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Geometric and algebraic reduction for singular momentum maps.
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This paper concerns symplectic reduction in the singular case. In 1974 J. E. Marsden and A. Weinstein [Rep. Math. Phys. **5** (1974), no. 1, 121–130; MR0402819 (53 #6633)] formalized the notion of symplectic reduction. One begins with a Hamiltonian G-action on a symplectic manifold P. J, a function on P with values in $(\mathrm{Lie}(G))^*$, denotes its momentum map (generalized Noether-conserved quantities). The symplectic reduced space is the subquotient $P_0 = \{J = 0\}/G$, of P. The definition works admirably provided 0 is a regular value of J. (Actually, it works fine if 0 is "weakly regular".)

What if 0 is a singular value of J? Then there are several competing definitions. J. Śniatycki and Weinstein's [Lett. Math. Phys. **7** (1983), no. 2, 155–161; MR0708438 (85c:58046)] algebraic reduction proceeds by mimicking algebraic geometry. They replace the geometric operations of restricting to a submanifold and of dividing by a group action in the original definition by their function-theoretic counterparts, namely dividing the ring of smooth functions on P by an ideal (the functions vanishing on $\{J = 0\}$) and restricting to the set of G-invariant functions. The endproduct is a Poisson algebra which may not be the function algebra of any topological space. In another definition of singular reduction, extracted from Dirac's work in the present paper, the reduced space is constructed by dividing the space $\{J = 0\}$ by an equivalence relation generated by a certain class of functions termed "observable". The final definition, the "geometric reduction" of the present paper, is to divide $\{J = 0\}$ by a stronger equivalence relation than Dirac's. The authors declare two points q and p, with $J(q) = J(p) = 0$, to be equivalent if there exists a piecewise smooth curve c joining them, which lies in $\{J = 0\}$ and whose derivative, when it exists, satisfies $W(dc/dt, v) = 0$ for all vectors v tangent to some (other) curve which also lies entirely within $\{J = 0\}$. Here W is the symplectic form.

The main result of this paper is a collection of theorems in Section 6 giving conditions under which the various reductions agree. They assume G is compact and connected. The main tools of proof are a normal form for the momentum map, the quadratic singularity theorem of Arms, Marsden and V. E. Moncrief [Comm. Math. Phys. **78** (1980/81), no. 4, 455–478; MR0606458 (82m:58028)], and some real algebraic geometry. The quadratic singularity theorem states that the singularities of $\{J = 0\}$ are at worst conical, meaning that there are local coordinates in which the set $\{J = 0\}$ is the product of a hyperplane with the zero set of a homogeneous quadratic function from one vector space to another. The authors give an alternate, very direct proof of this theorem using their normal form. (This normal form can be found buried in several places, for instance in the text of Guillemin and Sternberg, but nowhere as clearly worded.)

The end of the paper consists of a number of examples and counterexamples. The paper is well organized and well written.

Reviewed by *Richard Montgomery*