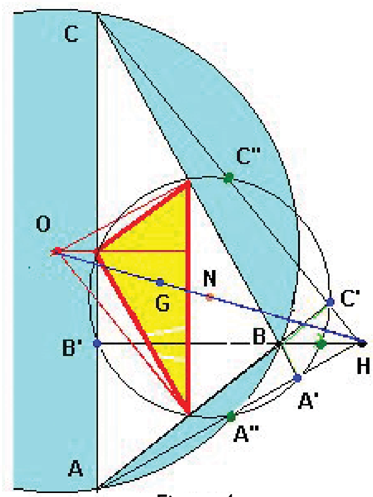


Figure 4

Transferring your attention from ABC to the triangle AHC, you will notice that B is the orthocentre of the latter (see?), in other words H and B have switched places. In fact, the vertices of the triangles A*B*C*, A'B'C', and A"B"C" associated with ABC play an interesting game of musical chairs with the corresponding points associated with AHC. For example, A' is the foot of the perpendicular from C onto AH, while C' is the foot of the perpendicular from A to CH. Hence A'B'C' is the *orthic triangle* of both ABC and AHC (though A' and C' switch positions).

For the triangles A*B*C* and A"B"C", things are not as simple: their vertices are permuted within the sextuple A*B*C*A"B"C", the corresponding one for AHC being C"B*A"C*B"A*. The reader is urged to identify, at least partially, what goes where—it's fun. But please don't expect the Euler lines of ABC and AHC to coincide: the only point they have in common is N.

Upshot: to show that A*B*C*, A'B'C', A"B"C" lie on a single circle, we never need to consider obtuse triangles like ABC, because we can transfer the drama (with the same cast of characters) to the acute triangle AHC. Why is that acute? Because of the right angles AC'C and AA'C, the angles HCA and CAH are acute; so is AHC, since it is caught in the quadrilateral A'HC'B with two right angles and one obtuse angle. Why do we want it acute? Because our arguments will rest on the principle that any vertex is as good as any other.

To prove the promised identity K*=K'=K", we have to reach into geometric prehistory for a theorem which seems to have been known forever—almost certainly to Thales, who was (with Pythagoras) one of the founders of Greek geometry. In school texts, it goes by the handle of "Angles Subtended by a Chord" or "Circle Theorems", and it says:

Consider a segment PQ and points R and N on the same side of the line PQ. Then R lies on the circumcircle of the triangle QPN if and only if the angles PNQ and PRQ are congruent.

You may look this up in a book or an electronic resource to get an idea of what it is saying. But in many of these places, the reasoning is horribly convoluted, and it is much more fun to figure it out by counting up angles in the isosceles triangles PQO, PON, and NOQ. For now, we require only two *corollaries* of this useful old-timer:

(1) A point R lies on some circle of diameter PQ if and only if PRQ is a right angle.

(2) If the diagonals of a trapezoid are congruent, all its vertices lie on one circle.

Corollary (1) follows from the theorem by taking N to be the apex of an isosceles right triangle with hypotenuse PQ. For Corollary (2), you could let PQ and NR be the two parallel sides of the trapezoid. Then there are several ways of concluding the congruence of the triangles PRQ and QNP from that of the segments PR and QN. It is worthwhile to reflect on these and to fill in all details of the proof.

The fact that A', B', and C' lie on K* does not seem to be explicitly mentioned in Euler's paper (unless my Latin is too weak to detect it), but it *was* noted decades later by the distinguished French geometers Brianchon and Poncelet.

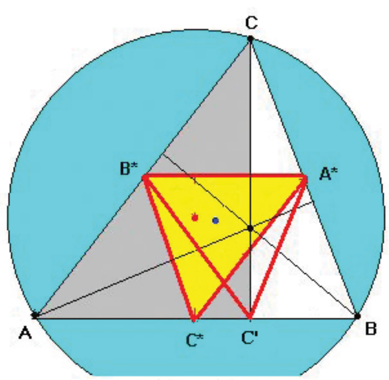


Figure 5

To see that C' lies on K*, imagine the circumcircle (not shown in Figure 5) of the right triangle AC'C. Corollary (1) implies that AC is one of its diameters, whence B* is its centre and B*C' one of its radii. This makes B*C' congruent to B*A and hence to A*C* (see?). Corollary 2 now says that all vertices of the trapezoid A*B*C*C' lie on a single circle—which can only be the circumcircle K* of A*B*C*.

Analogous arguments, using different right triangles (which ones?), will pull B' and A' onto K*.

For the initially promised result, it remains to bring K" into the story. This was done by a contemporary of Brianchon and Poncelet, a French mathematician with the remarkable name of Olry Terquem. People who know how to pronounce that name therefore tend to refer to the circle K*=K'=K" as *Terquem's Circle*, while Terquem himself had named it the *circle of the nine points* (in French). In English this has become the *nine-point circle*, a puzzling label, since everyone knows that a circle has many more than nine points.

Figure 6 is meant to show why A" lies on K*=K (this equation having been proved already). If you draw a

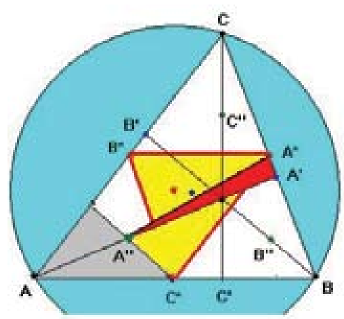


Figure 6

parallel to BH through C*, you will create a triangle AC*A" (gray) which is half the size, by linear measure, of the triangle ABH (since C* is the midpoint of AB). Being parallel to BH and hence to BB', the segment C*A" is perpendicular to AC as well as to C*A*. In other words, A"C*A* is a right triangle with hypotenuse A"A*—but so is the triangle A"A'A*, for even more obvious reasons. Corollary (1) says that A"A* is a diameter of the circumcircles of both A"C*A* and A"A'A*, which must therefore coincide. Now we have A", C*, A', and A* on a single circle which can only be the circumcircle of C*A'A*, i.e., the circle K*=K'.

Again, analogous arguments also put B" and C" on this circle K*, which has O* (the midpoint of the Euler line) as its centre and half the radius of K.

This seems to be sufficient reason to call it "Euler's circle", especially since it was Euler who began to study triangles in these terms. Other names like "Terquem" or "Feuerbach" are used only by those who know how to pronounce them, and "nine-point" is somewhat misleading—though it explains why the midpoint of the Euler line is usually labelled N.

And where did Feuerbach—whoever that was—come in? Young Karl Feuerbach put the icing on this cake in 1822, by showing that K* was tangent to the incircle as well as to the three "exocircles" of ABC. But thereby hangs another tale.

Leonhard Euler: A Biography

by Alexander Litvak and Alina Litvak

adapted from an article in *Pi in the Sky* issue 6 (March 2003)

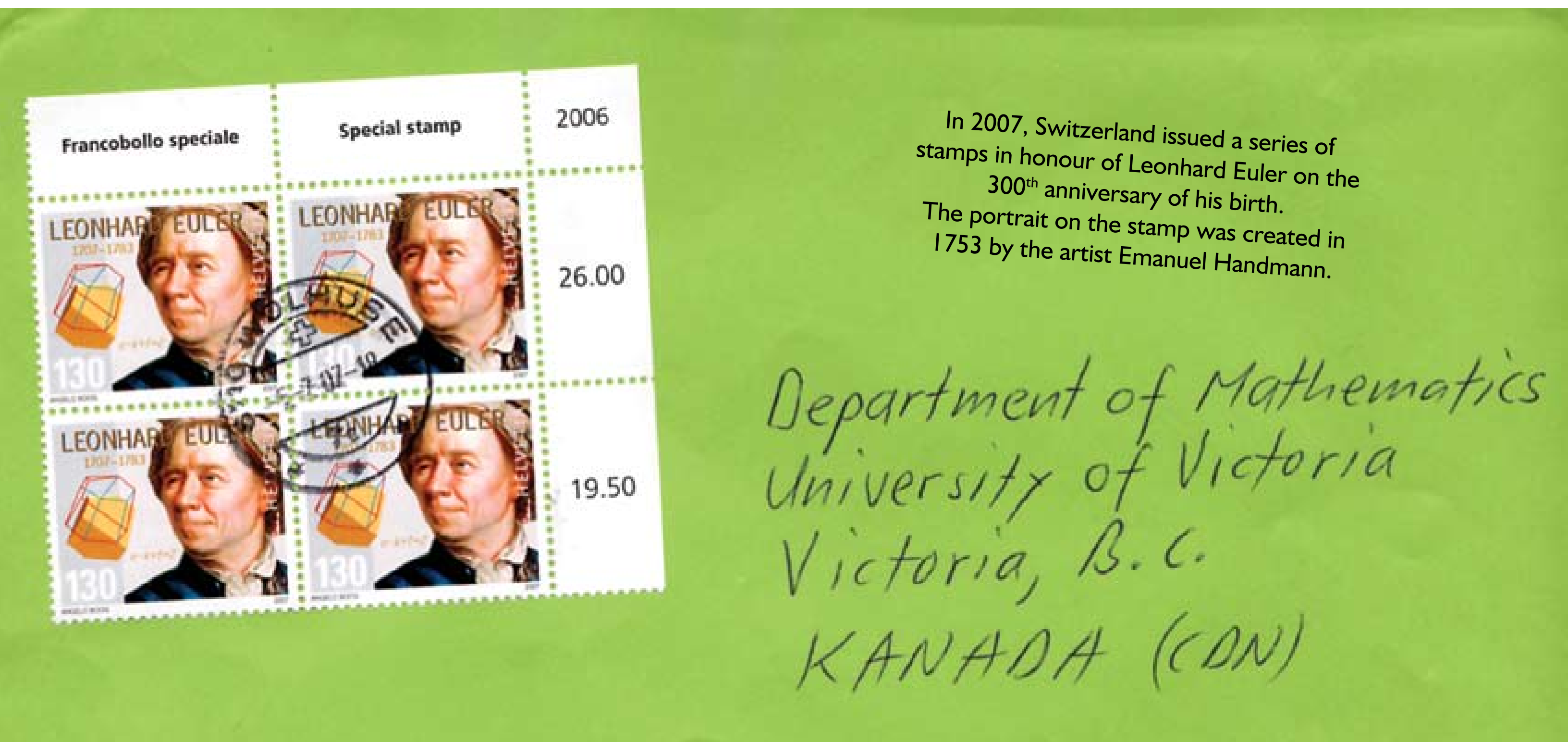
Leonhard Euler was born in Basel, Switzerland on April 15, 1707. His father was a pastor and wanted his son to become a minister as well. Euler was sent to the University of Basel to study theology. However, it turned out that the young man had a gift for mathematics. Swiss mathematician Johann Bernoulli paid attention to the talented student and convinced the elder Euler to allow his son to change his specialization to mathematics. Euler finished his studies at the University of Basel in 1726. He published his first research paper in 1726 and his second in 1727. His work on the best arrangement of masts on a ship was submitted for the Grand Prize of the Paris Academy of Science and won second place. That was a big achievement for the young mathematician. In 1726, Euler was offered, and accepted, the physiology post at the Russian Academy of Science in St. Petersburg.

Portrait of Leonhard Euler by Johann Georg Brucker

Euler arrived in St. Petersburg in 1727. D. Bernoulli and J. Hermann, who were already working at the Russian Academy, helped Euler to join the mathematics–physics division, which meant that he also became a full member of the Academy. The same year, Euler married Katherine Gzel, daughter of a Swiss painter who worked in St. Petersburg.

In 1736, Euler published the two-volume work "*Mechanica, sive motus scientia analytice exposita*," where he applied mathematical analysis methods to the problems of motion in a vacuum and in a resisting environment. This work earned him world fame. Euler developed some of the first analytical methods for the exact sciences; he started to apply differentiation and integration to physical problems. By 1740, Euler had attained a very high profile, having won the Grand Prize of the Paris Academy of Science in both 1738 and 1740. He had also written the wonderful

The Russian Academy of Science in St. Petersburg



In 2007, Switzerland issued a series of stamps in honour of Leonhard Euler on the 300th anniversary of his birth. The portrait on the stamp was created in 1753 by the artist Emanuel Handmann.

"*Direction to Arithmetic*," which was later translated into Russian. It was the first Russian book to represent arithmetic as a mathematical science.

In 1740, after the death of the Empress Anna Ioanovna, two-month-old Ioan IV was declared Emperor of Russia. As he was too young to rule, his mother, Anna Leopoldovna, became regent. Living in Russia became dangerous, especially for foreigners, and Euler decided to accept the invitation of Frederick the Great, the King of Prussia, to work in Berlin. There, Euler was met with great respect and was given the freedom to pursue his research as he wished. However Euler didn't completely end his work for Russian Academy. He was still partially paid by Russia, and he continued to write reports for the Academy and teach young Russians who arrived in Berlin.

Frederick the Great

Euler's 25 years in Berlin were very busy and productive. He enjoyed great mathematical success and also found time to accomplish all kinds of social work. For example, he served on the Library and Scientific Publications Committee of the Berlin Academy and was a government adviser on state lotteries, insurance, annuities and pensions, and artillery.

Euler wrote nearly 380 articles while he was in Berlin. He also wrote many scientific and popular science books, including his famous "*Letters to a Princess of Germany*," which was translated into many languages. He also led the Berlin Academy of Science after the death of Maupertuis in 1759, although he never held the formal title of President.

Euler's phenomenal ability to work is demonstrated by the fact he produced about 800 pages of text per year. That would be a significant number even for a novelist; for a mathematician, it is hardly believable. Euler made a big contribution to analysis, geometry, trigonometry, and number theory, and introduced such notation as $f(x)$ for function, Σ for sum, e for the base of natural logarithm, π for the ratio of the length of a circle to its diameter, and i for imaginary unit. Euler proved the following formula for a convex polyhedron: $V+F=2+E$, where V is number of vertexes of the polyhedron, F is number of faces of the polyhedron, and E is number of edges of the polyhedron. This formula has the extension, very important in topology, called Euler characteristics. In addition to his work in mathematics, Euler published works in philosophy, astronomy, physics, and mechanics. Using the graph theory that he introduced, Euler solved the following famous problem, the so-called "Königsberg's Bridges Problem." *(See related article on page 7).*

Catherine the Great

After ascending the Russian throne in 1762, Empress Catherine II offered Euler an important post in the mathematics department, conference-secretary of the Academy. She instructed her representative to agree to Euler's terms if he didn't like her first offer, to ensure that he would arrive in St. Petersburg as soon as possible.

In 1766, Euler returned to St. Petersburg. Soon after, he became almost blind due to a cataract in his left eye (his right eye was already very poor). However, that didn't stop him from working—Euler continued his work through dictation.

In September 1771, Euler had surgery to remove his cataract. The surgery took only three minutes and was successful—the mathematician's vision was restored. Doctors advised Euler to avoid bright light and overloading his eyes; reading and writing were forbidden. Unfortunately, Euler didn't take care of his eyes; he continued to work and after a few days lost his vision again, this time without any hope of recovery. Amazingly, his productivity only increased. Despite his blindness, Euler wrote almost half of his articles after his return to St. Petersburg.

After his wife's death in 1773, Euler continued to work diligently, having others read to him. He worked until Sept. 18, 1783, when Euler gave a mathematics lesson and discussed recently discoveries by astronomers. He died later that evening.

A painting of the Mikhalovsky Castle in St. Petersburg

Equations: The Game Of Creative Mathematics

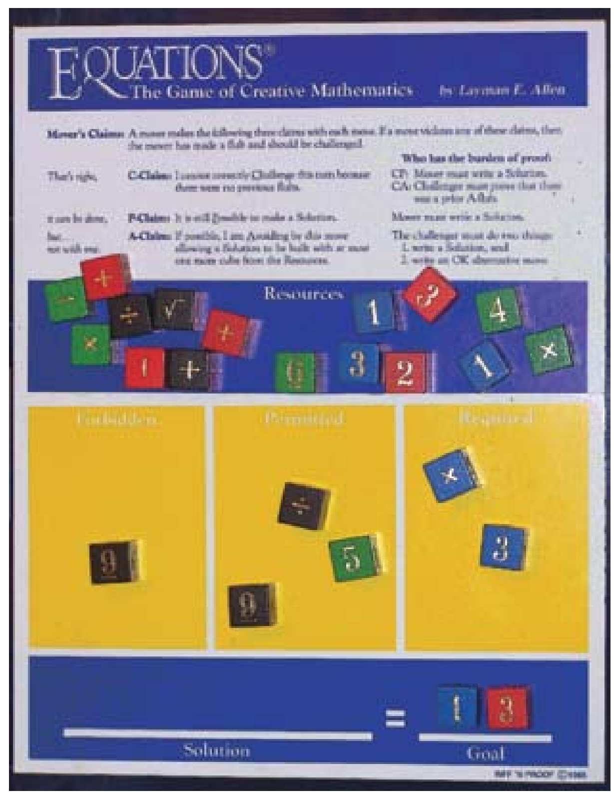
A math game for First Nations (and all nations)

by Campbell Ross, Grande Prairie Regional College, and
Brian Phung, Math Coordinator, Sturgeon Lake Band School

When the reporter from the local paper asked Keenan why he liked the math game 'Equations', he replied that it challenged him more than the textbook exercises. "It was real easy (before), now it's harder for me—it makes me think better." Keenan Goodswimmer is 11 years old and belongs to the Sturgeon Lake Cree Nation in northwest Alberta. He is in Grade 5 at his Band school on the reserve. Along with 15 other students from his own school and another in the nearby community of Grande Prairie, he was demonstrating this math game to teachers attending their annual Teachers Convention in March, 2007.

What is Equations: the game of Creative Mathematics?

It began when Layman E. Allen, then professor in the faculty of law in Yale University, surprised by student apprehension over a course that "looked like mathematics," developed a series of user friendly exercises in symbolic logic that culminated in the game WFF'N PROOF (WFF is an acronym for 'well formed formula'). Interest and financial support from the Carnegie Foundation expanded these efforts into creation of mathematics teaching materials, including Equations, for elementary and middle school students.



School classes play Equations in groups of three students of roughly equal ability, in a tournament format with each player representing a different team. This means that both the mathematically weak and the mathematically strong players have an equal chance of contributing to the overall points gained by their team, since each is playing with opponents at their own skill level and thus has a reasonable chance of emerging the winner (with most points) in their particular group. This stress on the cooperative element in team success is very motivating.

Equations is described by its creators as a resource allocation game. The resources are a set of cubes (students just call them dice) whose sides are imprinted with either integers (from 0 to 9) or operations symbols (addition, subtraction, multiplication, division, radicals or exponents).

The first player sets a goal: a number that can be arrived at by some combination of the numbers and operations appearing on the thrown dice. The players then take turns allocating the remaining resources into three categories—Forbidden, Permitted, and Required—while claiming that an equation that matches the goal can still be constructed from the assigned and unassigned resources. Players win, or lose, by challenging this claim.

Why Equations for First Nations?

Equations seemed to be successful with Keenan and his fellow students at Sturgeon Lake Band School because it made them think about math in ways that were challenging and fun. Both advantages have a particular relevance for ways of thinking that are important in First Nation culture.

Equations meets two needs for improving math achievement among First Nations students: it captures the subtle interaction of competition and cooperation in Aboriginal ethics and makes the transition from contextualized to de-contextualized mathematical thinking.

Research into the nature of mathematical thinking in traditional First Nation culture in Canada suggests that the kinds of numeracy being used were all context dependent: for example, the builder using the span of his own arm as a scale when giving directions to another on how to build a birch bark canoe. Another example of cultural context might be the spiritual significance given to certain numbers, such as four, or certain geometric shapes, such as the circle.

As a result, the design of most support materials in mathematics for Native students is based on creating linkages with specific contexts within traditional culture (the geometry of the tipi) or traditional systems of production (measurement of materials for traditional dead-fall traps or rabbit snares).

Support materials in mathematics for Native students that rely upon visual (or easily visualized) linkages to elements of traditional culture might be extremely helpful in early stages of numeration, in lower elementary school, but may hinder the transition into the self-contained language and operations of the mathematics that the student will encounter in secondary school. Equations is about doing real math in a way that builds the familiarity and facility with math's own ways. It promotes thinking about numbers and their relations with the same ease as 'gossip', to use Keith Devlin's vivid metaphor for the mental readiness necessary for success in the math to come.

Another advantage of Equations is that it finds a way to meet the widespread Aboriginal respect for individual autonomy creating a discomfort about the possibility of direct contradiction of others in formal settings as might seem encouraged by Constructivist teaching methods. Consider the unease this may create for the child who has been raised to believe that it is rude and disrespectful to directly challenge the views of any other. But there is one kind of acceptable confrontation and that is in a game or tournament. This acceptability through game competition is enhanced in Equations by the cooperative design of that tournament format, as described above.

Equations is particularly appropriate to general and specific outcomes for grades 5–7 as outlined in the Western Canadian Protocol for mathematics.

The appeal of Equations for all students lies in the fun they have doing real math in this kind of classroom tournament. No one is eliminated. Every student has a fair chance of contributing to the success of her team, because after each round students are re-allocated so as to play against those of equal ability based on previous results. At the end of the month, the students themselves can produce an Equations newsletter to parents, telling them of everybody's success.

Equations: The Game of Creative Mathematics, created by Layman E. Allen, Professor of Law and Senior Research Scientist, University of Michigan, is available from Accelerated Learning Foundation, 402 E. Kirkwood Avenue, Fairfield, IA 52556, USA.

Below are some references which may be of interest to those who may wish to look at some of the available research on mathematics and First Nations culture.

References

Rupert Ross: Dancing with a Ghost, 1996

Diamond Jenness: Indians of Canada, 1932

Ron Scollon and Suzanne Scollon Narrative, Literacy and Face in Interethnic Communication, 1981

J. Peter Denny: "Cultural Ecology of Mathematics: Ojibway and Inuit Hunters" in Native American Mathematics Michael P. Closs ed., University of Texas Press, 1986

F. David Peat: "Sacred Mathematics" in Blackfoot Physics, 2002

Keith Devlin: The Math Gene: How Mathematical Thinking Developed, 2000

Math Jokes

Q: What does a mathematician say at the door on Halloween?

A: Trig or treat!!!

Q: What does a mathematician say when she comes home and doesn't find her parrot?

A: Polly gone!

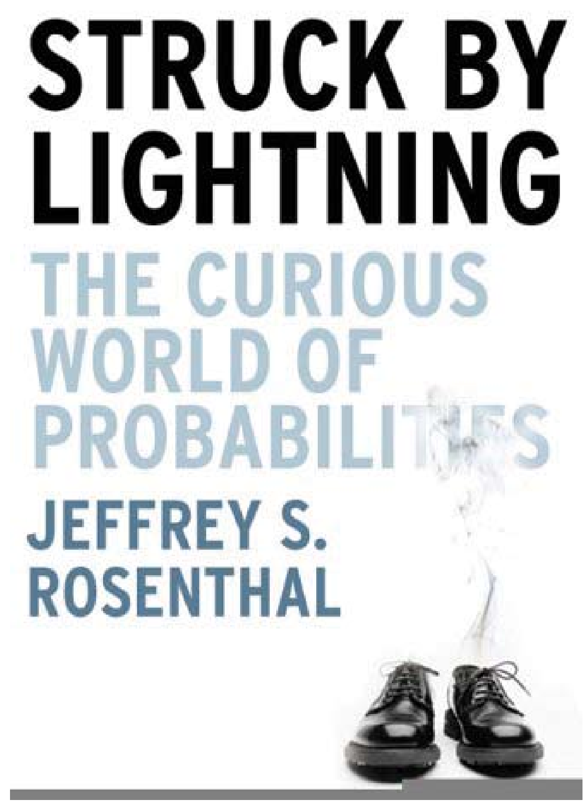
Book Reviews

Struck by Lightning: the Curious World of Probabilities by Jeffery S. Rosenthal

by David Leeming

Jeffrey Rosenthal's entertaining book highlights topics where a knowledge of probability adds insight. It takes us on a tour of casinos and gambling (Chapter 3), murder (Chapter 5), polls, 'margin of error' (Chapters 10 and 11), spam (Chapter 15), and much more!

In addition to clear explanations of the chapter topics, Rosenthal uses both liberal amounts of humour and 'between the lines' fictional vignettes to illustrate the topics. For example, on page 17 is 'Musical Mayhem'. Here is an excerpt:



> You're excited because you have just purchased the Super Special Song Spooler digital music device. Eagerly you download 4,000 favourite songs, and press the button to play in random order. The headphones pulsate as guitar riffs mingle with drum solos in a scintillating symphony of sounds. You look forward to musical variety and bliss for the rest of your days.
>
> As the 75th song begins, you are shocked to hear that it is a repeat of the 42nd song... How could this be? The device must be defective!

Is it defective? Read the book to find out.

Here are three sections of the book that are among my favourites: Chapter 2 gives the clearest explanation of the 'birthday problem' that I have seen. This problem, whose solution belies our intuition, has its mystery revealed. Chapter 5 investigates (among other topics) homicide trends. Are we in more danger of being murdered in Canada now than in 1992? (Again, read the book to find out!). Chapter 13 explores randomness in biology—including both genetics ('designer blue genes') and viruses. In my edition of the paperback version there is an 'update'. Following the Index comes "The Probabilities Just Keep Flowing" which includes a look at the popular television game show *Deal Or No Deal.*

I would highly recommend this book. It is both entertaining and informative on the subject of probability. It is certainly accessible to almost anyone—even to a Grade 11 student who has been paying attention in math class.

Few books end with a final exam. This one does. It tests to find out if the reader now has achieved the 'Probability Perspective'. (I probably passed).

Jeffrey S. Rosenthal is Professor of Statistics at the University of Toronto. *Struck by Lightning* first appeared in hard cover in 2005 (ISBN-13: 978-0-00-639495-2), then in paperback in 2006 (ISBN-10: 0-00-639945-7). It is published in Canada by HarperCollins Publishers, Ltd. and has been translated into seven other languages.

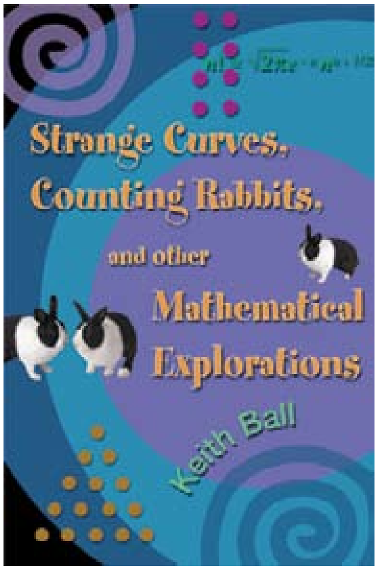
http://probability.ca/sbl/

Strange Curves, Counting Rabbits and Other Mathematical Explorations by Keith Ball

by Gordon Hamilton

Yeah, I liked this book!

Keith Ball is that rare breed of mathematician who is also a great teacher and author. He has an engaging way of introducing each chapter, taking readers to a rich mathematical area, and guiding them through its beautiful proofs. He does not hold the reader's hand throughout, but sets some appropriate problems along the way. Their complete solutions appear at the end of each chapter. From start to finish, the reader is in capable hands.



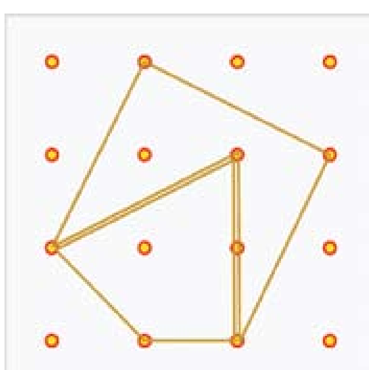
If the book does suffer from anything, it is in the breadth of audiences that the chapters target. The chapters accessible to the recreational reader and a good Grade 9 student include those on Hamming codes (Chapter 1), Pick's Theorem (Chapter 2), Peano's area-filling curve (4) and arguably Chapter 3: Fermat's Little Theorem and Infinite Decimals. The rest of the book (Chapters 5-10) assumes a comfort level with logarithms, calculus and complex algebraic formulae. Their inclusion makes the book more suitable as an award for top Grade 12 students than for students of the lower grades.

Let's look at a few examples that demonstrate why I liked this book so much. All 10 chapters are eloquent and so the examples could have been chosen from any chapter.

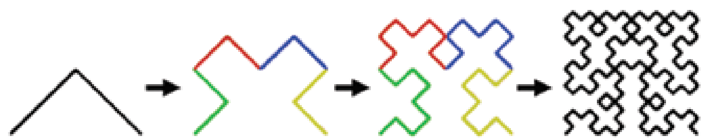
1) Pick's theorem (Chapter 2) is that the area of a lattice polygon is given by the formula

(# of Interior Points) + (# of Boundary Points) -1

Keith proves Pick's theorem by first showing that the formula has the additive property: Arbitrarily split a lattice polygon into two smaller lattice polygons; the sum of the areas of the two smaller lattice polygons equals the area of the original lattice polygon. Secondly, he decomposes any polygon into triangles—following the 1985 proof by Dale Varberg in *American Mathematical Monthly*. Pick's theorem is not new to me, but I have never seen it proved so elegantly. Bravo!



2) One of my favourite problems in the book is asked about one of Hilbert's area-filling curves (Chapter 4): where would you be if you started at the lower right and traveled a third of the way along his recursively defined curve?



(This is one of Hilbert's area-filling curves. The limiting curve has infinite length and completely fills a square. The central two images are coloured to allow the reader to see the recursive pattern more easily.)

3) For anyone who has enjoyed exploring Taylor series approximations, Padé approximations (Chapter 9) are a must-see. Elsewhere I have seen the end product of these calculations—specifically the $\tan(x)$ formula:

$$\tan(x) = x\,(1 - x^2/\,(3 - x^2/\,(5 - x^2/\,(7 - x^2 \ldots))))$$

but again, Keith's treatment is the best I've read. He expertly guides the reader to these derivations in an effortless, way with plenty of diagrams to compare the accuracy of successive Padé and Taylor approximations. Bravo!

The book does not suffer from any glaring mistakes… except on the first paragraph of the first page where Keith, in preparation for a discussion on error-correcting codes, confidently asserts that the ISBN sequence of the book is 0-691-11321-1. It's not. It is 0-691-12797-2. I have the soft-cover edition. Presumably the quoted ISBN number is that of the hard-cover edition… so in order to get fewer mistakes in your copy – make sure to order the hard cover edition!

Solutions to the Problems Published in the January, 2007 Issue of Pi in the Sky:

Problem 1

Prove there are infinitely many rational numbers r such that $\sqrt{r}$ and $\sqrt{r+1}$ are simultaneously rational numbers.

Solution by Lisa Lajeunesse, Capilano College, British Columbia:

Let n be any odd integer. Then n^2 is also odd and $n^2=2k+1$ for some integer k.
Define $r=\frac{k^2}{n^2}$ for which both r and $\sqrt{r}$ are rational. Then,

$$\sqrt{r+1}=\sqrt{\frac{k^2}{n^2}+1}=\sqrt{\frac{k^2+n^2}{n^2}}=\sqrt{\frac{k^2+2k+1}{n^2}}=\sqrt{\frac{(k+1)^2}{n^2}}=\frac{k+1}{n}$$

is also rational.

Finally, each choice of odd n yields a distinct rational number $r=\frac{k^2}{n^2}$. Indeed if $m\neq n$ is another odd integer such that $m^2=2h+1$ and $\frac{k^2}{n^2}=\frac{h^2}{m^2}$ then $k^2(2h+1)=h^2(2k+1)$ i.e. $k=h$ or $m=n$, a contradiction.

A similar solution was submitted by Tzvetalin Vassilev, Saskatoon.
Don Helvie, Montesano High School, Washington, submitted a different solution by taking r as above, with $k^2+n^2=h^2$ (Pythagorean triple). We could consider $k=a^2-b^2$, $n=2ab$, $h=a^2+b^2$ with a,b as positive integers.

Problem 2

Consider the number 3025. If we split the digits into 30 and 25, add these two numbers and square the result, we get 3025 back. Find another four–digit number, all four digits distinct, which has the same property.

Solution:
A four–digit number M can be written as $M=100b+a$, where a,b are integers such that $0\leq a\leq 99$, $10\leq b\leq 99$. The required numbers must verify $(a+b)^2=100b+a$ or equivalently.

$$(a+b)(a+b-1)=99b \qquad (*)$$

Thus $99\,|\,(a+b)(a+b-1)$. Since $a+b$ and $a+b-1$ are relatively prime we could have:

(i) $99\,|\,(a+b)$ or $99\,|\,(a+b-1)$
(ii) $9\,|\,(a+b)$ and $11\,|\,(a+b-1)$
(iii) $9\,|\,(a+b)$ and $9\,|\,(a+b-1)$

(i) If $a+b=99k$, with k any positive integer, by using (*) we get $k(99k-1)=b$.

Only $k=1$ works well and gives $b=98$ and $a=1$, hence $M=9801$.

If $a+b-1=99k$ then from (*) $(99k+1)k=b$ and no value for k is convenient.

(ii) If $9\,|\,(a+b)$ then $a+b=9k$. We should have $9k\equiv 1 \pmod{11}$ or $k\equiv 5 \pmod{11}$ i.e. $k=11h+5$, where h is any integer $h\geq 0$. Substituting in (*) we get:

$$(11h+5)(9h+4)=b$$

Only $h=0$ is convenient. Hence $b=20$, $a=25$ and thus $M=2025$.

(iii) Similarly as above we get $M=3025$.

We conclude that $M=9801$ is the only four–digit number having the required properties.

Several solvers mentioned the number 9801 but did not explain how they reached this number.

This problem can be found in J. T. Rogers' book, *The Calculating Book,* Random House (1974). An extension of this problem is proposed by Robert M. Hashway, *The American Mathematical Monthly* 812 (1976).

Problem 3

Let m, n be positive integers such that $n\sqrt{23} - m > 0$. Prove that $n\sqrt{23} - m > \frac{2}{m}$.

Solution:

If $m \in \{1,2\}$ then $n\sqrt{23} - m > \frac{2}{m}$ for any positive n, hence we may assume $m > 2$. Also, $23n^2 - m^2 \notin \{1,2,3,4\}$ for any positive integers m,n. Indeed, if by contradiction $23n^2 - m^2 \in \{1,2,3,4\}$ then $m^2 \pmod{23} \in \{19,20,21,22\}$. This is not possible since

$$m^2 \pmod{23} \in \{0,1,2,3,4,6,8,9,12,13,16,18\}$$

Hence, $23n^2 - m^2 \geq 5$. Since $1 > \frac{4}{m^2}$ we obtain

$$23n^2 \geq m^2 + 5 = m^2 + 4 + 1 > m^2 + 4 + \frac{4}{m^2} = \left(m + \frac{2}{m}\right)^2$$

that is $n\sqrt{23} > m + \frac{2}{m}$, as required.

Also solved by Kee–Wai Lau, Hong Kong.

Alternative Solution to Problem 3

Solution by Maria Binkowska, Poland:

Let us assume by contradiction that $n\sqrt{23} \leq m + \frac{2}{m}$. Then $m < n\sqrt{23} \leq m + \frac{2}{m}$ or equivalent $m^2 < 23n^2 \leq m^2 + 4 + \frac{1}{m}$.

Since $23n^2$ is an integer we must have $m^2 < 23n^2 \leq m^2 + 4$ hence $23n^2 \in \{m^2+1,\ m^2+2,\ m^2+3,\ m^2+4\}$.

1. Let $23n^2 = m^2 + 1$. Since $23n^2 - m^2 = 1$ then one of the number m, n is even and the other is odd. If n is even and m is odd, then $23n^2$ is divisible by 4 and m^2+1 is not divisible by 4. Hence the equality $23n^2 = m^2+1$ does not hold. When n is odd and m is even, then $23n^2 \equiv 3 \pmod 4$ while $(m^2+1) \equiv 1 \pmod 4$ and therefore we still have $23n^2 \neq m^2+1$.

2. Let $23n^2 \equiv 3m^2 + 2$. Then the numbers n,m must both be even or both odd. Let m,n both be even. Then $23n^2$ is divisible by 4 and m^2+2 is congruent to 2 modulo 4. Let m,n both be odd. Let $n = 2x+1$, $m = 2y+1$. We have

$$23 \cdot 4x^2 + 23 \cdot 4x + 23 = 4y^2 + 4y + 3 \Longleftrightarrow 23 \cdot x(x+1) + 5 = y(y+1).$$

But $23 \cdot x(x+1) + 5$ is odd and $y(y+1)$ is even. Hence the equality $23n^2 = m^2 + 2$ is impossible.

3. Let $23n^2 = m^2 + 3 \Rightarrow 23n^2 - m^3 = 3$. For every integer p, p^2 is divisible by 3 or p^2 is congruent to 1 modulo 3. If $23n^2 - m^2 = 3$ then we have n,m both divisible by 3 and $23n^2 - m^2$ is divisible by 9 and we have a contradiction.

4. Let $23n^2 = m^2 + 4 \Longleftrightarrow 23n^2 - m^2 = 4$. For any integer p, p^2 is divisible by 4 or p^2 is congruent to 1 modulo 4. It implies that m,n are both even. Let $m=2x$, $n=2y$. We have $23 \cdot 4x^2 = 4y^2 + 4 \Longleftrightarrow 23x^2 = y^2 + 1$ but the last equation is impossible (see 1.)

We have a contradiction, it means that $n\sqrt{23} > m + \frac{2}{m}$.

Maria Binkowska also successfully solved all other problems in *Pi in the Sky*, issue 10.

Problem 4

Find all real numbers $x_1, \dots, x_{2007}$ such that $x_1 + \dots + x_{2007} = 2007$ and $|x_1 - x_2| = |x_2 - x_3| = \dots = |x_{2007} - x_1|$.

Solution by Kee-Wai Lau, Hong Kong:
We show that the only real numbers $x_1, \dots, x_{2007}$ satisfying the conditions of the problem are $x_1 = x_2 = \dots = x_{2007} = 1$.

It suffices to show that $|x_1 - x_2| = |x_2 - x_3| = \dots = |x_{2007} - x_1| = 0$. Suppose, on the contrary, that

$$|x_1 - x_2| = |x_2 - x_3| = \dots = |x_{2007} - x_1| = k > 0$$

If n of the 2007 numbers $x_1 - x_2, x_2 - x_3, \dots, x_{2007} - x_1$ equal k and $2007 - n$ of them equal $-k$, where $0 \le n \le 2007$, then

$$\begin{aligned} 0 &= (x_1 - x_2) + (x_2 - x_3) + \dots + (x_{2007} - x_1) \\ &= nk + (2007 - n)(-k) \\ &= k(2n - 2007) \end{aligned}$$

which is impossible. Hence $|x_1 - x_2| = |x_2 - x_3| = \dots = |x_{2007} - x_1| = 0$ and this completes the solution.
A correct solution was received from Tzvetalin Vassilev, Saskatoon.

Problem 5

Solution:
Find all $a, b, c \in \mathbb{R}$ such that $|ax^2 + bx + c| \le 1$ for every $-0.5 \le x \le 0.5$ and $a^2+b^2+c^2$ is maximum.

We solve a more general problem:
Let $\alpha \in (0, 1], \beta > 0$. Find all quadratic polynomials $q(x) = ax^2+bx+c$ such that $|q(x)| \le \beta$ for every $x \in [-\alpha, \alpha]$ and $a^2+b^2+c^2$ is maximum.
Taking $x = \pm\alpha, x = 0$ in $|q(x)| \le \beta$ we get $|a\alpha^2 \pm b\alpha + c| \le \beta$ and $|c| \le \beta$.
Now

$$|a\alpha^2 \pm b\alpha| = |a\alpha^2 \pm b\alpha + c - c| \le |a\alpha^2 \pm b\alpha + c| + |c| \le 2\beta$$

Hence

$$|a\alpha \pm b| \le 2\beta/\alpha$$

On the other hand,

$$a^2 + b^2 + c^2 = \frac{1}{2\alpha^2}(a\alpha + b)^2 + \frac{1}{2\alpha^2}(a\alpha - b)^2 + c^2 - \frac{1-\alpha^2}{\alpha^2}b^2 \le \frac{4\beta^2}{\alpha^4} + \beta^2$$

The equality holds if and only if $b = 0, a = \pm 2\beta/\alpha^2$ and $c = \mp\beta$. The requested polynomials are $p(x) = \dfrac{\pm 2\beta}{\alpha^2}x^2 \mp \beta$.
Problem 5 is also retrieved if $\beta=1$, $\alpha=1/2$.
Also solved by Kee-Wai Lau, Hong Kong.

Problem 6

Two altitudes of a triangle are of length 1 and 2. If h denotes the length of the third altitude, then show that $2/3<h<2$.

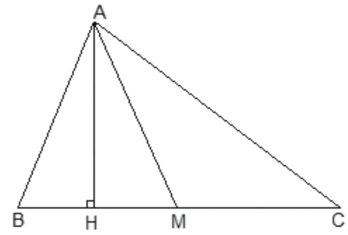
Solution by Kee-Wai Lau, Hong Kong:
Let A be the area of the triangle so that the lengths of the three sides of the triangle are $2A$, A and $2A/h$. Since the sum of any two sides of a triangle is greater than the third side, we must have $2A+A>2A/h$ and $A+2A/h>2A$. Hence $2/3<h<2$ as required.
A similar solution was received from Tzvetalin Vassilev, Saskatoon.

A Geometry Challenge Problem

by Danesh Forouhari

Let AM, BN, and CP be the medians in the triangle ABC. Prove $3(AB^2 + AC^2 + BC^2) = 4(AM^2 + BN^2 + CP^2)$.

Solution



Consider the following triangle where AH and AM are the height and median to side BC, respectively.

In the right triangle AHM, we have

$$AM^2 = AH^2 + HM^2.$$

We replace AH and HM in the above equation with their equivalents TWICE, i.e., first

$$\begin{aligned} 1.\ AM^2 &= AH^2 + HM^2 \\ &= AB^2 - BH^2 + (MB - BH)^2 \\ &= AB^2 - BH^2 + MB^2 + BH^2 - 2MB \cdot BH \\ &= AB^2 + MB^2 - 2MB \cdot BH \\ &= AB^2 + (BC/2)^2 - 2 \cdot BC/2 \cdot BH \\ AM^2 &= AB^2 + BC^2/4 - BC \cdot BH \qquad \text{(A)} \end{aligned}$$

and second

$$\begin{aligned} 2.\ AM^2 &= AH^2 + HM^2 \\ &= AC^2 - HC^2 + (HC - MC)^2 \\ &= AC^2 - HC^2 + HC^2 + MC^2 - 2HC \cdot MC \\ &= AC^2 + MC^2 - 2 \cdot HC \cdot MC \\ &= AC^2 + (BC/2)^2 - 2\, HC \cdot BC/2 \\ AM^2 &= AC^2 + BC^2/4 - HC \cdot BC \qquad \text{(B)} \end{aligned}$$

Adding equations (A) and (B), we get

$$\begin{aligned} 2AM^2 &= AB^2 + BC^2/4 - BC \cdot BH + AC^2 + BC^2/4 - BC \cdot CH \\ &= AB^2 + AC^2 + BC^2/2 - BC\,(BH + CH) \\ &= AB^2 + AC^2 + BC^2/2 - BC^2 \\ 2AM^2 &= AB^2 + AC^2 - BC^2/2. \qquad \text{(C)} \end{aligned}$$

Performing the same steps for medians BN and CP, we get

$$2\,BN^2 = AB^2 + BC^2 - AC^2/2 \qquad \text{(D)}$$

$$2\,CP^2 = AC^2 + BC^2 - AB^2/2 \qquad \text{(E)}$$

Adding equations (C), (D) and (E), we get

$$2\,(AM^2 + BN^2 + CP^2) = 3/2\,(AB^2 + AC^2 + BC^2)$$

So

$$4\,(AM^2 + BN^2 + CP^2) = 3\,(AB^2 + AC^2 + BC^2) \qquad \text{QED}$$

Have a solution to the Pi in the Sky challenges?
A challenge for our readers?

Send them to us:
Pi in the Sky - Math Challenges
200 - 1933 West Mall
University of British Columbia
Vancouver, BC V6T 1Z2
Canada

Or by email:
pi@pims.math.ca

Pi in the Sky Math Challenges

Problem 1

Find all four–digit numbers $abcd$ such that $abcd - a^4 - b^4 - c^4 - d^4$ has a maximum value.

Problem 2

Let $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\} = \{1, 2, \ldots, 2n\}$ such that $a_1 < \ldots < a_n$ and $b_1 > \ldots > b_n$. Prove that

$$|a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n| = n^2$$

Problem 3

Let A be a set of positive integers such that for every $m, n \in A$,

$$|m - n| \geq \frac{1}{20}(m+1)(n+1).$$

What is the maximal number of elements that A could have?

Problem 4

Prove that any positive integer n coprime to 10 is a divisor of a repunit.

- Two integers are called coprime (or relatively prime) if they do not have a common divisor greater than 1.
- A repunit is an integer consisting of copies of the single digit 1.

Problem 5

Let n be a positive integer and let $\varphi(n)$ denotes the number of positive integers less than or equal to n that are coprime to n. Prove that for any positive integers m and n

$$\varphi(mn) \leq \sqrt{\varphi(m^2)\varphi(n^2)}.$$

Problem 6

R and r are respectively the radii of the spheres circum–scribed about, and inscribed in, a tetrahedron. Prove that $R \geq 3r$.

Problem 7

Prove that in any convex polyhedron

(a) there is either a triangle face or a vertex at which three edges meet and

(b) there is a face having less than six sides.

Pi *in the Sky* will award $100 for the most out-standing solution to each of Challenge Prob-lems 4-7 submitted by a high-school student. The decision will be made by a panel of judges appoint-ed by the Editor-in-Chief. Their decision is final.

Quickies

Q1. Let $x = \sqrt{3\sqrt{2\sqrt{3\sqrt{2\ldots\ldots}}}}$. Find x^3.

Q2. Find the positive integer n that satisfies the equality $\log_2 3 \log_3 4 \ldots\ldots \log_n(n+1) = 2007$.

Q3. If x, y are distinct real numbers such that $x^3 = 1 - y$ and $y^3 = 1 - x$, find all possible values of xy.

Q4. In the sequence obtained by omitting the squares and the cubes from the sequence of positive integers find the position on which 2007 sits.

Note: 'Quickies' solutions are not eligible for prizes.

Comments:

Problems 4–7 are related to some concepts and results of Euler.

(a) The function defined in **Problem 5** is called *Euler's phi function*, or *Euler's totient function*. To solve this prob–lem you might use the following formula

$$\varphi(n) = (p_1 - 1)p_1^{k_1 - 1} \ldots (p_s - 1)p_s^{k_s - 1}$$

where $n = p_1^{k_1} \cdot p_2^{k_2} \ldots p_s^{k_s}$ is the unique factorization of n as a product of prime numbers.

A celebrated result related to this function is *Euler–Fer–mat's Theorem* which states that if m and n are coprime positive integers, then n is a divisor of $m^{\varphi(n)} - 1$. You can use this result to solve **Problem 4**.

(b) *Euler's triangle identity* states that the distance d be–tween the circumcenter and incenter of a triangle is given by

$$d^2 = R(R - 2r)$$

where R and respectively r are the radii of the above circles. A direct consequence of this identity is so–called *Euler's triangle inequality*

$$R \geq 2r$$

which was published by Euler in 1765. **Problem 6** is a similar inequality for a tetrahedron.

(c) You can solve **Problem 7** by using the famous *Euler's polyhedral formula*

$$V - E + F = 2$$

where V, E and F, respectively, denote the number of ver–tices, edges and faces of any convex polyhedron.