The Pacific Institute for the Mathematical Sciences

π in the Sky

December 2000

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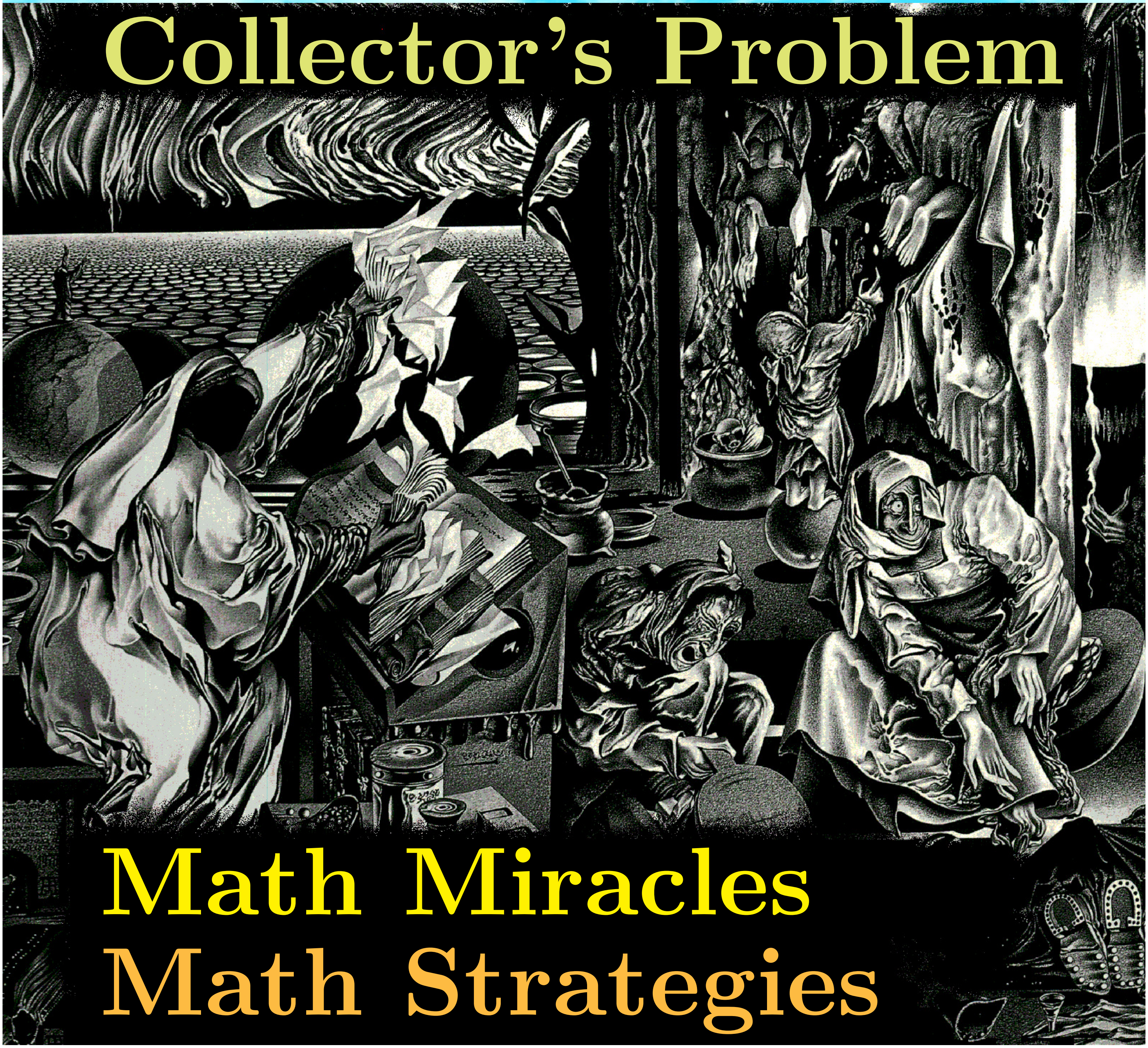

Math Miracles
Math Strategies

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This journal is devoted to cultivating mathematical reasoning and problem-solving skills and preparing students to face the challenges of the high-technology era.

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We also welcome **Letters to the Editor** from teachers, students, parents or anybody interested in math education (be sure to include your full name and phone number).

Cover page: The picture on the cover page is a fragment of a painting by prominent Russian mathematician Anatoly T. Fomenko, which was inspired by mathematical ideas. We feature other artwork by A.T. Fomenko at the web site: http://lnfm1.sai.msu.ru/lat/Zakh/alm-cat/index.html

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Math Opinion FREE FORUM*

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The Red Violin of Science

by Tomasz Kaczynski

The first scene of the Canadian film *The Red Violin,* directed by François Girard, takes place in Renaissance Italy. Master instrument maker Nicolo Bussotti inspects a violin just finished by his assistant. The instrument is fine but too ordinary—the master smashes it against a table and shouts at the poor boy, *"Put your anger into your work!"*

Recently I was struck by the following title of a talk at the Canadian Mathematical Society Winter 2000 Meeting: *Chinese excellence in mathematics teaching: can we match in North America?*

How fast the times change! Until a decade ago there was a similar fascination about the quality of mathematics teaching in the Soviet Union and in Eastern Europe. Let us recall that communist states created special schools at central locations for talented kids from all over the country. I remember one such school in my home city of Warsaw, but the most famous one was in Moscow. It was a real factory specializing in mass production of International Mathematical Olympiad winners. An intriguing observation is that those schools, besides their principal mission, were renown for accommodating children of high rank government or Communist Party officials. Now, with Soviet communism gone, special elitist schools have collapsed and the majority of Russian geniuses have emigrated to America. The centre of attention has now moved to the last bastion of communism left in the world. What is the secret of the system that imposed such high standards in the domain of science, while being so destructive for the economy and human rights? **Is it just the excellence in teaching, or is it the exceptional motivation of students that is needed for a high-quality mathematics education?**

Let me tell you my personal view on how college students of my generation were attracted to mathematics. In communist Poland, the life was grim, politics disgusting, communistic propaganda omnipresent and career prospects very obscure. **Choosing mathematics was an escape from the gruesome reality to an area that was out of reach of the bureaucrats and politicians, who were unable to understand it.**

But our dreams of finding a refuge in this abstract domain appeared to be just an illusion. The conscience did not let us stay indifferent to what was happening, and we became engaged in a political battle with the regime. This was not a unique Polish phenomenon. Prominent mathematicians were among the students rioting in Berkeley against the Vietnam War in 1967, on the barricades of Paris the same year and, certainly in large numbers, in Tiananmen Square in 1989. Of course, it would be foolish to assign to every mathematician of that époque the revolutionary label. First of all, we were fascinated by the challenge of solving problems and by the legends of famous conjectures; we tried to follow the examples of educational and scientific idols. What I want to say is that to study mathematics one needed passion, not a cold estimate of job opportunities.

Today, mathematics is seen as a means to benefit society through its applications to sciences, engineering, industry, and medicine. Mathematicians are under constant pressure to demonstrate the usefulness and potential for immediate pay-off from their research. There is a growing feeling that mathematics is being misunderstood as a set of cooking recipes and that its "spirit" is passing away. A dozen years ago at a conference in Corner Brook, Newfoundland, I listened to a talk by Professor Swaminathan of Dalhousie University about famous equations of mathematics. The first equation he presented was

$$2 + 2 = 4.$$

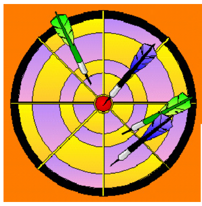
Is it too obvious? Less than it looks. An obvious equation would be 2 fingers + 2 fingers = 4 fingers or $2\,\mathrm{kg} + 2\,\mathrm{kg} = 4\,\mathrm{kg}$ or $\$2 + \$2 = \$4$. What distinguishes the first equation written above from the others? It is the absence of contextual units. The first man who discovered that we may write this equation without any units and it will still hold true in any practical context was the greatest mathematician of all time. This is exactly what makes mathematics so powerful and universal, and why many mathematical discoveries have preceded their practical applications by more than a century. In the present world of quick profits, our favorite equation is put on trial.

Towards the end of *The Red Violin*, a highly sophisticated scientific researcher helps to uncover the tragic mystery of the instrument—I will not explain more since you might want to see the film. Maybe one day a genetics researcher will help to uncover the mystery of the first man who wrote $2 + 2 = 4$ without contextual units.

Tomasz Kaczynski was born in Poland, where he obtained his Master Degree in Mathematics at Warsaw University. He obtained his PhD from McGill University and presently works at Université de Sherbrooke. You can view a personal web page of Tomasz Kaczynski at:

http://www.dmi.usherb.ca/personnel/prof_maths/tomasz.kaczynski/index.html

or send him an E-mail message at: kaczyn@dmi.usherb.ca



Math Games

Counting with Base Two and the Game of Nim

by Akbar Rhemtulla

Here we will learn to count using base two arithmetic, and then play a game of Nim where base two math will be a powerful tool!

Counting with Base Two

The usual way of expressing numbers involves ten symbols $\{0, 1, 2, \ldots, 9\}$. Some say this evolved historically because we have ten fingers on our two hands. When we want to express a number greater than 9, we use combinations of these symbols with their place values. So, for example, ten is 10, and

$$2542 = 2 \times 10^3 + 5 \times 10^2 + 4 \times 10^1 + 2.$$

Mathematicians call this the decimal system. But there is nothing special about using ten symbols. Indeed, in computers we have only two positions for switches 'off' and 'on', so we count using only two symbols $\{0, 1\}$. This is called base two.

How then do you write the number 2? Well, following the decimal system, where 10 comes after 9, we write 10 for the number that follows 1. Here are the numbers from zero to ten:

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010.$$

Let us see what the decimal format is for the number that is written in base two as 10011.

$$11011 = 1\times2^4+1\times2^3+0\times2^2+1\times2^1+1 = 16+8+2+1 = 27.$$

Here are some more numbers in base two and base ten,

$$1001 \longleftrightarrow 9 \qquad 111 \longleftrightarrow 7 \qquad 10011 \longleftrightarrow 21.$$

Addition is similar to the base ten system. So:

$$\begin{array}{lrcr}
 & \textit{Base two} & \textit{convert} & \textit{Base ten} \\
 & 111 & \longleftrightarrow & 7 \\
+ & 101 & \longleftrightarrow & 5 \\
+ & 1101 & \longleftrightarrow & 13 \\
= & 11001 & \longleftrightarrow & 25
\end{array}$$

Game of Nim

This game is played by two players. A bunch of match sticks are arranged in several rows and each player alternately picks some (at least one) from any one row. The player who picks the last match stick(s) is the winner. Let us try a game.

$$\begin{array}{ll}
|\;|\;|\;|\;|\;|\;| & (111 = 7\ \textit{sticks}) \\
|\;|\;|\;| & (100 = 4\ \textit{sticks}) \\
|\;|\;|\;|\;| & (101 = 5\ \textit{sticks}) \\
|\;|\;| & (011 = 3\ \textit{sticks})
\end{array}$$

A and B are the players and A has the first move. He takes three sticks from row two.

$$\begin{array}{ll}
|\;|\;|\;|\;|\;|\;| & (111 = 7\ \textit{sticks}) \\
| & (001 = 1\ \textit{stick}) \\
|\;|\;|\;|\;| & (101 = 5\ \textit{sticks}) \\
|\;|\;| & (011 = 3\ \textit{sticks})
\end{array}$$

Now it is the turn of clever player B. She is in the losing position (we will see why), but she hopes that A will make a false move. She removes one stick from row one.

$$\begin{array}{ll}
|\;|\;|\;|\;|\;| & (110 = 6\ \textit{sticks}) \\
| & (001 = 1\ \textit{stick}) \\
|\;|\;|\;|\;| & (101 = 5\ \textit{sticks}) \\
|\;|\;| & (011 = 3\ \textit{sticks})
\end{array}$$

Player A now removes three sticks from row three.

$$\begin{array}{ll}
|\;|\;|\;|\;|\;| & (110 = 6\ \textit{sticks}) \\
| & (001 = 1\ \textit{stick}) \\
|\;| & (010 = 2\ \textit{sticks}) \\
|\;|\;| & (010 = 2\ \textit{sticks})
\end{array}$$

This is just what B is hoping for. She now removes all six sticks from row one.

$$\begin{array}{ll}
 & (00 = 0\ \textit{stick}) \\
| & (01 = 1\ \textit{stick}) \\
|\;| & (10 = 2\ \textit{sticks}) \\
|\;|\;| & (11 = 3\ \textit{sticks})
\end{array}$$

Player A is doomed! If he takes the stick from row two, B will take one stick from row four and then copy A's move onwards. If he takes two sticks from row three, then B will remove two sticks from row four. He decides to take one stick from row three.

(00 = 0 *stick*)
| (01 = 1 *stick*)
| (01 = 1 *stick*)
| | | (11 = 3 *sticks*)

B now removes all three sticks from row four to put A in the following position.

(0 = 0 *stick*)
| (1 = 1 *stick*)
| (1 = 1 *stick*)
(0 = 0 *stick*)

A removes the stick in row two and B clears row three and wins.

Strategy

Player B was counting the sticks in each row using the base two system. Here are the numbers and totals in columns at each stage, starting from initial setup with A to move.

Initialsetup	*A moves*	*B moves*	*A moves*
111	111	110	110
100	001	001	001
101	101	101	010
011	011	011	011
3, 2, 3	2, 2, 4	2, 2, 3	1, 3, 2

B moves	*A moves*	*B moves*	*B wins*
000	000	000	000
001	001	001	000
010	001	001	001
011	011	000	000
0, 2, 2	0, 1, 3	0, 0, 2	0, 0, 1

B added the number of ones in each column (we will use the decimal system to count the total). In the first column, the numbers (in base two format) are 111, 100, 101 and 011. Three of these end in digit '1' and two have '1' as their second digit and three have '1' as the third digit. The bottom numbers 3,2,3 give a count of these. Her strategy was to make every one of these an even number.

She could do it if she were to start. For example, removing enough sticks in row one to get the number from 111 to 010 would do, since it would bring the bottom count from 3,2,3 to 2,2,2. She could even achieve this by removing enough sticks from row two to change the number from 100 to 001 (making the bottom line change from 3,2,3 to 2,2,4). Finally she could attack row three, removing all the matches to bring the number from 101 to 000 (and the bottom line from 3,2,3 to 2,2,2). However, she could not make all the columns even by attacking row four.

She would like to start, but it is player A's turn to start. The first move of A is great! He manages to bring the number of 1s in each of the three columns to even numbers. B can only hope for A to make an error. The rule of the game is that you must remove at least one match stick and you cannot remove the sticks from more than one row. So whatever B does, one of the 1s in at least one of the columns will change from an even to an odd number. So she removes one stick and prays.

Well, A goofs. All he had to do was remove one stick from either of rows two, three or four. Instead, he removes three sticks from row three and the count of 1s changes to 1,3,2.

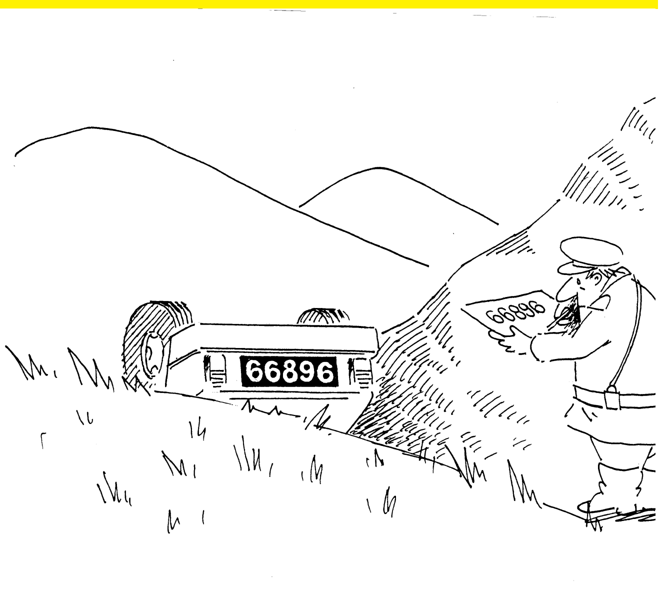
B has the game under her control at this point. Notice she brings the totals to 0,2,2. A's move results in a count of 0,1,3, and again B makes all those digits even. The game ends when B finally makes the bottom count all even with 0,0,0.

This is the game of Nim. Play with your friends and show the power of mathematical tools. Could you have done this if you were counting in powers of ten (decimal system)?

There are versions of the game where the last person to remove a match stick is the loser. The strategy is very similar to above. On her second move, player B had complete control of the game and if she had chosen to lose the game, there would be nothing A could do to stop her!

Note: If you know any other game with a winning strategy, or if you are the author of a computer game involving such a feature, please send us the details. We would definitely give consideration to your submission and possibly include it in one of our upcoming issues.

Find more about the author at the following web site: http://www.math.ualberta.ca/People/Facultypages/Rhemtulla.A.html
You can also send your comments directly to the author by E-mail at ar@ulberta.ca



EVERYDAY MATH

Relating Mathematical Ideas to Simple Observations

by Jim G. Timourian

I went to college in the northeastern part of the U.S. The highways there are often carved through rocky, hilly areas, so that as you drive by you can see various geological features. I envied my friends who were taking an elementary geology course since they could pass the traveling time by explaining to us how the layers of rock were formed. At the time, it didn't seem as if I could relate anything I saw in the world to the mathematics I was taking.

I soon learned that there are mathematical features illustrated all around us. I thought that most were too sophisticated to be identified by beginning students. More recently, I realized that there are in fact many simple observations that demonstrate relatively elementary mathematical ideas. I started assembling a list of these, with the hope that as the years go by many more can be added. Here is my start.

Exercise: Each observation below is associated with a basic mathematical idea. Describe it.

a) You jog for an hour on a motorized treadmill and go nowhere.

b) The mirror image of a mirror image of the letter R looks like the letter R again.

c) You study twice as hard for a test as your friend, but you do not learn twice as much.

d) An instructor raises all grades on a test by 20 points.

e) An instructor raises all grades on a test by 20%.

f) Two teams of equal strength fight a tug of war to a tie.

g) A broken clock displays the correct time twice a day.

Answers:

a) If you go nowhere, your running speed must be the same as that of the treadmill, but in the opposite direction. On a straight line, speed can be given a plus or minus sign to distinguish the direction. We call the combination of speed and direction *velocity*. In this case, if your velocity is x, then the velocity of the treadmill belt is $-x$ and the sum of the two velocities is 0. Numbers x and $-x$ are called *additive inverses* of each other, and each number on the number line has such an additive inverse.

b) Draw horizontal and vertical axes on a sheet of paper, and write the letter R on the sheet. When you hold the paper up to a mirror you get a backwards R. The mirror image of that backwards R is again the original R. If you label the axes x and y, then each point in the R you have drawn has an address (x, y). When you hold the paper to the mirror and look at the mirror image, each point (x, y) in the R you drew moves to the point $(-x, y)$ in the mirror. When you take the mirror image of the mirror image, you get back to (x, y). This illustrates the mathematical property that multiplying two minuses equals a plus.

c) If the graph of a function is a straight line, the function is particularly simple. It is called a *linear* function (technically the function must have value zero at zero) and has a certain proportionality property. For example, if the amount you learn is measured as a function of the time you spend studying, and if the relationship is linear, it would be true that if you studied for 0 hours you would learn 0, if you studied for x hours you would learn twice as much as what you would learn if you studied for only half of that time, and three times as much as what you would learn if you studied for only one third of that time. Most processes in real life are not linear (they are formally called *nonlinear*). A little experience with learning any skill, whether it is mathematics, playing a musical instrument or developing a golf swing, demonstrates that the "learning function" is nonlinear.

d) Suppose the original grades are marked as numbers on a straight line. If an instructor raises all grades by 20 points, one can imagine that all of the points on the line are being *translated* 20 units to the right. That is, each point x is moved to the point $x + 20$. Such a translation can be identified with the idea of "adding by 20", with the feature that the relative distances between the grades stay the same. A translation such as this is an example of a *rigid motion* (of the line), an important idea in Euclidean geometry.

e) Suppose the original grades are marked as numbers on a straight line. If an instructor raises all grades by 20%, one can imagine that the point 0 is fixed while the rest of the line is stretched to the right and left, so that each point x is moved to the point $(1.20)x$. This movement would be represented by a *magnification* of the line and can be identified with the idea of "multiplying by 1.20." A magnification such as this is an example of a motion that is not rigid since relative distances change among the grades.

f) This example is similar to a), except there are two forces that are equal in intensity, but in opposite directions. A force has both intensity and direction, so if one team pulls with force x, the other team must be pulling with force $-x$, and the sum of the two forces is zero.

g) A clock works with *modular arithmetic*. Imagine that the numbers on a number line represent time. Two times are equivalent if their difference is divisible by 12. We re-label all the numbers just using numbers x where $0 < x \leq 12$. Define *addition mod* 12 by saying that if y is another number, then $x \oplus y$ is equal to the remainder you get if you divide the usual $x + y$ by 12. This puts $x \oplus y$ into the interval from 0 to 12. In this system, the number 12 is the *additive identity*: $x \oplus 12 = x$ for any x. If the time the clock broke is x, then every 12 hours it will again show the correct time x.

* Find more about the author at the following web site:
http://www.math.ualberta.ca/People/Facultypages/Timourian.J.html
You can also send your comments directly to the author by E-mail at
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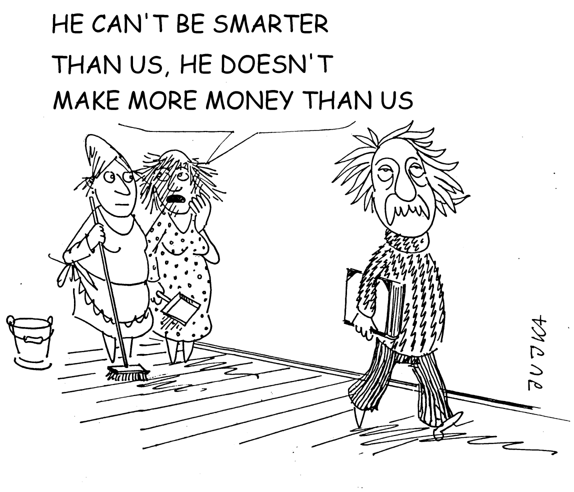
Math Jokes



There really are only two types of people in the world, those that DON'T do MATH, and those that take care of them. (Larry James)

Philosophy is a game with objectives and no rules. Mathematics is a game with rules and no objectives. (Ian Ellis)

Theorem: *All numbers are equal to zero.*

Proof: Suppose that $a = b$. Then $a = b$, so $a^2 = ab$. By subtracting b^2 from the last equality we get $a^2 - b^2 = ab - b^2$, hence $(a+b)(a-b) = b(a-b)$, thus $a + b = b$, therefore finally $a = 0$. (Benjamin J. Tilly)

Theorem: *All positive integers are interesting.*

Proof: Assume the contrary. Then there is the lowest non-interesting positive integer. But, hey, that's pretty interesting! A contradiction.

In earlier times, they had no statistics, and so they had to fall back on lies. (Stephen Leacock)

Two mathematicians walk into a restaurant for lunch. One challenges the other to a wager, loser pays the tab. The challenger says, "The waiter will not know the correct formula for $(a + b)^2$."

"You're on!" is the reply.

They place their order and the waiter is asked the formula for $(a + b)^2$. The waiter replies,

"Obviously, $(a + b)^2 = a^2 + b^2$."

"Provided, of course, that a and b are anticommutative!"

(Oscar Lanzi III)

Mathematics is the systematic misuse of a nomenclature developed for that specific purpose. (Poul-Henning Kamp)

The four branches of arithmetic—ambition, distraction, uglification and derision. (Lewis Caroll: "Alice in Wonderland")

Algebraic symbols are used when you do not know what you are talking about.

Logic is a systematic method for getting the wrong conclusion... with confidence.

Surely 'statistics' is a systematic method for getting the wrong conclusion... with 95% confidence.(Rafy Marootians)

One attractive young businesswoman to another, over lunch: "My life is all math. I am trying to add to my income, subtract from my weight, divide my time and avoid multiplying."

How about the apocryphal story about the MIT student who cornered the famous John von Neumann in the hallway:

Student: "Er, excuse me, Professor von Neumann, could you please help me with a calculus problem?"

John: "Okay, sonny, if it's real quick—I'm a busy man."

Student: "I'm having trouble with this integral."

John: "Let's have a look." (brief pause) "All right, sonny, the answer's two-pi over 5."

Student: "I know that, sir, the answer's in the back—I'm having trouble deriving it, though."

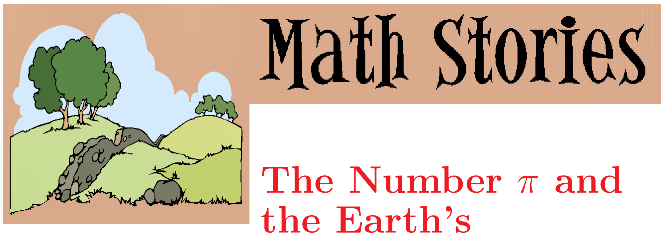
John: "Okay, let me see it again." (another pause) "The answer's two-pi over 5."

Student (frustrated): "Uh, sir, I know the answer, I just don't see how to derive it."

John: "Whaddya want, sonny, I worked the problem in two different ways!"

(Mark A. Thomas)

Someone who had begun to read geometry with Euclid, when he had learned the first proposition, asked Euclid, "But what shall I get by learning these things?" whereupon Euclid called in his slave and said, "Give him three pence since he must make gain out of what he learns." (Stobaeus)



The Number π and the Earth's Circumference

by W. Krawcewicz

The most famous quantity in mathematics is the ratio of the circumference of a circle to its diameter, which is also known as the number *pi* and denoted by the Greek letter π :

$$\frac{\text{circumference}}{\text{diameter}} = \pi.$$

The symbol π was not introduced until just over two hundred years ago. The ancient Babylonians estimated this ratio as 3 and, for their purposes, this approximation was quite sufficient. According to the Bible*, the ancient Jews used the same value of π. The earliest known trace of an approximate value of π was found in the Ahmes Papyrus written in about 16th century B.C. in which, indirectly, the number π is referred to as equal to 3.1605. Greek philosopher and mathematician Archimedes, who lived about 225 B.C., estimated the value of π to be less than $3\frac{1}{7}$ but more than $3\frac{10}{71}$. Ptolemy of Alexandria (c. 150 B.C.) gives the value of π to be about 3.1416. In the far East, around 500 A.D., a Hindu mathematician named Aryabhata, who worked out a table of sines, used for π the value 3.1416. Tsu Chung-Chih of China, who lived around 470, obtained that π has a value between 3.1415926 and 3.1415927, and after him no closer calculation of π was made for one thousand years. The Arab Al Kashi about 1430 obtained the amazingly exact estimated value for π of 3.1415926535897932. There were several attempts made by various mathematicians to compute the value of π to 140, then 200, then 500 decimal places. In 1853, William Shanks carried the value of π to 707 decimal places. However, nobody seemed to be able to give the exact value for the number π.

What is the exact value of the number π? I talked to my daughter about this problem and we decided to find our own estimation of the number π by an experiment. For this purpose, we used an old bicycle wheel of diameter 63.7 cm. We marked the point on the tire where the wheel was touching the ground and we rolled the wheel straight ahead by turning it 20 times. Next, we measured the distance traveled by the wheel, which was 39.69 meters. We divided the number 3969 by 20×63.7 and obtained 3.115384615 as an approximation of the number π. Of course, this was just our estimate of the number π and we were aware that it was not very accurate.

The problem of finding the exact value of the number π inspired scientists and mathematicians for many centuries before it was solved in 1761 by Johann Heinrich Lambert (1728-1777). Lambert proved that the number π cannot be expressed as a fraction or written in a decimal form using only a finite number of digits. Any such representation would always be only an estimation of the number π. Today, we call such numbers *irrational*. The ancient Greeks already knew about the existence of irrational numbers, which they called *incommensurables*. For example, they knew that the length of the diagonal of a square, with side of length equal to one length unit, is such a value. This value, which is denoted $\sqrt{2}$ and is equal to the number x such that $x^2 = 2$, cannot be expressed as a fraction.

Today in schools we use the estimation 3.14 for the number π, and of course this is completely sufficient for the type of problems we discuss in class. However, it was quickly noticed that in real life we need a better estimate to find more accurate measurements for carrying out construction projects, sea navigations and military applications. For most practical purposes, no more than 10 digits of π are required. For mathematical computation, even with astronomically precise calculations, no more than fifty exact digits of π are really necessary: 3.1415926535 8979323846 2643383279 5028841971 6939937510. However, with the power of today's supercomputers, we are able to compute more than hundreds of billions of digits of the number π. You can download at the Web site

http://www.verbose.net/

files with the exact digits of the number π up to 200 million decimals. We also have the following approximations of the number π:

3.1415926535	8979323846	2643383279	5028841971	6939937510
5820974944	5923078164	0628620899	8628034825	3421170679
8214808651	3282306647	0938446095	5058223172	5359408128
4811174502	8410270193	8521105559	6446229489	5493038196
4428810975	6659334461	2847564823	3786783165	2712019091
4564856692	3460348610	4543266482	1339360726	0249141273
7245870066	0631558817	4881520920	9628292540	9171536436
7892590360	0113305305	4882046652	1384146951	9415116094
3305727036	5759591953	0921861173	8193261179	3105118548
0744623799	6274956735	1885752724	8912279381	8301194912
9833673362	4406566430	8602139494	6395224737	1907021798
6094370277	0539217176	2931767523	8467481846	7669405132
0005681271	4526356082	7785771342	7577896091	7363717872
1468440901	2249534301	4654958537	1050792279	6892589235
4201995611	2129021960	8640344181	5981362977	4771309960
5187072113	4999999837	2978049951	0597317328	1609631859
5024459455	3469083026	4252230825	3344685035	2619311881
7101000313	7838752886	5875332083	8142061717	7669147303
5982534904	2875546873	1159562863	8823537875	9375195778

There is a lot of information on the number π available on the Internet. Check out our Math Links section for sites related to the number π (page 28).

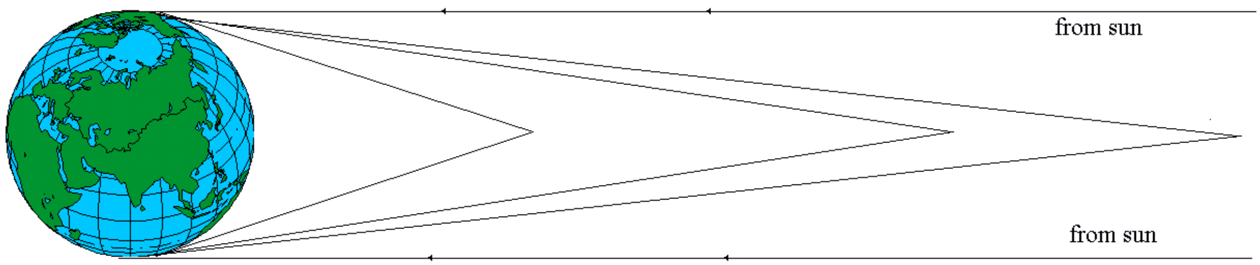
Since the number π is the ratio of the circumference of a circle to its diameter, we can write a formula for the circumference of a circle, which is

$$C = \pi d,$$

where C denotes the circumference and d denotes the diameter of the circle. If r denotes the radius of the circle than $d = 2r$, and we can rewrite the formula for the circumference as

$$C = 2\pi r.$$

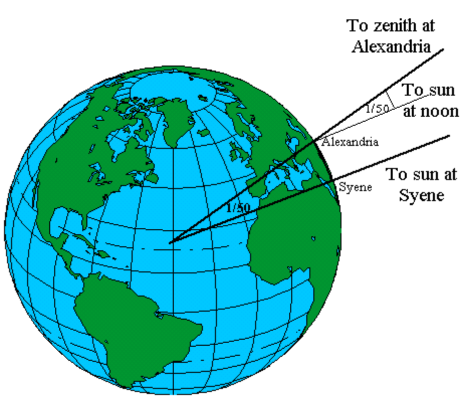
* Kings 7:23 He [Solomon] made the Sea of cast metal, circular in shape, measuring ten cubits from rim to rim [diameter = 10] and five cubits high. It took a line of thirty cubits to measure around it [circumference = 30].



The more distant the light source, the more nearly parallel are the rays of light coming from it.

Of the mathematical achievements of the Greek astronomers none is more interesting than the measurement of the circumference and the diameter of the earth by Eratosthenes (born at Cyrene in 274 B.C and died in 196 B.C). Eratosthenes was an astronomer of the Alexandrian school who had the idea the sun is so distant from the Earth compared with its size, that the sun's rays intercepted by all parts of the Earth approach it along seemingly parallel lines. This is a simple fact based on the observation that the more distant the source of light, the smaller the angle between the rays (see diagram above). For a source *infinitely* distant, the rays travel along parallel lines. We know that the sun is not infinitely far away from the Earth, but the angle between the rays of light coming from the sun at any place on the Earth is less than one third of a minute ($\frac{1}{3}'$), which is very small. As a consequence, at every location on the Earth, people who can see the sun or stars are all looking in the same direction. This concept was also used in celestial navigation at sea.

Eratosthenes' method of determining the size of the Earth

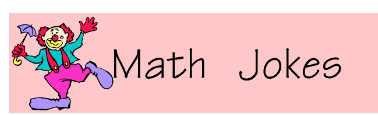


Eratosthenes noticed that at Syrene in Egypt (now called Aswân), on the first day of summer, sunlight struck the bottom of a vertical well at noon, which indicated that Syrene was on a direct line from the center of the Earth to the sun. At the corresponding time and date in Alexandria, which was 833 km from Syrene (in the time of Eratosthenes Greeks were using *stadium* as a length unit, which was equal to about $\frac{1}{6}$ km), he observed that the sun was not directly overhead but slightly south of zenith, so that its rays made an angle with the vertical equal to $\frac{1}{50}$ of a circle. Therefore, Alexandria must be one-fiftieth of the Earth circumference north of Syrene, and the Earth circumference must be 50×833 km, or $41,666$ km (see diagram at left). The correct value of the Earth circumference is $40,000$ km. The diameter of the Earth is found from the circumference by dividing the latter by π, so it is equal to $12,738$ km.

We can only speculate what would happen if these facts were known to Columbus sixteen centuries later. However, it is very unfortunate that for almost one thousand years Western civilization was living in complete darkness, unaware of the great scientific discoveries of the ancient Greeks. This one thousand years was a great loss for mankind.[†]

By using geometrical construction, the last Greek astronomer of antiquity—Claudius Ptolemy, who lived around 140 A.D.—was able to obtain a nearly correct value for the distance to the moon, which is $382,680$ km. His construction, which requires more knowledge about the properties of triangles, will be discussed in an upcoming issue.

If you find this story interesting and would like to learn more about the number π, its history and how to compute its exact digits, you definitely should read the article *"The Amazing Number π,"* published on our web site http://www.pims.math.ca/pi. The article was written by Peter Borwein, who is one of the best experts on this issue.



Mathematician's bakery: House of Pi.

Q: How does a mathematician support himself?
A: With brackets.

This isn't really a joke—it supposedly happened in a UK GCSE exam some years ago, but it may amuse you:

Q: How many times can you subtract 7 from 83, and what is left afterwards?

A: I can subtract it as many times as I want, and it leaves 76 every time. (Peter Taylor)

In modern mathematics, algebra has become so important that numbers will soon have only symbolic meaning. (Peter Bengtsson)

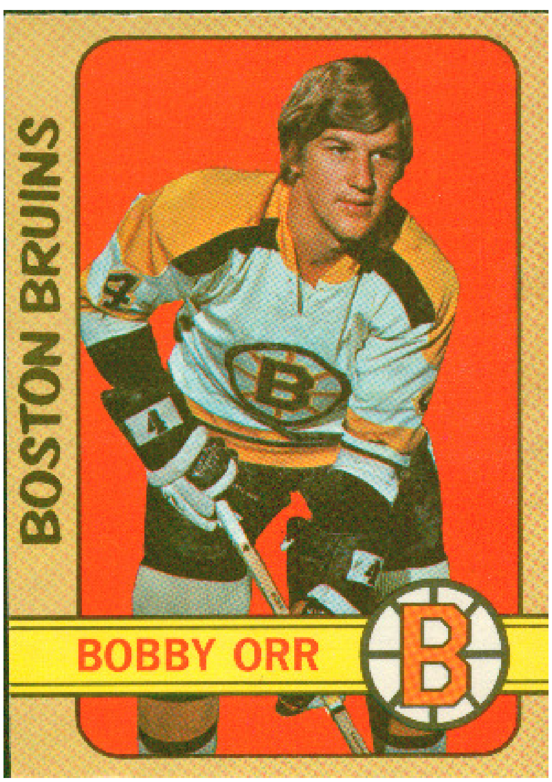
[†] We should mention that a group of prominent Russian mathematicians, after extensive studies of the numerial and astronomical data contained in historical documents available today, denies the existence of the so called "dark ages." They claim that the ancient history was incorrectly dated and should be shifted at least 1000 years closer to our present time. This is another story that we may feature in one of the upcoming issues of π *in the Sky*.

Collector's Problem

by Byron Schmuland

If you've ever collected sports cards, you know what it's like. At first, every pack you buy adds to your rapidly growing collection, but pretty soon you are collecting a lot of duplicates, and in a short time your duplicate pile is bigger than your collection. Eventually, getting a new card for your collection is a rare event; most of the time the whole pack goes straight into the duplicate pile. Finally, you need only one last card, the elusive Bobby Orr, and you spend several weeks buying pack after pack before you finally finish off your collection.

Are you being ripped off? Does it make sense that your duplicate pile should end up being two or three times bigger than your collection? Is the card company artificially creating rare cards to increase sales?



Let's try to understand the mathematics of the collector's problem. For simplicity, we'll assume that you buy cards one at a time, and that every player's card has an equal chance of turning up. This says that the card company is not cheating—there are as many Bobby Orr's as there are Bobby Schmautz's. We let n be the total number of different cards. When I collected hockey cards, n was about 250. This is hard to handle, so let's warm up with a simpler problem where $n = 4$.

Start with a well-shuffled deck of cards. Randomly pull out a card, replace and repeat. How long does it take, on average, to get all four suits? Try it a few times, and you'll see that you very rarely get all four suits with the first four cards, usually it will take eight or nine, and occasionally over a dozen cards before all four suits show up. Here are some sample results:

7 cards:	$\overset{\downarrow}{\spadesuit}\spadesuit\overset{\downarrow}{\clubsuit}\overset{\downarrow}{\diamondsuit}\clubsuit\diamondsuit\overset{\downarrow}{\heartsuit}\heartsuit\clubsuit\ldots$
5 cards:	$\overset{\downarrow}{\clubsuit}\overset{\downarrow}{\heartsuit}\heartsuit\overset{\downarrow}{\diamondsuit}\overset{\downarrow}{\spadesuit}\diamondsuit\ldots$
14 cards:	$\overset{\downarrow}{\clubsuit}\clubsuit\overset{\downarrow}{\heartsuit}\heartsuit\clubsuit\heartsuit\clubsuit\clubsuit\overset{\downarrow}{\spadesuit}\clubsuit\clubsuit\spadesuit\heartsuit\overset{\downarrow}{\diamondsuit}\clubsuit\ldots$

I've marked each new suit with a little arrow, and the fourth new suit means we're done. To do the mathematics, it is convenient to divide the total into the four pieces T_1, T_2, T_3, and T_4 between new suits. Let's call

$$\begin{aligned}
E(T_1) &= 1,\\
E(T_2) &= \text{average number of cards between 1}^{\text{st}}\text{ and 2}^{\text{nd}}\text{ new suit},\\
E(T_3) &= \text{average number of cards between 2}^{\text{nd}}\text{ and 3}^{\text{rd}}\text{ new suit},\\
E(T_4) &= \text{average number of cards between 3}^{\text{rd}}\text{ and 4}^{\text{th}}\text{ new suit}.
\end{aligned}$$

The average number of cards to get all four suits is $E(T_1) + E(T_2) + E(T_3) + E(T_4)$.

> **Flip it over!**
>
> If the chance of a random outcome is p, then on average you need $1/p$ trials until this outcome occurs.
>
> For instance, if you toss a fair coin, you need 2 throws on average to get a head, and with a fair die you need 6 throws on average to get a ⚄. I guess this makes sense. The rarer the outcome, the longer it takes to see it.

Back to the card problem. The first equation $E(T_1) = 1$ is easy since the first card always gives a new suit. Now after you have one suit, the chance of a new suit is $3/4$ and the average number of cards until this happens is (flip it over!) $E(T_2) = 4/3$. After you have two suits, the chance of a new suit is $1/2$ and the average number of cards until this happens is $E(T_3) = 2$. Finally, after you have three suits, the chance of a new suit is $1/4$, so $E(T_4) = 4$. This gives the final answer of

$$E(T_1) + E(T_2) + E(T_3) + E(T_4) = 4\left(\tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{1}\right) = 8\,\tfrac{1}{3}.$$

The nice thing is that the same pattern works no matter how big the problem. Going back to the hockey cards,

Find more about the author at the following web site:
http://www.stat.ualberta.ca/people/schmu/dept_page.html
You can also send your comments directly to the author at
schmu@stat.ualberta.ca

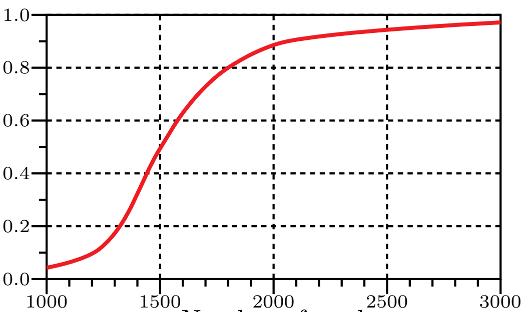
if there are a total of n cards to be collected, then the average number of random selections until all n different cards appear is

$$n\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{1}\right)\approx n(\log n+0.577),$$

where $\log n$ means the natural log of n. Plugging in $n=250$, I find that the average number of cards purchased to get a full collection is about 1525. By the time you're done, the average duplicate pile should be five times larger than the collection. Well, that explains a lot!

However *average* doesn't mean *typical*. Some hockey card collectors will need to buy 1525 cards or so, but some lucky people will get away with less, and some unlucky people will need more. Further analysis of the collector's problem gives us the following chart.

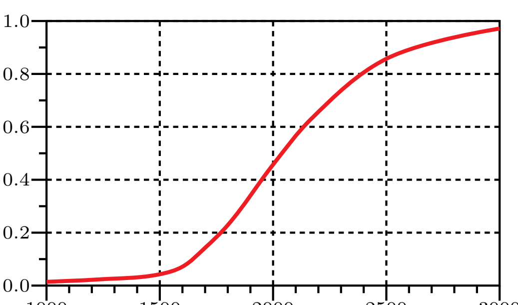
Probability to complete a collection of size $n=250$:



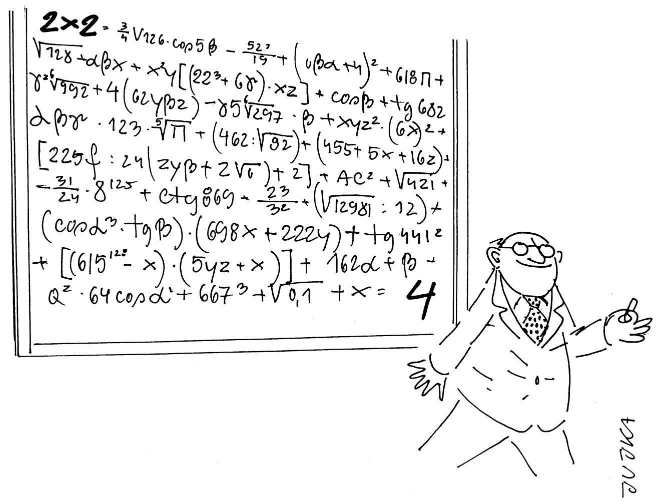
From this chart, we can see that virtually everybody will need to buy between 1000 and 2500 cards to complete their collection. The card companies don't need to create shortages deliberately—as with casino owners, random chance alone guarantees big sales.

So what's a poor hockey card collector to do? The answer: trade with your friends! Some of the cards you want are in your friend's duplicate pile and vice versa. The mathematics is more difficult, but if two collectors cooperate, the average number of purchases to obtain two complete collections is about $n(\log n+\log(\log n)+1.09)$. If $n=250$, you and your friend can expect to buy about 2080 cards before getting two full collections. That's about 1040 purchases per person, considerably fewer than the 1525 when you go it alone. In fact, the more friends you have trading, the cheaper you *all* get your complete collections.

Probability to complete two collections of size $n=250$:



Math Jokes



My mother is a mathematician, so she knows how to induce good behaviour. "If I've told you n times, I've told you $n+1$ times...." (Quantum Seep)

Hungarian mathematician Frigyes Riesz needed two assistants for his lectures: one was reading aloud his (Riesz's) book, the second was writing everything on the board, while Riesz was standing next to the board nodding. (Aniko Szabo)

Mathematics is the art of giving the same name to different things. (Jules Henri Poincare (1854-1912)—French mathematician)

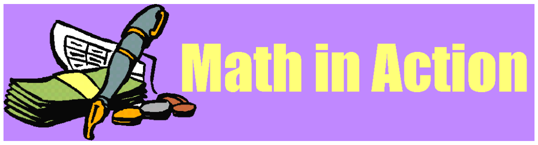
A student at our high school a few years back, having had his fill of drawing graph after graph in senior high math class, told his teacher, "Mrs. Smith, I'll do algebra, I'll do trig, and I'll even do statistics, but graphing is where I draw the line!" (Kevin Carver)

If God is perfect, why did He create discontinuous functions?

During an oral examination by the Polish mathematician M. Kac, a student was asked about the behaviour of the Rieman zeta function $\zeta(s)$ at $s=1$. When the student had no idea, Kac gave the hint: "Think of me." The answer came immediately: "Aah, it has a simple pole."

"The number you have dialed is imaginary. Please rotate your phone 90 degrees and try again." (Mark Chrisman)

You're aware the boy failed my grade school math class, I take it? And not that many years later he's teaching college. Now I ask you, is that the sorriest indictment of the American educational system you ever heard? [pauses to light cigarette.] No aptitude at all for long division, but never mind. It's him they ask to split the atom. How he talked his way into the Nobel prize is beyond me. But then, I suppose it's like the man says, "It's not what you know..." (Karl Arbeiter—former teacher of Albert Einstein)



Mathematics is Everywhere: But Whom is it Reaching?

by Krisztina Vasarhelyi

People may be divided into two groups: there are those who are instinctively attracted to mathematics and those whose interest is not easily awakened. Reaching both groups lie at the hearts of many programmes on the PIMS Education Agenda. The **Mathematics is Everywhere** poster campaign is no exception in this regard.

Each month, the poster features a snapshot of the familiar world around us: a sunflower, a child playing the violin, a soccerball, the full moon above the city skyline. Nothing is out of the ordinary. Or perhaps it is? *The moon at Equinox rising over Vancouver. What time was the photo taken?* Not necessarily the sort of thing that moongazers contemplate on a clear spring evening.

The motivating force behind this PIMS project, conceived and created by Klaus Hoechsmann, is the desire to increase public awareness of the surprising ways mathematics touches many aspects of our lives. At the very least, the hope is to catch the attention of many, capture the imagination of some, and convince a few to dig deeper.

The eye catching pictures with thought-provoking questions are displayed monthly on selected bus lines in Vancouver and Victoria and in secondary schools in Calgary. They are also posted on the PIMS web site to allow anyone around the world to enter a solution and have a chance at winning the $100 prize. In fact, it is hoped that by the end of 2000, which is UNESCO World Mathematical Year, participation in the monthly contest will become truly international.

The questions are designed to highlight a wide range of mathematical topics, such as combinatorics, probability, logarithmic curves, Fibonacci numbers, and more. They also vary in difficulty to stimulate public interest among all age groups, from elementary school students to adults. The questions are posed in such a way that unambiguous numerical answers can be given. Each month, one winner is drawn among the correct answers submitted through the web site. However, it is nourishing the mind, rather than nourishing a competitive spirit, which is the primary goal of this project. This is most evident to those who venture to probe the intricate connections the sometimes deceivingly simple questions conceal. This exploration can be initiated by browsing the informative web site and links made available with each month's question.

Whom is the poster campaign actually reaching? Evidently, a growing number of people are connecting to the poster web site. In February, only 193 individuals looked at the "sunflower" page, but about 1800 unique connections were made to the "equinox" page in June. While most connections and entries to the contest were initially from western Canada, the poster campaign appears to be gaining an international following.

April Poster

May Poster

June Poster

Visit the *Mathematics is Everywhere* web site

http://www.pims.math.ca/education/everywhere

Here are some of the winners:

Pam Liem, winner of the February contest "The Sunflower Spiral Count," is an active 14-year-old student at Vancouver Technical School, with sports and tambourine dancing as her favourite hobbies. She hopes to pursue studies in Commerce at UBC.

Stefan Lukits, winner of the March contest "The Violin String," is a 27-year-old pastor at Emmanuel Baptist Church in Vancouver. Prior to studying theology at Regent College, Stefan received training in mathematics in Graz, Austria. He enjoys literature, bicycling, and talking with friends in his free time.

Katy Cheng, winner of the April contest "Soccerball Symmetries," is a 35-year-old accounting contractor in Vancouver.

Jordan Wan, winner of the May contest "Chance and Randomness," is a 15-year-old high school student at Aden Bowman Collegiate in Saskatoon. He loves to participate in sports such as basketball and volleyball, play the piano, and work on his computer. Jordan says his future plans are not yet clear, but will likely include both math and sciences.

In addition to these and future winners, it is safe to say that all those whose curiosity is aroused and those inspired to investigate further will have won something too—perhaps an insight into a small corner of a mysterious and fascinating world.

If you would like to have the *Mathematics is Everywhere* poster displayed in your school, contact the PIMS office in Vancouver: Pacific Institute for the Mathematical Sciences, 1933 West Mall, University of British Columbia, Vancouver, B.C., V6T 1Z2, Canada.

Math Secrets

The Banach–Tarski Paradox or What Mathematics and Miracles Have in Common

by Volker Runde

As he went ashore he saw a great throng; and he had compassion on them, and healed their sick. When it was evening, the disciples came to him and said: "This is a lonely place, and the day is now over; send the crowds away to go into the villages and buy food for themselves." Jesus said: "They need not go away; you give them something to eat." They said to him: "We have only five loaves here and two fish." And he said: "Bring them here to me." Then he ordered the crowds to sit down on the grass; and taking the five loaves and the two fish he looked up to heaven, and blessed, and broke and gave the loaves to the disciples, and the disciples gave them to the crowds. And they all ate and were satisfied. And they took up twelve baskets full of the broken pieces left over. And those who ate were about five thousand men, besides women and children.

Mt 14:14–21

Why does an article that is supposed to be about mathematics start with the feeding of the five thousand?

In the 1920s two Polish mathematicians—Stefan Banach and Alfred Tarski—proved a mathematical theorem that sounds a lot like the feeding of the five thousand. In their honor, it is called the *Banach–Tarksi paradox*†. Consequences of the Banach–Tarski paradox are, for example:

> An orange can be chopped into a finite number of chunks, and these chunks can then be put together again to yield *two* oranges, each of which *has the same size as the one that just went into pieces*.

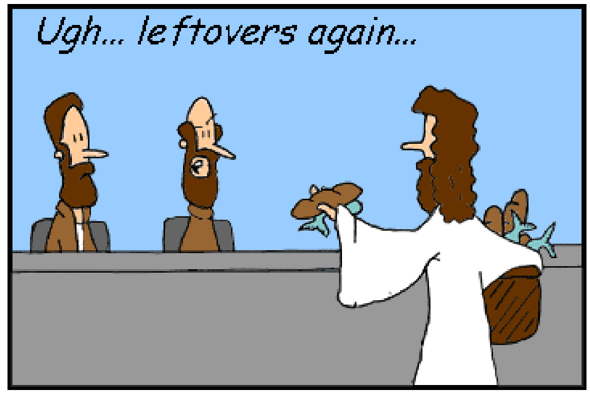
Another, even more bizarre consequence is:

> A pea can be split into a finite number of pieces, and these pieces can then be reassembled to yield a solid ball whose diameter is larger than the distance of the Earth from the sun.

More generally, whenever you have a three-dimensional body (with a few strings attached), you can obtain any other such body by breaking the first into pieces and reassembling the parts. To turn five loaves and two fish into enough food to feed a crowd of more than five thousand then appears to be a minor exercise.

If you have read this far, your attitude will presumably be one of the following:

- Your belief in the absolute truth of mathematical theorems is so strong that it makes you swallow the Banach–Tarski paradox.
- You are a staunch skeptic, so that you neither take the feeding of the five thousand nor the Banach–Tarski paradox at face value.



If you fall into the first category, there is probably little incentive for you to read any further. Otherwise, I guess, your attitude is best described as follows: You may believe in the story of the feeding of the five thousand but not take it literally, and if you hear of a mathematical theorem whose consequences are obviously nonsense, you tend to think that the theorem is wrong.

Take an orange, a sharp knife and a chopping block. Chop the orange into pieces, and try to form two globes of approximately the same size out of the orange chunks. If the chunks are small enough, each of these two globes will bear reasonable resemblance to a ball, but, of course, each has a volume that is only about half of that of the original orange. Perhaps you just didn't chop up the orange in the right way? Give it another try. The result will be the same. You can try your luck on hundreds of oranges: you will produce tons of orange pulp, but no corroboration of the Banach–Tarski paradox. Doesn't this show that the Banach–Tarski paradox is wrong?

The Banach–Tarski paradox is a so-called existence theorem: there is a way of splitting up a pea such that the pieces can be reassembled into, say, a life-sized statue of Stefan Banach. The fact that you haven't succeeded in finding such a way doesn't mean that it doesn't exist—you just might not have found it yet. Let me clarify with an example from elementary arithmetic. A positive integer p is called *prime* if 1 and p itself are its only divisors; for example, 2, 3, and 23 are prime, whereas $4 = 2 \cdot 2$ and $243 = 3 \cdot 81$ aren't. The ancient Greeks knew that every positive integer has a *prime factorization*: if n is a positive integer, then there are prime numbers $p_1, \ldots, p_k$ such that $n = p_1 \cdot \cdots \cdot p_k$. For small n, such a prime factorization is easy to find: $6 = 2 \cdot 3$, $243 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$, and $6785 = 5 \cdot 23 \cdot 59$, for example. There is essentially only one way of finding a prime factorization —trying. Already finding the prime factorization of 6785—armed only with pencil and paper—would have taken you some time. And now think of a large number, I mean, *really large*:

$$7380563434803675764348389657688547618099805.$$

This is a perfectly nice positive integer, and the theorem tells you that it has a prime factorization, but—please!—don't waste hours, days or even years of your

† The theorem is proved in the article: S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parts respectivement congruents*. *Fund. Math.* **6** (1924), 244–277.

life trying to find it. You might think: what were computers invented for? It is easy to write a little program that produces the prime factorization of an arbitrary positive integer (and it may even produce one of 7380563434803675764348389657688547618099805 in a reasonable amount of time). However, the average time such a program would take to find the prime factorization of an integer n goes up dramatically as n gets large: for sufficiently large n, even the fastest super-computer available today would —on average—take longer to find the prime factorization of n *than the age of the universe*.

So, although a prime factorization of a positive integer always exists, it may be impossibly hard to find. In fact, this is a good thing—it is at the heart of the public key codes that make credit card transactions on the internet safe, for example. Now, think again of the Banach–Tarski paradox. Just because you couldn't put it to work in your kitchen (just as you couldn't find the prime factorization of some large integer), this doesn't mean that the theorem is false (or that this particular integer doesn't have a prime factorization).

Let's try to refute the Banach–Tarski paradox with the only tool that works in mathematics: pure thought. What makes the Banach–Tarski paradox defy common sense is that, apparently, the volume of something increases out of nowhere. You certainly know a number of formulae to calculate the volumes of particular three-dimensional bodies. For example, if C is a cube whose edges have length l, then its volume $V(C)$ is l^3; if B is a ball with radius r, then its volume $V(B)$ is $\frac{4}{3}\pi r^3$. But what's the volume of an arbitrary three-dimensional body? No matter how the volume of a concrete body is calculated, the following are certainly true about the volumes of arbitrary, three-dimensional bodies:

- If the body $\tilde{B}$ is obtained from the body B simply by moving B in three-dimensional space, then $V(\tilde{B}) = V(B)$;
- If $B_1, \ldots, B_n$ are bodies in three-dimensional space, then the volume of their union is less than or equal to the sum of their volumes, i.e.,

$$V(B_1 \cup \cdots \cup B_n) \leq V(B_1) + \cdots + V(B_n);$$

- If $B_1, \ldots, B_n$ are bodies in three-dimensional space such that any two of them have no point in common, then the volume of their union is equal to the sum of their volumes, i.e.;

$$V(B_1 \cup \cdots \cup B_n) = V(B_1) + \cdots + V(B_n).$$

So, let B be an arbitrary three-dimensional body, and let $B_1, \ldots, B_n$ be subsets of B such that any two of them have no point in common and $B = B_1 \cup \cdots \cup B_n$. Now, move each B_j in three-dimensional space, and obtain $\tilde{B}_1, \ldots, \tilde{B}_n$. Finally, put the $\tilde{B}_j$ together and obtain another body $\tilde{B} = \tilde{B}_1 \cup \cdots \cup \tilde{B}_n$. Then we have for the volumes of B and $\tilde{B}$:

$$\begin{aligned} V(B) &= V(B_1 \cup \cdots \cup B_n) \\ &= V(B_1) + \cdots + V(B_n), && \text{by (iii)}, \\ &= V(\tilde{B}_1) + \cdots + V(\tilde{B}_n), && \text{by (i)}, \\ &\geq V(\tilde{B}_1 \cup \cdots \cup \tilde{B}_n), && \text{by (ii)}, \\ &= V(\tilde{B}). \end{aligned}$$

This means that the volume of $\tilde{B}$ must be less than or equal to the volume of B—it can't be larger. Banach and Tarski were wrong! Really?

Our refutation of Banach–Tarski seems to be picture perfect. All we needed were three very basic properties of the volume of three-dimensional bodies. But was this really all? Behind our argument, there was a hidden assumption—every three-dimensional body has a volume. If we give up that assumption, our argument suddenly collapses. If only one of the bodies B_j has no volume, our whole chain of (in)equalities makes no longer sense. But why shouldn't every three-dimensional body have a volume? Isn't that obvious? What is indeed true is that every orange chunk you can possibly produce on your chopping block has a volume. For this reason, you will never be able to use the Banach–Tarski paradox to reduce your food bill. A consequence of the Banach–Tarski paradox is therefore that there is a way of chopping up an orange so that you can form, say, a gigantic pumpkin out of the pieces—but you will never be able to do that yourself using a knife. What kind of twisted logic can make anybody put up with that?

Perhaps, you are more willing to put up with the *axiom of choice*:

> If you have a family of non-empty sets S, then there is a way to choose an element x from each set S in that family.

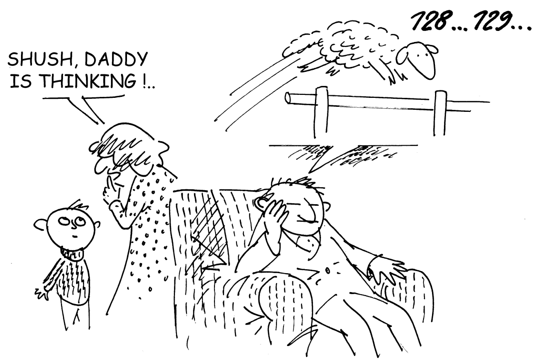
That sounds plausible, doesn't it? Just think of a finite number of non-empty sets $S_1, \ldots, S_n$: Pick x_1 from S_1, then proceed to S_2, and finally take x_n from S_n. What does the axiom of choice have to do with the Banach–Tarski paradox? As it turns out, a whole lot: *If* the axiom of choice is true, then the Banach–Tarski paradox can be derived from it and, in particular, there must be three-dimensional bodies without volume. So, the answer to the question of whether the Banach–Tarski paradox is true depends on whether the axiom of choice is true.

Certainly, the axiom of choice works for a finite number of non-empty sets $S_1, \ldots, S_n$. Now think of an infinite sequence $S_1, S_2, \ldots$ of non-empty sets. Again, pick x_1 from S_1, then x_2 from S_2, and just continue. You'll never come to an end, but eventually you'll produce some element x_n from each S_n. So, the axiom of choice is true in this case, too. But what if we have a truly arbitrary family of sets? What if we have to deal with the family of all non-empty subsets of the real line? It can be shown that this family of sets can't be written as a sequence of sets. How do we pick a real number from each set? There is no algorithm that enables us to pick one element from one set, a second element from another set and, eventually, to pick an element from every set in the family. Nevertheless, the axiom of choice still seems plausible—each set S in our family is non-empty and therefore contains some element x—why shouldn't there be a way of choosing a particular element from each such set?

On the other hand, accepting the axiom of choice implies strange phenomena like the Banach–Tarski paradox. If it's true, we have to put up with the mysterious duplication of oranges. If it's false, then why? Please, don't try to prove or to refute the axiom of choice—you won't succeed either way. The axiom of choice is beyond proof or refutation. We can *suppose* that it's true, or we can *suppose* that it's false. In other words, we have to *believe* in it or leave it. Most mathematicians these days are believers in the axiom of choice for a simple reason—with the axiom of choice, they can prove useful theorems, most of which are much less baffling than the Banach–Tarski paradox.

Are you disappointed? Instead of elevating the feeding of five thousand from a matter of belief to a consequence of a bullet-proof mathematical theorem, the Banach–Tarski paradox demands that you accept another article of faith—the axiom of choice—before you can take the theorem for granted. After all, the Banach–Tarski paradox is not that much removed from the feeding of the five thousand…

Math Jokes



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Most prime numbers are even. **Proof**: Pick up any math text and look for a prime number. The first one you find will probably be even.

Never say "n factorial"; simply scream "n" at the top of your lungs.

(Mark David Biesiada)

Some engineers are trying to measure the height of a flag pole. They have only a measuring tape and are quite frustrated trying to keep the tape along the pole; it falls down all the time. A mathematician comes along and asks what they are doing. They explain it to him. "Well, that's easy..." He pulls the pole out of the ground, lays it down and measures it easily. After he has left, one of the engineers says, "That's so typical of these mathematicians! What we need is the height - and he gives us the length!"

A mathematician, an engineer, and a computer scientist are vacationing together. They are riding in a car, enjoying the countryside, when suddenly the engine stops working. The mathematician says, "We drove past a gas station a few minutes ago. Someone should go back and ask for help."

The engineer says, "I should have a look at the engine. Perhaps I can fix it."

The computer scientist says, "Why don't we just open the doors, slam them shut and see if everything works again?"

Two men are sitting in the basket of a balloon. For hours, they have been drifting through a thick layer of clouds, and they have lost their orientation completely. Suddenly, the clouds part, and the two men see the top of a mountain with a man standing on it. "Hey! Can you tell us where we are?!" The man doesn't reply. The minutes pass as the balloon drifts past the mountain. When the balloon is about to be swallowed again by the clouds, the man on the mountain shouts: "You're in a balloon!"

"That must have been a mathematician."

"Why?"

"He thought long and thoroughly about what to say. What he eventually said was irrefutably correct. And it was of no use whatsoever..."

In the old days of the cold war, when it was very hard for Westerners visit the Soviet Union, a British mathematician traveled to Moscow to speak in the seminar of a famous Russian professor. He started his talk by writing a theorem on the board. When went to prove it, the professor interrupted him: "This theorem is clear!" The speaker was, of course, annoyed, but managed to conceal it. He continued his talk with a second theorem, but, again, when he went to start with the proof, he was interrupted by his host: "This theorem is also clear!" With a stern face, he wrote a third theorem on the board and asked, "Is this theorem clear, too?!" His host nodded. The visitor grinned and said, "This theorem—is false..."

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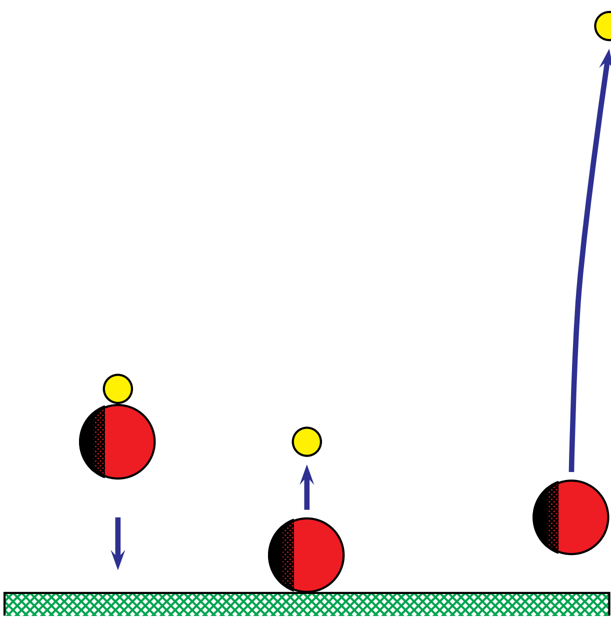
Why did the calculus student have so much trouble making Kool-Aid? Because he couldn't figure out how to get a litre of water into the little package.

Math Discovery

The Slingshot Effect of Celestial Bodies

by Florin Diacu

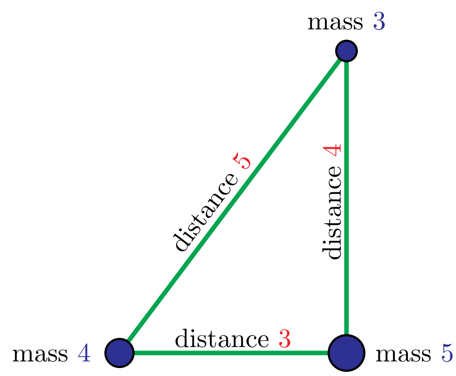
Try the following outdoor experiment: Place a tennis ball on top of a basketball and let them fall from shoulder height. The outcome is unexpected—the bounce of the basketball will send the tennis ball at high speed well above your head (see diagram below). This surprising energy-transfer phenomenon has recently been under careful mathematical investigation. Full understanding will help us find new fuel-free acceleration techniques for space shuttles and ways of travelling astronomical distances faster.



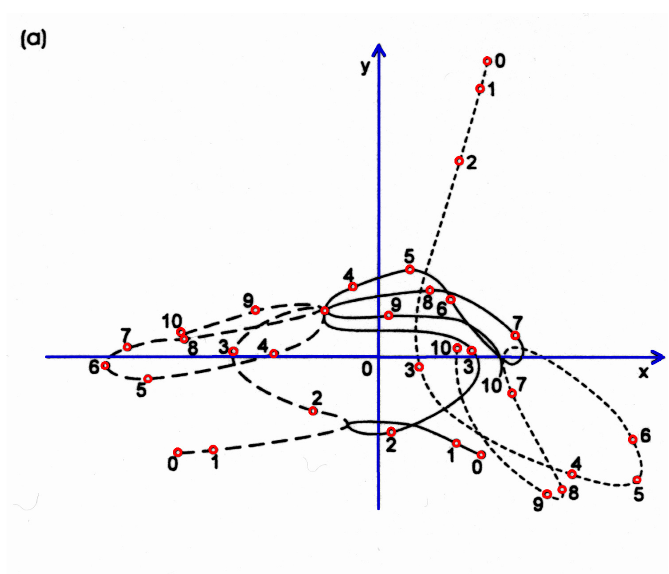
Everything started in 1687, when Isaac Newton published his masterpiece *Principia*, in which he founded several branches of science, including calculus (studied today by freshmen students), the theory of differential equations (which is part of the sophomore science curriculum) and celestial mechanics. The first two apply to many fields of human activity ranging from physics and economics to psychology and art. Based on these mathematical theories, celestial mechanics aims to understand the gravitational motion of stars, planets, asteroids and comets, and to compute the orbits of spacecrafts.

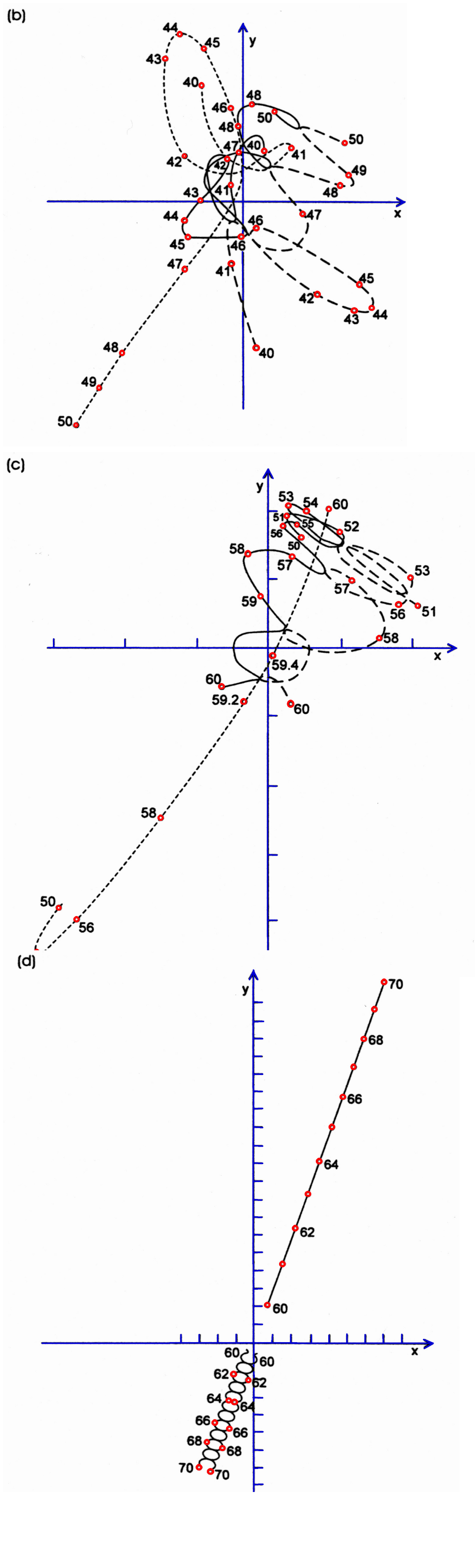
Since Newton, many famous mathematicians, including Bernoulli, Euler, Lagrange, Laplace, Gauss, Jacobi, Poincaré, Birkhoff and others, tried to predict the trajectories of celestial bodies. But the differential equations describing the orbits are so complicated that any hope of obtaining a complete solution was abandoned long ago. Still, interesting results appear from time to time.

One of them shows that the close encounter of three celestial bodies leads to a slingshot effect as in the earth-tennis-basketball experiment. This property was discovered in 1966 through computer simulations done by Victor Szebehely and Myles Standish at Yale University and by Eduard Stiefel at ETH-Zurich. From the vertices of a triangle having 3-, 4- and 5-length-unit sides, they released bodies of 3-, 4- and 5-mass units. Gravitation first led to an erratic behaviour, but then made 2 bodies orbit around each other and the third move away at high speed (see the picture below).



These computer results can be followed in the graphs represented below, in which the orbits of the 3-, 4- and 5-mass-unit bodies are drawn as dotted, dashed and continuous lines, respectively. The numbers along each line denote time units; they allow us to locate each particle. The graphs (a) and (b) show a complicated motion without any pattern, graph (c) indicates that the heavier particles tend to move around each other and graph (d) makes clear how the lighter particle is expelled with high velocity away from the other two after a close triple encounter (which takes place between the time units 59 and 60).





In 1973, Jörg Waldvogel from ETH-Zurich initiated computer investigations of the general motion of three bodies. Again, after coming close to each other, the bodies first moved unpredictably and then followed the slingshot pattern. Ignorant of these results, Richard McGehee of the University of Minnesota in Minneapolis rigorously proved in 1974 that, if on a line, the bodies must encounter a slingshot effect near triple collision.

These results led to the theoretical discovery of motions that become unbounded in finite time. In other words, celestial bodies could move so fast that they reach infinity in a couple of seconds. How could this happen? Imagine that a body is accelerated through consecutive slingshot effects such that it travels the first mile in a second, the second mile in half a second, the third mile in a quarter of a second, etc. This means that after $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ seconds the body travels infinitely many miles. But mathematicians know that the above infinite sum has the value 2. So as inconceivable as it may be, a body could reach infinity in only 2 seconds.

The difficulty was to prove that a proper sequence of slingshot effects can lead to the above scenario. Success came in 1992 when Zhihong Xia, a young Ph.D. student at Northwestern University, published his thesis. In his work he showed that, if properly positioned, five bodies can move under the influence of gravitation such that four of them escape to infinity in finite time, while the fifth oscillates back and forth among the others. At a recent meeting in Vancouver, Xia received the Blumenthal Award, made every four years in recognition of distinguished achievements in mathematics.

Ed Belbruno (on the left) and Zhihong Xia (on the right) at a conference in Seattle in 1995.

In the meantime, space scientists tried to use the slingshot effect for practical purposes. Through related work done in 1991, Edward Belbruno, a consultant with Jet Propulsion Laboratory in Pasadena, California, managed to find a Japanese satellite lost several months earlier and proposed a mathematical solution for rescuing it. The Japanese used his solution and succeeded in their mission, an event that made headlines in 1994.

The slingshot effect is an intriguing phenomenon, still under the scrutiny of mathematicians and space scientists. It is amazing how a simple observation of a ball experiment and dedicated mathematical research can lead to such spectacular achievements.

Florin Diacu is a mathematics professor at the University of Victoria and the University of Victoria Site Director of the Pacific Institute for the Mathematical Sciences. His award-winning book *Celestial Encounters*, co-authored with Philip Holmes of Princeton University, is a runaway bestseller. You can find more information at:
http://www.math.uvic.ca/faculty/diacu/index.html

Math Jokes

This is a guide to translating the language of math textbooks and professors.

1) *"It can be proven... "*
This may take upwards of a year, and no fewer than four hours, and may require something like five reams of scratch paper, 100 pencils, or 100 refills (for those who use mechanical pencils). If you are only an undergraduate, you need not bother attempting the proof as it will be impossible for you.

2) *"It can be shown..."*
Usually, this would take the teacher about one hour of blackboard work, so he/she avoids doing it. Another possibility, of course, is that the instructor doesn't understand the proof himself/herself.

3) *"It is obvious..."*
Only to PhDs who specialize in that field, or to instructors who have taught the course 100 times.

4) *"It is easily derived..."*
Meaning that the teacher figures that even the student could derive it. The dedicated student who wishes to do this will waste the next weekend in the attempt. Also possible that the teacher read this somewhere, and wants to sound like he/she really has it together.

5) *"It is obvious..."*
Only to the author of the textbook, or Carl Gauss. More likely only Carl Gauss. Last time I saw this was as a step in a proof of Fermat's last theorem.

6) *"The proof is beyond the scope of this text."*
Obviously this is a plot. The reader will never find any text with the proof in it. The proof doesn't exist. The theorem just turned out to be useful to the author.

7) *"The proof is left up to the reader."*
...sure, let us do all the work. Does the author think that we have nothing better to do than sit around with THEIR textbook, and do the work that THEY should have done? (Michael J. Bauers)

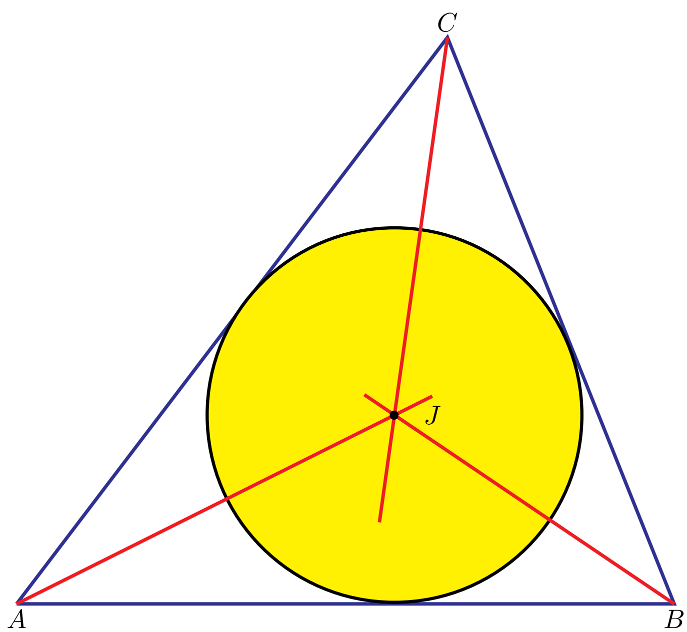
Little Bits of Geometry

Anatomy of Triangles

by Klaus Hoechsmann[†]

A triangle has, of course, three vertices, three sides and three angles. It also has further threesomes that do not meet the eye quite as readily. If you draw three lines at random, chances are that they will yield the sides of some triangle—it would be unusual if they met at a single point. But that is what happens with the following triples of lines, which are associated with any triangle.

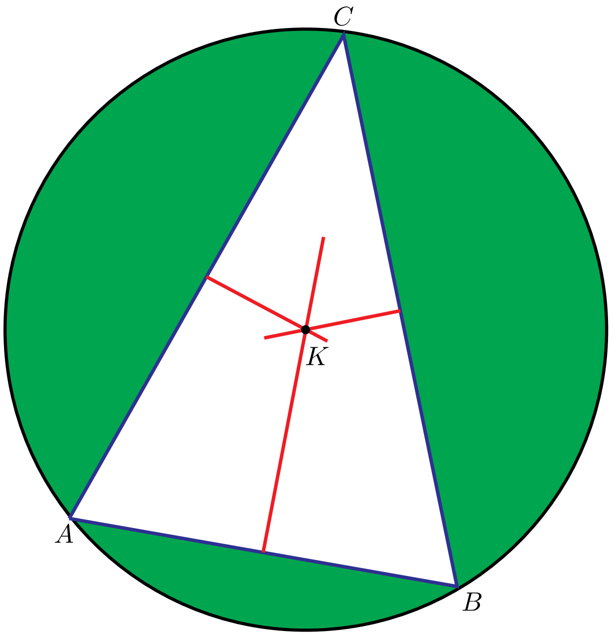
If you move from the vertex A into the triangle ABC, always making sure that your distance to the side AB is the same as that to AC, you are cruising along the *angle bisector* of A. Similarly, every point on the angle bisector of B is equidistant from the sides AB and BC. The point J of intersection of these two angle bisectors is therefore equidistant from all three sides. Hence the angle bisector of C must also go through J — see?—and J is the centre of the *inscribed circle* of the triangle.



If you move from the side AB into the triangle ABC, always making sure that your distance to the vertex A is the same as that to B, you are cruising along the *perpendicular bisector* of AB. Similarly, every point on the perpendicular bisector of BC is equidistant from the vertices B and C. The point K of intersection of these two perpendicular bisectors is therefore equidistant from all

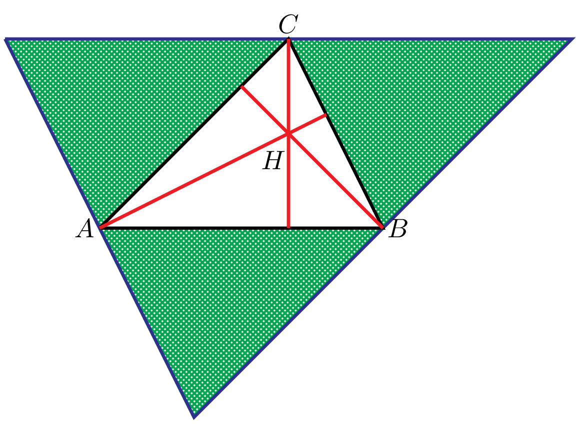
[†] You can find more information about the author and other interesting articles at:
http://www.math.ubc.ca/~hoek/Teaching/teaching.html

three vertices. Hence the perpendicular bisector of CA must also go through K—see?—and K is the centre of the *circumscribed circle* of the triangle.

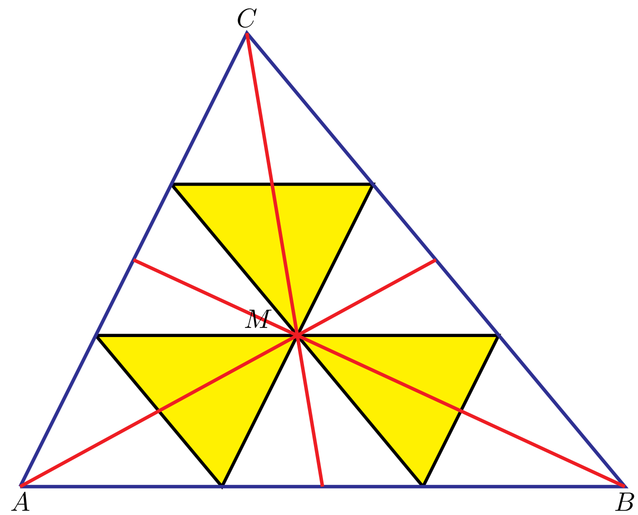


A perpendicular line segment from a vertex to the opposite side is called an *altitude* of the triangle; it runs inside the triangle if and only if the angles at the other two vertices are both acute.

The altitudes of a given triangle ABC are just the perpendicular bisectors of its fourfold enlargement, as shown. Hence the three of them meet in a single point H, namely the *circumcentre* of the enlargement.



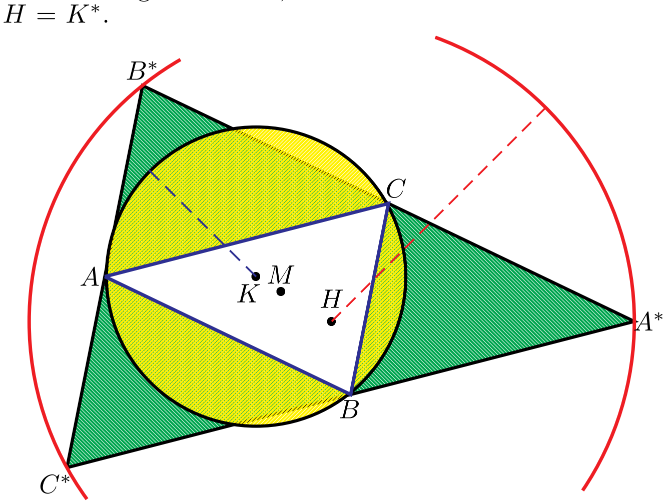
By trisecting all sides, any triangle can be subdivided into nine smaller ones, as shown. If you move from the vertex A toward the intersection M of the three yellow triangles, you are cruising along the diagonal of a little white and yellow parallelogram, and if you continue beyond M, you echo the first half of your trip (through the white triangle at A). You will therefore arrive at the *midpoint* of the side BC.



The line connecting a vertex to the midpoint of the opposite side is called the *median* of that side, and the description just given shows that the three medians go through the same point M, which lies two thirds of the way along each median. M is known as the *centroid* of the triangle.

Bisectors, medians, and altitudes occur in many parts of the theory of triangles. So do the three points (namely K, M and H) that are the subject of our next and last example—but the most surprising fact about them seems to be more of a curiosity: *K, M and H lie on a single line, with MH twice as long as KM.* The line containing them is named after the great eighteenth century mathematician Leonhard Euler.

The point M is the centroid of both the triangle ABC and its enlargement $A^*B^*C^*$. As we have seen above, the intersection H of the altitudes is also the circumcentre of the triangle $A^*B^*C^*$, so it would make sense to write $H = K^*$.



Consider the following transformation of the plane: with M as the centre, turn the plane through 180 degrees and simultaneously double the distance of every point from M. How would this affect the points A, B, and C? How does it explain the mystery of Euler's line?

Diploma Exams Around the World

by Wieslaw Krawcewicz

All the problems discussed in this section come from actual diploma exams that have been given in European or Asian countries in recent years.

Problem 1†. Solve the following inequality

$$\lim_{n\to\infty}\left[2^{-\sin 3x}+4^{-\sin 3x}+\ldots+(2^n)^{-\sin 3x}\right]\le 1.$$

Solution. Notice that (geometric progression)

$$\begin{aligned}2^{-\sin 3x}&+4^{-\sin 3x}+\ldots+(2^n)^{-\sin 3x}\\&=2^{-\sin 3x}+\left(2^{-\sin 3x}\right)^2+\ldots+\left(2^{-\sin 3x}\right)^n\\&=2^{-\sin 3x}\left(\frac{1-2^{-n\sin 3x}}{1-2^{-\sin 3x}}\right).\end{aligned}$$

Thus, for the existence of the limit

$$\begin{aligned}&\lim_{n\to\infty}2^{-\sin 3x}\left(\frac{1-2^{-n\sin 3x}}{1-2^{-\sin 3x}}\right)\\&=\frac{2^{-\sin 3x}}{1-2^{-\sin 3x}}-\frac{1}{1-2^{-\sin 3x}}\lim_{n\to\infty}\left(2^{-\sin 3x}\right)^n,\end{aligned}$$

we need $2^{-\sin 3x}<1$ and, in this case, we have

$$\lim_{n\to\infty}2^{-\sin 3x}\left(\frac{1-2^{-n\sin 3x}}{1-2^{-\sin 3x}}\right)=\frac{2^{-\sin 3x}}{1-2^{-\sin 3x}}.$$

Indeed, for every $0<a<1$ we have

$$\lim_{n\to\infty}a^n=\lim_{n\to\infty}e^{n\ln a}=e^{-\infty}=0.$$

Consequently, we obtain the following two inequalities

$$2^{1-\sin 3x}\le 1,\quad 2^{-\sin 3x}<1,$$

which are equivalent to

$$1-\sin 3x\le 0,\quad -\sin 3x<0,$$

so

$$\begin{aligned}1\le\sin 3x&\iff 3x=\frac{\pi}{2}+2k\pi,\quad k=0,\pm1,\pm2,\ldots\\&\iff x=\frac{\pi}{6}+\frac{2k\pi}{3},\quad k=0,\pm1,\pm2,\ldots.\end{aligned}$$

† Problem from a May 1998 European diploma exam

Problem 2†. Solve the following equation

$$\sin^8 x+\cos^8 x=\frac{17}{32}.$$

Solution. Notice that we have the following identities

$$\begin{aligned}\sin^8 x+\cos^8 x&=(\sin^4 x+\cos^4 x)^2-2\sin^4 x\cos^4 x\\&=\left[(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x\right]^2\\&-\ 2\sin^4 x\cos^4 x\\&=(1-2\sin^2 x\,\cos^2 x)^2-2\sin^4 x\,\cos^4 x.\end{aligned}$$

By applying the substitution $t=\sin^2 x\cos^2 x$ we obtain that the given equation is equivalent to

$$(1-2t)^2-2t^2=\frac{17}{32},$$

where $0\le t\le 1$ (indeed, the product $\sin^2 x\cos^2 x$ must always be non-negative and smaller than or equal to 1), and after expanding it we obtain the quadratic equation

$$2t^2-4t+\frac{15}{32}=0,$$

or equivalently

$$64t^2-128t+15=0,$$

for which we compute the roots using the usual formulas

$$t_{12}=\frac{64\pm\sqrt{64^2-64\cdot 15}}{64},$$

i.e.,

$$t_1=\frac{1}{8},\quad t_2=\frac{15}{8}.$$

Therefore, we obtain that

$$\sin^2 x\,\cos^2 x=\frac{1}{8},$$

so

$$\sin x\,\cos x=\pm\frac{1}{2\sqrt{2}}.$$

Since $\sin x\,\cos x=\frac{1}{2}\sin 2x$, we have

$$\sin 2x=\pm\frac{\sqrt{2}}{2};$$

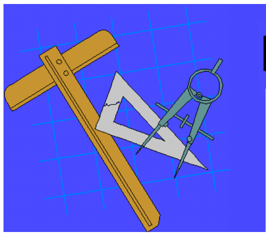
thus, the solution set for $2x$ is

$$\left\{\pm\frac{\pi}{4}+k\pi : k=0,\pm1,\pm2,\pm3,\ldots\right\},$$

so that the solution set for x can be expressed as

$$\left\{\frac{\pi}{8}+k\frac{\pi}{4} : k=0,\pm1,\pm2,\pm3,\ldots\right\}.$$

† Problem from a 1993 European university entrance exam



Math Strategies

The Rearrangement Inequality

by Dragos Hrimiuc

In this note we reveal a nice result that provides a very simple but powerful inequality that can be used for proving many other inequalities.

Let's consider two triplets (a_1,a_2,a_3) and (b_1,b_2,b_3) of real numbers. If we take all rearrangements (permutations) (x_1,x_2,x_3) of (b_1,b_2,b_3) we can generate $3! = 1\cdot 2\cdot 3 = 6$ sums of the following form:

$$S = a_1x_1 + a_2x_2 + a_3x_3. \tag{1}$$

Question: Which one of the above sums is the largest and which one is the smallest?

Before answering this question, let's introduce a simple concept:

Definition: Two triplets (a_1,a_2,a_3) and (b_1,b_2,b_3) are said to be:

- *similarly arranged* if both are increasing (i.e., $a_1 \le a_2 \le a_3$ and $b_1 \le b_2 \le b_3$) or both are decreasing (i.e., $a_1 \ge a_2 \ge a_3$ and $b_1 \ge b_2 \ge b_3$).
- *oppositely arranged* if one is increasing and the other is decreasing.

Examples

1. $(-1,1,3)$ and $(2,5,7)$ are similarly arranged while $(-1,1,3)$ and $(7,5,2)$ are oppositely arranged.
2. If $0 < a \le b \le c$, then (a,b,c) and $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ are oppositely arranged, while (a,b,c) and $\left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$ are similarly arranged.
3. If $0 < a \le b \le c$ and m is a positive real number, then (a,b,c) and (a^m,b^m,c^m) are similarly arranged while (a,b,c) and $\left(\frac{1}{a^m},\frac{1}{b^m},\frac{1}{c^m}\right)$ are oppositely arranged.
4. If $a \le b \le c$ and n is an odd integer, then (a,b,c) and (a^n,b^n,c^n) are similarly arranged.

The Rearrangement Inequality

Let (a_1,a_2,a_3), and (b_1,b_2,b_3) be two triplets of real numbers and (x_1,x_2,x_3) a permutation of (b_1,b_2,b_3).

- If (a_1,a_2,a_3) and (b_1,b_2,b_3) are similarly arranged, then

$$a_1b_1 + a_2b_2 + a_3b_3 \ge a_1x_1 + a_2x_2 + a_3x_3. \tag{2}$$

- If (a_1,a_2,a_3) and (b_1,b_2,b_3) are oppositely arranged, then

$$a_1b_1 + a_2b_2 + a_3b_3 \le a_1x_1 + a_2x_2 + a_3x_3. \tag{3}$$

Proof: Let's take two triplets (a_1,a_2,a_3) and (b_1,b_2,b_3) increasingly arranged and let (x_1,x_2,x_3) be a permutation of (b_1,b_2,b_3). Assume that $x_1 \ge x_2$.

Let $S = a_1x_1+a_2x_2+a_3x_3$ and $S' = a_1x_2+a_2x_1+a_3x_3$. S' is obtained from S by interchanging x_1 and x_2. We have

$$\begin{aligned} S' - S &= a_1x_2 + a_2x_1 - a_1x_1 - a_2x_2 \\ &= a_2(x_1-x_2) - a_1(x_1-x_2) \\ &= \underbrace{(x_1-x_2)}_{+}\,\underbrace{(a_2-a_1)}_{+} \ge 0. \end{aligned}$$

Hence $S' \ge S$. That is, interchanging x_1 and x_2 can only increase the value of the sum S. Therefore, if we interchange all pairs (x_i,x_j) so that $x_i \ge x_j$ for $i<j$, the sum can only get larger. The largest sum is that one that corresponds to (b_1,b_2,b_3), that is $a_1b_1+a_2b_2+a_3b_3$.

The above argument works similarly if (a_1,a_2,a_3) and (b_1,b_2,b_3) are both decreasing. We may also use the same argument to prove (3).

Notice: The equality in (2) or (3) occurs if and only if $b_1=b_2=b_3$ or $a_1=a_2=a_3$.

Now, let's apply the Rearrangement Inequality in some examples:

Example 1: Let $a,b,c \in \mathbb{R}$. Then

(i) $a^2+b^2+c^2 \ge ab+bc+ca$,

(ii) $a^n+b^n+c^n \ge a^{n-1}b+b^{n-1}c+c^{n-1}a$ for every even positive integer n.

Solution:
(i) is a particular case of (ii). Let's show (ii). Assume $a \le b \le c$. Since the triplets (a,b,c) and $(a^{n-1},b^{n-1},c^{n-1})$ are similarly arranged by using (2) we get:

$$aa^{n-1} + bb^{n-1} + cc^{n-1} \ge ab^{n-1} + bc^{n-1} + ca^{n-1},$$

which is just (ii).

Example 2: If $a,b,c>0$, then

(i) $\frac{a+b+c}{abc} \le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$,

(ii) $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \ge \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$,

(iii) $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge a+b+c$.

Solution: (i) The triplets $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ and $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ are similarly arranged (we may assume that $a \le b \le c$). Thus

$$\frac{1}{a}\frac{1}{a}+\frac{1}{b}\frac{1}{b}+\frac{1}{c}\frac{1}{c} \ge \frac{1}{a}\frac{1}{b}+\frac{1}{b}\frac{1}{c}+\frac{1}{c}\frac{1}{a},$$

that is (i).

(ii) The triplets $\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ and $\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ are similarly arranged. Hence

$$\frac{a}{b}\frac{a}{b}+\frac{b}{c}\frac{b}{c}+\frac{c}{a}\frac{c}{a} \ge \frac{a}{b}\frac{b}{c}+\frac{b}{c}\frac{c}{a}+\frac{c}{a}\frac{a}{b},$$

which is (ii).

(iii) The triplets (a^2, b^2, c^2) and $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ are oppositely arranged. Hence

$$a^2\frac{1}{a} + b^2\frac{1}{b} + c^2\frac{1}{c} \le a^2\frac{1}{b} + b^2\frac{1}{c} + c^2\frac{1}{a},$$

that is, just (iii).

Notice: In the above inequalities, the equality occurs if and only if $a = b = c$.

Example 3: If $a, b, c > 0$, then

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \ge \frac{3}{2}.$$

Solution: The triplets (a, b, c) and $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ are similarly arranged (we may assume $a \le b \le c$).

Thus

$$a\frac{1}{b+c} + b\frac{1}{c+a} + c\frac{1}{a+b} \ge a\frac{1}{c+a} + b\frac{1}{a+b} + c\frac{1}{b+c},$$

and also

$$a\frac{1}{b+c} + b\frac{1}{c+a} + c\frac{1}{a+b} \ge a\frac{1}{a+b} + b\frac{1}{b+c} + c\frac{1}{c+a}.$$

Adding these two inequalities we find:

$$\begin{aligned} 2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) &\ge \frac{a}{c+a} + \frac{b}{a+b} + \frac{c}{b+c} \\ &\quad + \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} = 3. \end{aligned}$$

Therefore, the required inequality is obtained. Equality holds if and only if $a = b = c$.

The Rearrangement Inequality may be used to prove some classical inequalities.

Example 4: (Chebyshev's Inequality) If (a_1, a_2, a_3), (b_1, b_2, b_3) are similarly arranged, then

$$\frac{a_1b_1 + a_2b_2 + a_3b_3}{3} \ge \left(\frac{a_1 + a_2 + a_3}{3}\right)\left(\frac{b_1 + b_2 + b_3}{3}\right).$$

Solution: Using the Rearrangement Inequality we get:

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 &= a_1b_1 + a_2b_2 + a_3b_3 \\ a_1b_1 + a_2b_2 + a_3b_3 &\ge a_1b_2 + a_2b_3 + a_3b_1 \\ a_1b_1 + a_2b_2 + a_3b_3 &\ge a_1b_3 + a_2b_1 + a_3b_2. \end{aligned}$$

Adding these inequalities we obtain

$$\begin{aligned} 3(a_1b_1 \ + \ & a_2b_2 + a_3b_3) \\ &\ge a_1(b_1 + b_2 + b_3) + a_2(b_1 + b_2 + b_3) \\ &\quad + a_3(b_1 + b_2 + b_3), \end{aligned}$$

which is the desired result. Equality holds if $a_1 = a_2 = a_3$ or $b_1 = b_2 = b_3$.

Remark: If (a_1, a_2, a_3) and (b_1, b_2, b_3) are oppositely arranged, then

$$\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{3}\right) \le \left(\frac{a_1 + a_2 + a_3}{3}\right)\left(\frac{b_1 + b_2 + b_3}{3}\right).$$

Example 5: (Root Mean Square—Arithmetic Mean Inequality)

Let a_1, a_2, a_3 be real numbers. Then

$$\frac{a_1 + a_2 + a_3}{3} \le \sqrt{\frac{a_1^2 + a_2^2 + a_3^2}{3}}.$$

Solution: We may assume that $a_1 \le a_2 \le a_3$. Then the triplets (a_1, a_2, a_3) and (a_1, a_2, a_3) are similarly arranged; the required inequality follows from using Chebyshev's Inequality.

Example 6: (The Arithmetic Mean—Geometric Mean Inequality)

If a_1, a_2, a_3 are positive numbers then

$$\frac{a_1 + a_2 + a_3}{3} \ge \sqrt[3]{a_1a_2a_3}.$$

Solution: Let

$$x_1 = \frac{a_1}{P}, \quad x_2 = \frac{a_1a_2}{P^2}, \quad x_3 = \frac{a_1a_2a_3}{P^3} = 1$$

and

$$y_1 = \frac{1}{x_1}, \quad y_2 = \frac{1}{x_2}, \quad y_3 = \frac{1}{x_3} = 1,$$

where $P = \sqrt[3]{a_1a_2a_3}$.

We may assume without loss of generality (by relabelling if necessary) that (x_1, x_2, x_3) is increasing; then (y_1, y_2, y_3) is decreasing.

Hence

$$x_1y_1 + x_2y_2 + x_3y_3 \le x_1y_3 + x_2y_1 + x_3y_2;$$

that is,

$$1 + 1 + 1 \le \frac{a_1}{P} + \frac{a_2}{P} + \frac{a_3}{P},$$

which is just the required inequality.

The equality holds if and only if $x_1 = x_2 = x_3$, or equivalently, $a_1 = a_2 = a_3$.

Now try to solve the following problems yourself:

Problem 1.

(i) If (a_1, a_2) and (b_1, b_2) are similarly arranged, then

$$a_1b_1 + a_2b_2 \ge a_1b_2 + a_2b_1.$$

(ii) If (a_1, a_2) and (b_1, b_2) are oppositely arranged, then

$$a_1b_1 + a_2b_2 \le a_1b_2 + a_2b_1.$$

(iii) In (i) and (ii), the equality occurs if and only if $a_1 = a_2$ or $b_1 = b_2$.

(iv) State and prove Chebyshev's Inequality for two pairs of real numbers.

(v) Prove the inequality

$$\frac{a^n+b^n}{a+b} \geq \frac{1}{2}\left(a^{n-1}+b^{n-1}\right)$$

for a and b positive real numbers, and n a positive integer.

Problem 2. If $a, b \geq 0$, then

(i) $2(a^5+b^5) \geq (a^3+b^3)(a^2+b^2)$.

(ii) $a^9+b^9 \geq a^2b^2(a^5+b^5)$.

(iii) $(a+b)^n \leq 2^{n-1}(a^n+b^n)$.

Problem 3. If $a, b, c > 0$, then

(i) $ab+bc+ca \geq a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}$.

(ii) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$.

(*Hint:* Use Chebyshev's Inequality or the AM–GM Inequality of Example 6.)

(iii) $\frac{a+b+c}{3} \leq \sqrt[n]{\frac{a^n+b^n+c^n}{3}}$.

(*Hint:* See Example 5.)

Problem 4. If $a, b, c > 0$ and n is a positive integer, then

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b} \geq \frac{a^{n-1}+b^{n-1}+c^{n-1}}{2}.$$

(*Hint:* If we assume that $a \leq b \leq c$, then (a^n, b^n, c^n) and $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ are similarly arranged. See also the solution of Example 3. You may also use (v) from Problem 1.)

Problem 5. If $a, b, c > 0$, then

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

(*Hint:* If $a \leq b \leq c$, then (a, b, c) and $(\log a, \log b, \log c)$ are similarly arranged, use Chebyshev's Inequality and some properties of the logarithmic function.)

Problem 6. Let A, B, C be the angles (measured in radians) of a triangle with sides a, b, c and $p = \frac{1}{2}(a+b+c)$. Then

$$\frac{A}{p-a}+\frac{B}{p-b}+\frac{C}{p-c} \geq \frac{3\pi}{p}.$$

(*Hint:* We may assume that $A \leq B \leq C$. Then (A, B, C) and $\left(\frac{1}{p-a}, \frac{1}{p-b}, \frac{1}{p-c}\right)$ are similarly arranged. Use Chebyshev's Inequality and then the inequality of Problem 3(ii).)

Problem 7. Let a, b, c be positive real numbers such that $abc = 1$. Prove that[†]

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

(*Hint:* Set $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$. Since $abc = 1$ we obtain $xyz = 1$. With this new notation the required inequality transforms into

$$\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y} \geq \frac{3}{2}.$$

This inequality follows easily from Problem 4, combined with AM–GM Inequality.)

[†] This problem was proposed at the International Mathematical Olympiad in 1995.

Project #1.

- Investigate the Rearrangement Inequality for three or more positive triplets of real numbers.
- Write a proof and find some interesting examples. Can you extend Chebyshev's Inequality?

Project #2.

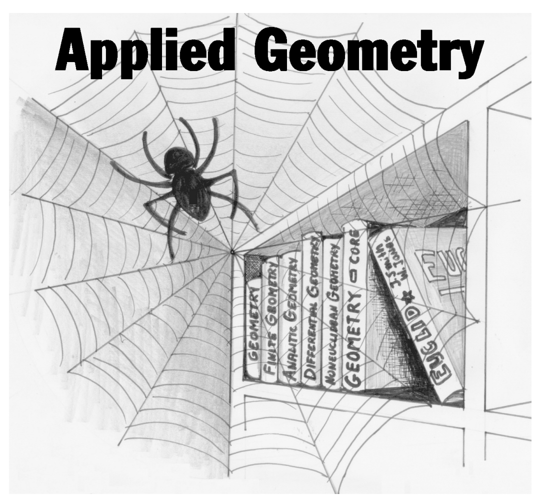
- Investigate the Rearrangement Inequality for two n-tuples $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ of real numbers.
- Find some interesting examples. (You may extend some from this article.)
- Write a proof of the AM–GM inequality and Chebyshev's inequality in this general case.

If you proceed with either of the projects, please send your results to us. We are going to publish the best notes in the next issues of π *in the Sky*.

Math Jokes

John Napier (1550-1617), who was an engineer, physicist and the mathematician considered to be the inventor of logarithms, was also regarded by his contemporaries as a dealer in black magic. One day he announced that his coal black rooster would identify for him which of his servants was stealing from him. The servants were sent one by one into a darkened room with instructions to pat the rooster on the back. Unknown to the servants, Napier had coated the bird's back with lampblack, and the guilty servant, fearing to touch the rooster, returned with clean hands.

Although Euler is pronounced *oil-er*, it does not follow that Euclid is pronounced *oi-clid*.



TEACHER'S STUDIO

Some Remarks on the Quadratic Polynomial†

Consider the quadratic polynomial

(1) $$f(x) = ax^2 + bx + c, \quad a \neq 0.$$

Denote as usual

(2) $$\Delta = b^2 - 4ac \quad \text{(the discriminant)}.$$

The following results for the roots x_1 and x_2 of the equation $f(x) = 0$ are known:

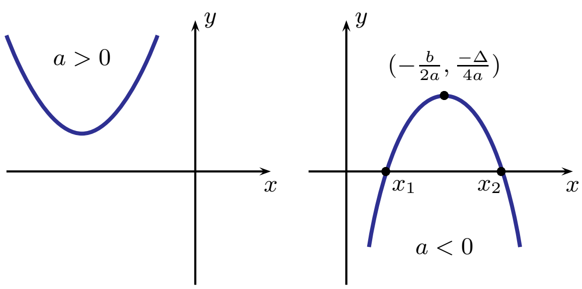
- The roots are given by the quadratic formula

(3) $$x_1 = \frac{-b - \sqrt{\Delta}}{2a}; \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a},$$

- $x_1, x_2 \in \mathbb{R} \iff \Delta \geq 0$; $x_1 \neq x_2 \iff \Delta > 0$,
- The sum $S = x_1 + x_2$ and product $P = x_1 x_2$ are given by

(4) $$S = -\frac{b}{a}; \quad P = \frac{c}{a}.$$

The graph of the function $y = f(x)$ is a parabola.



- The parabola opens upward if $a > 0$.
- The parabola opens downward if $a < 0$.
- $(-\frac{b}{2a}, \frac{-\Delta}{4a})$ is the vertex.
- The x axis intercepts (if exist) are $(x_1, 0)$ and $(x_2, 0)$.

An useful observation: Let A, B be two real numbers. Then

(i) $A > 0,\ B > 0 \iff AB > 0$ and $A + B > 0$,

(ii) $A < 0,\ B < 0 \iff AB > 0$ and $A + B < 0$,

(iii) $A < 0,\ B > 0 \iff AB < 0$.

† Topics presented in this section can be used by teachers in math classes and with students of different ability levels. We also recommend it to all motivated students to improve their understanding of mathematical concepts and ideas. This article was contributed by the teacher Venera Hrimiuc (vhrimiuc@ualberta.ca). We invite other teachers to send us their suggestions or articles for this section.

In some situations, the coefficients a, b, c of (1) do not have numerical values. In this case, the expressions of x_1 and x_2 as written in (3) may be complicated and very difficult to manipulate. However, we still can investigate some properties of x_1 and x_2 without using (3).

Example 1. Let $m \in \mathbb{R}$ be a parameter (the numerical value of m is not specified). Consider the quadratic equation

$$x^2 + 2(m+1)x + m(m-1) = 0,$$

with solutions x_1, x_2. Find all the values of m for which $x_1, x_2 \in \mathbb{R}$ satisfy the condition

$$\text{(i) } 0 < x_1 < x_2; \quad \text{(ii) } x_1 < x_2 < 0; \quad \text{(iii) } x_1 < 0 < x_2.$$

Solutions: x_1 and x_2 have to be in each case distinct and real, hence $\Delta > 0$. We have

$$\begin{aligned} \Delta &= 4(m+1)^2 - 4m(m-1) \\ &= 4(3m+1). \end{aligned}$$

Hence

$$\begin{aligned} \Delta &> 0 \iff 3m + 1 > 0 \\ &\iff m > -\frac{1}{3} \iff m \in \left(-\frac{1}{3}, \infty\right) \end{aligned}$$

(i) $0 < x_1 < x_2 \iff \Delta > 0$ and $x_1 x_2 > 0$ and $x_1 + x_2 > 0 \iff \Delta > 0$ and $P > 0$ and $S > 0$.

Since

$$P = \frac{c}{a} = m(m-1), \quad S = -\frac{b}{a} = -2(m+1),$$

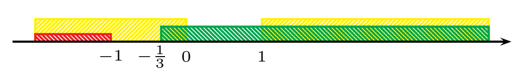
and

$$\begin{aligned} m(m-1) &> 0 \iff m \in (-\infty, 0) \cup (1, \infty) \\ -2(m+1) &> 0 \iff m + 1 < 0 \iff m \in (-\infty, -1), \end{aligned}$$

the condition (i) holds if simultaneously

$$m \in \left(-\frac{1}{3}, \infty\right), \quad m \in (-\infty, 0) \cup (1, \infty),$$

$$m \in (-\infty, -1).$$



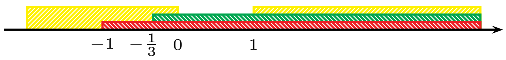
As you can see, there is no $m \in \mathbb{R}$ that satisfies all the requirements.

(ii) $x_1 < x_2 < 0 \iff \Delta > 0$ and $x_1 x_2 > 0$ and $x_1 + x_2 < 0 \iff \Delta > 0$ and $P > 0$ and $S < 0$. Since

$$\begin{aligned} P &> 0 \iff m(m-1) > 0 \iff m \in (-\infty, 0) \cup (1, \infty) \\ S &< 0 \iff -2(m+1) < 0 \iff m \in (-1, \infty), \end{aligned}$$

therefore m should satisfy simultaneously

$$m \in \left(-\frac{1}{3}, \infty\right), \quad m \in (-\infty, 0) \cup (1, \infty), \quad m \in (-1, \infty).$$



We obtain

$$m \in \Big(-\frac{1}{3}, 0\Big) \cup (1, \infty).$$

(iii) $x_1 < 0 < x_2 \iff \Delta > 0$ and $x_1 x_2 < 0 \iff \Delta > 0$ and $P < 0$. Since

$$P < 0 \iff m \in (0, 1),$$

we must have

$$m \in \Big(-\frac{1}{3}, \infty\Big) \quad \text{and} \quad m \in (0, 1),$$

thus

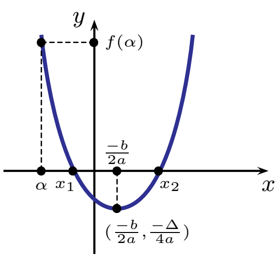
$$m \in (0, 1).$$

We can now work with a more general problem.

Problem 1. Let $\alpha \in \mathbb{R}$ be a given number. If x_1, x_2 are solutions of (1), find necessary and sufficient conditions expressed in terms of a, b, c such that:

(i) $\alpha < x_1 < x_2$, (ii) $x_1 < x_2 < \alpha$, (iii) $x_1 < \alpha < x_2$.

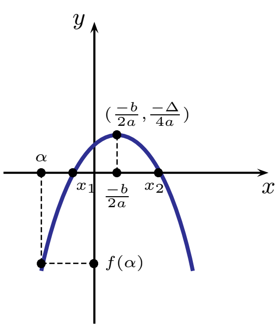
Solution. (i) Let's assume $a > 0$. The parabola $y = f(x)$ opens upward and intersects the x axis at x_1 and x_2.



$\alpha < x_1 < x_2 \iff x_1, x_2$ are real and distinct, α is outside the interval $[x_1, x_2]$ and on the left of $[x_1, x_2]$.

We can translate these requirements with mathematical tools:

$$\Delta > 0, \quad f(\alpha) > 0, \quad \alpha < -\frac{b}{2a}.$$



If $a < 0$ the parabola opens downward and
$\alpha < x_1 < x_2 \iff \Delta > 0, f(\alpha) < 0$, $\alpha < -\frac{b}{2a}$.

We remark that in each case $f(\alpha)$ has the sign of a, that is, equivalently, $af(\alpha) > 0$.

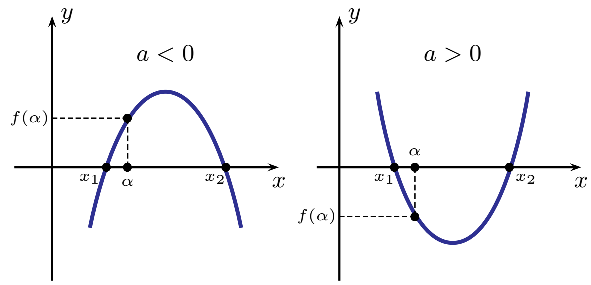
We conclude

$$\alpha < x_1 < x_2 \iff \begin{cases} \Delta > 0 \\ af(\alpha) > 0 \\ \alpha < -\frac{b}{2a}. \end{cases}$$

(ii) As above

$$x_1 < x_2 < \alpha \iff \begin{cases} \Delta > 0 \\ af(\alpha) > 0 \\ -\frac{b}{2a} < \alpha. \end{cases}$$

(iii)



$$x_1 < \alpha < x_2 \iff af(\alpha) < 0$$

Notice: If $af(\alpha) < 0$, then automatically $\Delta > 0$. *Indeed*, if $\Delta \leq 0$, the parabola is above or below the x axis or it is tangent to the x axis. In each of these cases, $af(\alpha) \geq 0$ for every $\alpha \in \mathbb{R}$.

Example 2. Let $m \in \mathbb{R}$, $m \neq -1$ be a parameter and x_1, x_2 the real roots of the equation

$$f(x) = (m+1)x^2 + 2mx + m - 2 = 0.$$

For which values of m does each of the following conditions hold?

(i) $x_1 < 1 < x_2$, (ii) $-1 < x_1 < x_2 < 1$.

Solution. Using the results of Problem 1 we get

$$\text{(i)} \qquad x_1 < 1 < x_2 \iff (m+1)f(1) < 0$$
$$\iff (m+1)(4m-1) < 0$$
$$\iff m \in \Big(-1, \frac{1}{4}\Big).$$

$$\text{(ii)} \quad -1 < x_1 < x_2 < 1 \iff \Delta > 0,\ af(-1) > 0,$$
$$af(1) > 0 \qquad \text{and} \quad -1 < -\frac{b}{2a} < 1.$$

$$\begin{aligned} \Delta > 0 &\iff m + 2 > 0 \\ &\iff m \in (-2, \infty), \\ af(-1) > 0 &\iff (m+1)(-1) > 0 \\ &\iff m \in (-\infty, 1), \\ af(1) > 0 &\iff (m+1)(4m-1) > 0 \\ &\iff m \in (-\infty, -1) \cup \Big(\frac{1}{4}, \infty\Big), \\ -1 < -\frac{b}{2a} < 1 &\iff -1 < \frac{m}{m+1} < 1 \\ &\iff m \in (-\infty, -1). \end{aligned}$$

The intersection of the above intervals is $(-2, -1)$; therefore

$$-1 < x_1 < x_2 < 1 \iff m \in (-2, -1).$$

Example 3.
Let $m \in \mathbb{R}$, $m \neq 0$ and

$$f(x) = mx^2 - 2(m-2)x - m + 2.$$

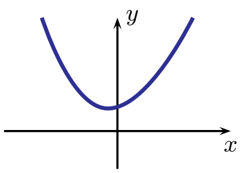
Find the values of m such that each of the following conditions holds:

(i) $f(x) > 0$ for every $x \in \mathbb{R}$,

(ii) $f(x) > 0$ for every $x \in (0, \infty)$,

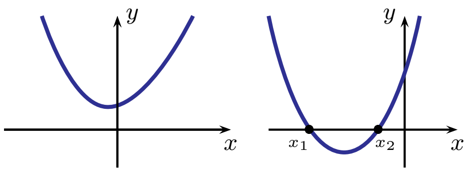
(iii) $f(x) > 0$ for every $x \in (0, 1)$.

Solution. (i)



The parabola should be above x axis. This is equivalent to $a > 0$, $\Delta < 0 \iff m > 0$, $(m-2)(m-1) < 0 \iff m \in (1, 2)$.

(ii) There are two possibilities: (a) $a > 0$ and $\Delta < 0$ or (b) $x_1 < x_2 < 0$.

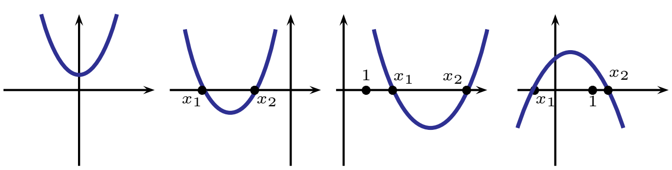


(a) $a > 0$ and $\Delta < 0 \iff m \in (1, 2)$ (see (i)).

(b) $a > 0$, $\Delta \geq 0$, $x_1 \leq x_2 \leq 0 \iff a > 0$, $\Delta \geq 0$, $af(0) \geq 0$, $\frac{-b}{2a} \leq 0 \iff m > 0$, $m \in (-\infty, 1] \cup [2, \infty)$, $m \in [0, 2]$, $m \in (0, 2]$ $\iff m = 2$.

The condition (ii) is satisfied for $m \in (1, 2]$.

(iii)



As the above picture shows, there are four possibilites:
(a) $a > 0$, $\Delta < 0 \iff m \in (1, 2)$ (see(i));
(b) $a > 0$, $\Delta > 0$ $x_1 < x_2 < 0 \iff m \in (1, 2]$ (see (ii));
(c) $a > 0$, $\Delta > 0$ $1 < x_1 < x_2$;
(d) $a < 0$, $\Delta > 0$ $x_1 < 0$ $1 < x_2$.

(c) $a > 0$, $\Delta \geq 0$, $1 \leq x_1 \leq x_2 \iff a > 0$, $\Delta \geq 0$, $af(1) \geq 0$, $-\frac{b}{2a} \geq 1 \iff m > 0$, $m \in (-\infty, 1] \cup [2, \infty)$, $m \in [0, 2]$, $m < 0$ $\iff m \in \emptyset$.

(d) $a < 0$, $\Delta > 0$, $x_1 < 0$, $1 < x_2 \iff a < 0$, $\Delta > 0$, $af(0) < 0$, $af(1) < 0 \iff m < 0$, $m \in (-\infty, -1) \cup (2, \infty)$, $m \in (-\infty, 0) \cup (2, \infty) \iff m \in (-\infty, -1)$.

Therefore, from (a), (b), (c), and (d) the condition (iii) is satisfied for $m \in (-\infty, -1) \cup (1, 2]$.

Example 4. For what values of the parameter m does the equation

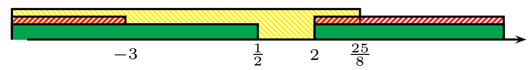
$$(m-2)x^3 - (2m-1)x^2 + (m+3)x = 0$$

have exactly three non-negative real roots?

Solution. We have that $x_3 = 0$ is always a root, and x_1 and x_2 roots of

$$f(x) = (m-2)x^2 - (2m-1)x + (m+3) = 0,$$

where $\Delta > 0$, $S > 0$, $P > 0$ i.e. $-8m + 25 > 0$, $\frac{2m-1}{m-2} > 0$ and $\frac{m+3}{m-2} > 0$.



Consequently, the required condition is satisfied for

$$m \in (-\infty, -3) \cup (2, \frac{25}{8}).$$

Example 5†. For what values of the parameter m does the equation

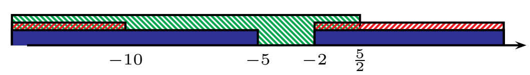
$$(2+m)\log_2^2(x+4) + 2(1-m)\log_2(x+4) + m - 2 = 0$$

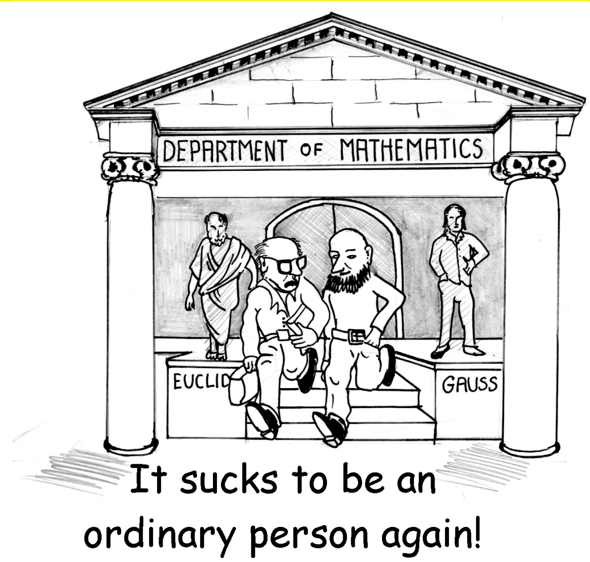
have two different negative solutions?

Solution. $-4 < x < 0 \iff 0 < x+4 < 4 \iff \log_2(x+4) < 2$. Therefore, there are two different negative solutions x_1 and x_2 if and only if the equation

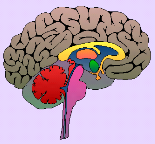
$$f(X) = (2+m)X^2 + 2(1-m)X + m - 2 = 0$$

has two solutions $X_1 = \log_2(x_1 + 4)$ and $X_2 = \log_2(x_2 + 4)$, satisfying $X_1 < X_2 < 2$. We have
$X_1 < X_2 < 2 \iff \Delta = 20 - 8m > 0$, $(2+m)f(2) > 0$, $\frac{-2(1-m)}{2(2+m)} < 2 \iff \frac{5}{2} > m$, $(m+2)(m+10) > 0$, $\frac{m+5}{m+2} > 0 \iff$ $m \in (-\infty, -10) \cup (-2, \frac{5}{2})$.





† Problem from 1998 diploma exam in Poland.



Math Challenges

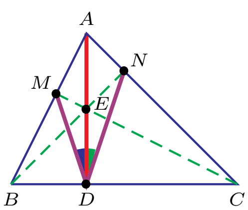
Problem 1. Let a, b, c be positive numbers. Show that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\leq\frac{1}{2}(a+b+c).$$

Hint: Read Math Strategies in this issue of π *in the Sky*.

Problem 2. In a tournament with n players, everybody plays against everybody else exactly once. Prove that at any moment during the tournament there are always at least two players who have played exactly the same number of games.
Hint: Read Math Strategies in the June issue of π *in the Sky*.

Problem 3. Let E be a point on the altitude AD of the triangle ABC. Assume that the line passing through C and E intersects AB at the point M and the line passing through B and E intersects AC at the point N. Prove that AD is a bisector of the angle MDN (see the diagram).



Problem 4. Let $f(x)=x^2-2(a+1)x+a^2$, $a\in\mathbb{R}$. Find all the values a such that $|f(x)|\leq\frac{17}{4}$ for every $x\in[-1,1]$.
Hint: Check out the Math Studio section in this issue of π *in the Sky*.

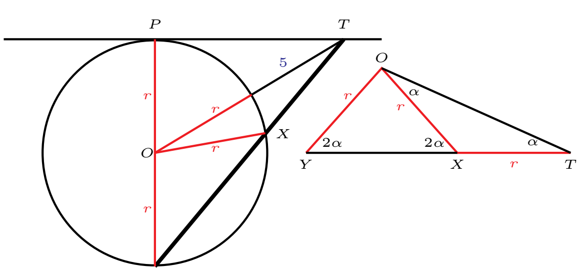
Problem 5. Prove that every positive integer n and every $x\in\mathbb{R}$ we have

$$\sin^{2n}x+\cos^{2n}x\geq\frac{1}{2^{n-1}}.$$

Send your solutions to π *in the Sky*: Math Challenges.

Solutions to the Problems Published in the June Issue of π *in the Sky*:

Problem 1: Substitute $X=\sqrt[4]{16x+1}$ in order to reduce the equation to $X^2-2X-3=0$, which has two solutions $X_1=3$ and $X_2=-1$. Therefore, $3=\sqrt[4]{16x+1}$, so $x=5$.
Problem 2: Notice that the mosaic of side length k is made of $1+3+5+\ldots+(1+2k)=(1+k)^2$ pieces. Since $\sqrt{15878}\approx 126.00783\ldots$, the maximal side of the mosaic is equal to 125 and there will be 2 pieces left over.
Problem 3: By cutting the square of side length $0<x<6$, we obtain the box of dimensions $(12-x)\times(18-2x)\times x$. The volume of this box is equal to $V(x)=(12-2x)(18-2x)x=4x^3-60x^2+216x$. Since the derivative $V'(x)$ is zero at $x_1=5-\sqrt{7}$ and $x_2=5+\sqrt{7}$, and $V(0)=V(6)=0$, the volume attains its maximum for $x=5-\sqrt{7}\approx 2.354248689\ldots$.
Problem 4:



We have that $r^2+10^2=(r+5)^2$ so $r=\frac{15}{2}$ and $T=X$, so the length x of XY can be determined from the equation $2\cos^2\alpha-1=\cos 2\alpha=\frac{x}{2r}$, i.e. $x=\frac{r^2+10r+25}{r}=\frac{35}{6}$.

Problem 5: Three cards: $K\spadesuit$ $Q\spadesuit$ $Q\heartsuit$

Problem 6: If the height this year is H, then the last year it was $H_o=\frac{H}{1.1}$ and the year before $H_*=\frac{H_o}{1.2}=\frac{H}{1.1\times 1.2}=\frac{H}{1.32}$, so your height increased 32% during the last two years.
Problem 7: (a) After n bets, if the gambler has k wins and $n-k$ losses, then his balance is $x=1+k-(n-k)=1+2k-n$. Thus, if $x+n$ is even, $P_n(x)=0$; otherwise

$$P_n(x)=\binom{n}{\frac{x-1+n}{2}}\left(\frac{1}{2}\right)^n.$$

(b) Instead of removing the gambler from the game, we can allow him to continue to play if we subtract the probability $Q_n(x)$ that a second 'daemon' player G' also has x dollars. Player G' starts with a balance of $-\$1$ dollar and wins $\$1$ whenever the gambler loses, and vice versa, so that when he has zero dollars, so does G'. Thereafter, the probability of a positive win x by the gambler after n bets equals the probability of a positive win x by G' since the probability of k losses equals the probability of k wins.
If $n+x$ is odd, the probability $P_n(x)-Q_n(x)$ is

$$\binom{n}{\frac{x-1+n}{2}}\left(\frac{1}{2}\right)^n-\binom{n}{\frac{x+1+n}{2}}\left(\frac{1}{2}\right)^n.$$

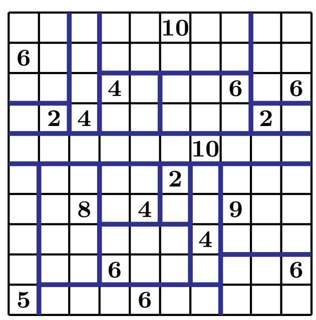
Hence, the probability that the gambler will still be in the game after n bets is

$$\sum_{x=1}^{n+1}P_n(x)-\sum_{x=1}^{n-1}Q_n(x)=\sum_{k=\lceil\frac{n}{2}\rceil}^{k=n}\binom{n}{k}\left(\frac{1}{2}\right)^n-\sum_{k=\lceil\frac{n}{2}\rceil+1}^{k=n}\binom{n}{k}\left(\frac{1}{2}\right)^n,$$

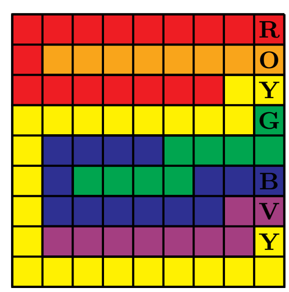
where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. On simplifying this telescoping series, we find that the probability that the gambler will still be in the game after n bets is

$$\binom{n}{\lceil\frac{n}{2}\rceil}\left(\frac{1}{2}\right)^n.$$

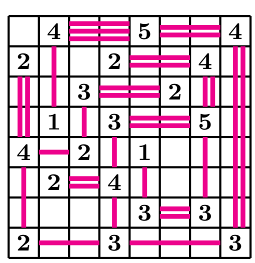
Problem 8:



Problem 9:



Problem 10:



These three problems were taken from Paul Vaderlind's book (in Swedish) "Är Detta Mathematik."

We received the solutions to all of the Math Challenge problems in the June issue from **Robert Wong**, a math teacher at Vernon Barford Junior High School. Good work!

Our Forum

I enjoyed the first issue of π *in the Sky*, but you might like to note that on p.14 the name of a great mathematician, Norbert Wiener, was incorrectly spelled (twice). Andy Liu's editorial was especially interesting. Peter Zvengrowski

We apologize for the mistake. Indeed, we spelled Norbert Wiener's name incorrectly.

—Editors

Dear Editors of π *in the Sky*,

I enjoy π *in the Sky* very much and am very happy with the content in it. However, I found a little mistake in one of the articles in the June 2000 issue, particularly in Dr. Dragos Hrimiuc's 'The Box Principle' article. In question 4, when you get the number 11...100...0 from subtracting, and cancelling all the zeroes, you get a number 11...1. This number, as said, is divisible by 1999, so when divided by 1999, gives remainder 0. However, this number is one of the numbers from 1, 11, 111, ..., 11...1. We assumed that the remainders of these numbers is not 0, but that number in question has remainder 0, hence it's a contradiction. This is not mentioned, and makes it seem that this contradiction is not realized.

Thanks again for the wonderful magazine. Jeffrey Mo

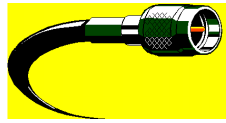
Yes, you are absolutely right. Alternatively, one can express the solution as a direct proof. Consider the integers $1, 11, \ldots, \underbrace{11\ldots1}_{2000\ digits}$. *If we divide these 2000 integers by 1999, we get the remainders* $0, 1, \ldots, 1998$. *At least two numbers have the same remainder (Box Principle) when they are divided by* 1999. *Therefore, their difference* $111\ldots10\ldots00$ *will be divisible by* 1999. *Canceling all zeros from the end, we get a number consisting of ones and divisible by* 1999.

—Editors

HYPATIA'S Street Theatre

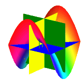
This play will be presented at 8:00 pm on **December 10, 2000** *at the* **Frederic Wood Theatre**, *on the University of British Columbia campus. Written by Klaus Hoechsmann and Ted Galay, it is organized around three mathematical skits. The principal ambition of this play is to show mathematics on stage—not just* talking *about it, but actually* doing *it—in whatever form the public can take.*

Hypatia, the last of the Alexandrian scholars recorded by history, was brutally murdered by a fundamentalist mob in March of 415 AD. Her father Theon, a mathematician, philosopher, and director of the University (called the "Museum") of Alexandria, had seen to it that his talented daughter received the best available training in all conceivable disciplines from rhetoric through music to mathematics. Blessed with physical strength and beauty, she was by all accounts a model of rectitude and modesty. It is difficult to exaggerate the esteem in which she was held by contemporaries, whether in Athens, Rome, or Alexandria itself.



Math Links

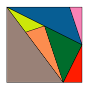
Math for Students with the Help of Students



This site contains a steadily growing library of math materials that are interactive and multimedia enhanced. Most of the programming and the multimedia work has been implemented by students.

http://www.ualberta.ca/dept/math/gauss/fcm

Puzzles



The site http://www.mathpuzzle.com *contains an exciting collection of dissection puzzles and problems. Here is an example. Divide a square into 7 triangles so that one triangle has edge lengths in the ratio 3:4:5 and the other 6 triangles can be arranged into another square! Can it be done with less than seven triangles?*

http://www.mathpuzzle.com/weihwatri.html

http://www.mathpuzzle.com/mine.html

How To Write Proofs

This site explains and provides examples of various proof techniques: Direct Proof, Proof by Contradiction, Proof by Contrapositive, Proof by Mathematical Induction, Proof Strategies, Constructive Versus Existential Proofs, Counterexamples, Proof by Exhaustion (Case by Case).

http://zimmer.csufresno.edu/~larryc/proofs/proofs.html

International Mathematical Olympiad

The annual International Mathematical Olympiad (IMO) is the world championship mathematics competition for high school students and is held each year in a different country. The first IMO was held in 1959 in Romania, with seven countries participating. It has gradually expanded to over 80 countries from all five continents. Learn more at these sites:

http://imo.math.ca/

http://www.cms.math.ca/Olympiads/

Famous Curves

An interactive site offering interesting facts about famous curves, their history, related names and other related curves.

http://www-history.mcs.st-andrews.ac.uk/history/Curves/Curves.html