Constraints, reduction, and quantization

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Theorems are proved that establish the unitary equivalence of the extended and reduced phase space quantizations of a constrained classical system with symmetry. Several examples are presented.

I. INTRODUCTION

Among classical dynamical systems, those which are "constrained" are often the most important and interesting. Typically, constraints arise when the equations of motion are overdetermined or when symmetries are present. In the first circumstance the constraints take the form of restrictions on the admissible initial data for the evolution equations of the system. The divergence constraints of electromagnetism and Yang–Mills theory and the super-Hamiltonian and supermomentum constraints of general relativity are standard examples. In the second case the constraints consist of *a posteriori* specifications of the constants of motion associated with the invariances of the system. The mass and charge constraints in the Kaluza–Klein formalism for a relativistic charged particle are of this type, as is, for example, fixing the angular momentum of a rotationally invariant system.

All constrained systems can be described naturally in terms of symplectic geometry.[1,2] Beyond this, however, many such systems have a rich group-theoretical structure. It is an amazing fact that the constraints are usually given by $J = \text{const}$, where J is a momentum mapping for an appropriately chosen group action.[3] These observations lead us to model a constrained dynamical system as follows.

Let (X,ω) be a symplectic manifold that represents the "extended" phase space of a system. Suppose that G is a Lie group which acts symplectically on (X,ω) and that $J: X \to \mathscr{g}^*$ is a momentum mapping for this action, where $\mathscr{g}$ is the Lie algebra of G. We interpret G as a "symmetry" or "gauge" group; J is the corresponding conserved quantity. A *constrained classical system with symmetry* is given by (X,ω,G,J) along with a fixed choice of $\mu \in \mathscr{g}^*$. The constraints are then $J = \mu$ and $J^{-1}(\mu) \subset X$ is the *constraint set*.

One may reduce the number of degrees of freedom of a constrained system by factoring out the symmetries of the constraint set. Subject to certain technical assumptions, Marsden and Weinstein[4] showed that the resulting orbit space $\overline{X}_\mu$ is a quotient manifold of $J^{-1}(\mu)$ and inherits a symplectic structure $\overline{\omega}_\mu$ from that on X. The symplectic manifold $(\overline{X}_\mu,\overline{\omega}_\mu)$ is the *reduced phase space* of invariant states of the system.

There are thus two symplectic manifolds associated to each constrained system: the extended and reduced phase spaces (X,ω) and $(\overline{X}_\mu,\overline{\omega}_\mu)$, respectively. *Classically*, there is no formal distinction between working on (X,ω) while carrying along the constraints versus solving the constraints, reducing the system and working on $(\overline{X}_\mu,\overline{\omega}_\mu)$. But these two approaches are not necessarily equivalent on the *quantum* level. This was recently emphasized by Ashtekar and Horowitz,[5] who showed that these two classical formalisms may engender real and significant physical differences in the quantum behavior of the system.

Recall that quantization associates to a phase space (X,ω) a Hilbert space $\mathscr{H}$ of quantum states and to some class of smooth functions f on X quantum operators $\mathscr{Q}f$ on $\mathscr{H}$. For a constrained classical system one may, as indicated above, quantize either the extended or the reduced phase space. The purpose of this paper is to determine under what conditions and in what sense these two quantizations will be equivalent.

We first consider the extended phase space quantization following Dirac.[6] The essential idea is that as the constraints have not been eliminated classically, they must be enforced quantum mechanically. This is possible if quantization provides a representation of $\mathscr{g}$ on $\mathscr{H}$. Since the constraints are given classically by $J = \mu$, it follows that the physically admissible quantum states are those which belong to the subspace $\mathscr{H}_\mu$ of $\mathscr{H}$ defined by

$$\mathscr{H}_\mu = \{\Psi \in \mathscr{H} \mid \mathscr{Q}J[\Psi] = \mu\Psi\}.$$

The situation is somewhat simpler for the reduced phase space $(\overline{X}_\mu,\overline{\omega}_\mu)$ as the constraints have already been solved and the symmetries divided out. There are no restrictions to be imposed on the quantum system and so, by construction, the associated Hilbert space $\overline{\mathscr{H}}_\mu$ consists of all the physically admissible states of the system.

These two quantizations each yield spaces of "physically admissible quantum states" which in general will not coincide. We may thus phase our question as follows: *When will $\mathscr{H}_\mu$ and $\overline{\mathscr{H}}_\mu$ be unitarily isomorphic*? There are three sets of obstructions to the existence of such an isomorphism, involving (i) the naturality of the extended phase space quantization, (ii) the compatibility of the extended and reduced phase space quantizations, and (iii) the unitary relatedness of the Hilbert space structures on $\mathscr{H}_\mu$ and $\overline{\mathscr{H}}_\mu$.

The first impediment is whether in fact the quantization of the extended phase space gives rise to a representation of the Lie algebra $\mathscr{g}$ of G on $\mathscr{H}$. A necessary condition is that $J^{-1}(\mu)$ be a coisotropic submanifold of (X,ω). This ensures the internal consistency of the quantization and effectively restricts the allowable values of $\mu \in \mathscr{g}^*$.

The next difficulty is to properly correlate the quantizations of the extended and reduced phase spaces. This can be accomplished by requiring that the auxiliary structures on (X,ω) necessary for quantization be G-invariant—provided this is possible—for they will then project to compatible quantization structures on $(\overline{X}_\mu,\overline{\omega}_\mu)$.

Physically, the above obstructions take the form of "quantization conditions" and/or "superselection rules" and place restrictions on the topology of G as well as the choice of quantization structures. Once they have been overcome, one obtains "smooth" quantizations of (X,ω) and $(\overline{X}_\mu,\overline{\omega}_\mu)$, i.e., linear spaces of C^∞ wave functions $\mathscr{H}$ and $\overline{\mathscr{H}}_\mu$, respectively. The final obstructions appear when one introduces the quantum inner products on these spaces. It may happen that $\overline{\mathscr{H}}_\mu$ does not inherit an inner product from $\mathscr{H}$ and, when it does, it must be checked that an equivalence of the underlying smooth quantizations extends to a unitary isomorphism of the corresponding Hilbert spaces.

Substantial progress towards answering this question has already been made by Gotay and Sniatycki,[7] Guillemin and Sternberg,[8] Puta,[9] Sniatycki,[10] Vaisman,[11] and Woodhouse.[12] Because of the intricacy of the problem and the vagaries of the quantization process, however, it is difficult to obtain results in a completely general setting that provide explicit information about concrete systems.

To rectify this, we concentrate in this paper on one specific class of constrained systems—those whose phase spaces are cotangent bundles and whose groups act by point transformations. There are numerous reasons for considering such systems.

(1) They are the most common and hence the most important physically. Indeed, all of the examples cited earlier—with the exception of the mass constraint in the Kaluza–Klein theory—fall into this class.

(2) Cotangent bundles, along with Kähler manifolds, are exceptional examples of symplectic manifolds as they have naturally defined polarizations (the vertical and antiholomorphic ones, respectively); this is a crucial advantage insofar as quantization is concerned. Guillemin and Sternberg[8] have studied the Kähler case and so the results we present here are, to some extent, complementary to theirs.

(3) Reduction keeps us within the cotangent bundle category: subject to certain assumptions (which are in any case necessary for quantization), the reduced phase space will also be a cotangent bundle. We may therefore quantize both the extended and reduced phase spaces using the corresponding vertical polarizations. This means, in physicists' terminology, that we always quantize in the "Schrödinger representation."

(4) We are able to obtain relatively "hard" results. Namely, we can explicitly identify and construct the momentum mapping, the reduced phase space, and all of the required quantization structures. The formalism we develop will also enable us to detail precisely the various obstructions discussed earlier as well as verify directly whether the assumptions we impose are satisfied in specific cases. Thus our general problem is reduced to a conceptually and computationally much simpler one.

The plan of attack is as follows. We consider systems of the form (T^*Q,ω,G,J,μ), where G acts on T^*Q by pullback and μ is Ad*-invariant. Applying the reduction technique of Kummer,[13] Satzer,[14] and Abraham and Marsden,[15] we show that the reduced phase space is symplectomorphic to $T^*(Q/G)$ (with a possibly noncanonical symplectic structure). These results are summarized in Sec. II.

Sections III and IV form the heart of the paper. After discussing some generalities on the quantization of constrained systems we quantize both the extended and reduced phase spaces. In particular, we show that quantization does indeed yield a representation of the symmetry algebra $\mathscr{g}$.

In the next section we construct a canonical unitary isomorphism between the two quantizations obtained in Sec. III. We also prove that it is possible to quantize invariant polarization-preserving functions in either formalism with equivalent results.

The following section presents several examples and we conclude with a discussion of possible generalizations of our results.

II. CONSTRAINED CLASSICAL SYSTEMS

We begin by reviewing some basic facts about group actions, momentum mappings, and reduction. The main references for what follows are Refs. 4, 15, and 16.

A. Hamiltonian *G*-spaces

Let G be a connected Lie group with Lie algebra $\mathscr{g}$ and let $\Phi: G\times Q\rightarrow Q$ be a smooth action of G on a manifold Q. For each $\zeta\in\mathscr{g}$, we denote by ζ_Q the corresponding infinitesimal generator on Q. The orbit of a point $q\in Q$ is written $G\cdot q$. Recall that when Φ is free and proper the orbit space $\overline{Q} = Q/G$ is a Hausdorff quotient manifold of Q and, furthermore, $\pi_Q: Q\rightarrow\overline{Q}$ is a left principal G-bundle.

Now suppose (X,ω) is a symplectic manifold on which G acts symplectically. A *momentum mapping* for this action is a map $J: X\rightarrow\mathscr{g}^*$ such that, for each $\zeta\in\mathscr{g}$, the associated function $J_\zeta(x) = \langle J(x),\zeta\rangle$ satisfies

$$\zeta_X \lrcorner\, \omega = -dJ_\zeta\ . \tag{2.1}$$

Then J is *Ad*-equivariant* provided

$$J(\Phi_g(x)) = \mathrm{Ad}^*_{g^{-1}}J(x)\ , \tag{2.2}$$

for all $g\in G$, where Ad* is the coadjoint action of G on $\mathscr{g}^*$. If an Ad*-equivariant momentum map J exists for the action Φ, we call (X,ω,G,J) a *Hamiltonian G-space*.

Let $\mu\in\mathscr{g}^*$ be a weakly regular value of J, so that the level set $J^{-1}(\mu)$ is a manifold with $TJ^{-1}(\mu) = \ker TJ$. The following result relates the geometry of $J^{-1}(\mu)$ with that of the orbits of G and G_μ, where G_μ is the isotropy group of μ under the coadjoint action.

Proposition (2.1): For $x\in J^{-1}(\mu)$,

(i) $T_x(G_\mu\cdot x) = T_x(G\cdot x)\cap T_xJ^{-1}(\mu)$,

and

(ii) $T_x(J^{-1}(\mu))^\perp = T_x(G\cdot x)$.

Here "⊥" denotes the ω-orthogonal complement.

By equivariance, $J^{-1}(\mu)$ is stable under the action of G_μ so that the orbit space $\overline{X}_\mu = J^{-1}(\mu)/G_\mu$ is well defined. Let $j_\mu: J^{-1}(\mu)\rightarrow X$ be the inclusion and $\pi_\mu: J^{-1}(\mu)\rightarrow\overline{X}_\mu$ the projection. The next result, due to Marsden and Weinstein,[4] is central to the theory.

Theorem (Marsden–Weinstein reduction): Let (X,ω,G,J) be a Hamiltonian G-space. If $\mu\in\mathscr{g}^*$ is a weakly regular value of J and the action of G_μ on $J^{-1}(\mu)$ is free and proper, then there exists a unique symplectic structure $\overline{\omega}_\mu$ on the manifold $\overline{X}_\mu$ such that $\pi_\mu^* \overline{\omega}_\mu = j_\mu^* \omega$.

Remark: If the Marsden–Weinstein reduction procedure fails [e.g., μ is not weakly regular or $(\overline{X}_\mu, \overline{\omega}_\mu)$ does not exist as a smooth symplectic manifold, as is the case in a number of important examples], one can still reduce $J^{-1}(\mu)$ on the level of Poisson algebras.[17]

B. The cotangent category

For the reasons cited in the Introduction, we restrict attention to constrained classical systems that belong to the "cotangent bundle category." Reduction in this category was first carried out by Satzer[14] for $\mu = 0$ and then extended to $\mu \neq 0$ by Abraham and Marsden[15] and Marsden.[16] Subsequently, Kummer[13] improved upon these results and put the theory into its present form. Our presentation is drawn from both Refs. 13 and 15.

Suppose that the system has configuration space Q and symmetry group G. We assume that G carries a bi-invariant metric and that Φ is a free and proper left action of G on Q. Let the extended phase space be $X = T^*Q$, where τ_Q is the cotangent bundle projection and ω and Θ denote the canonical two- and one-forms on T^*Q, respectively, with $\omega = d\Theta$. The induced symplectic action $T^*\Phi: G \times T^*Q \to T^*Q$, given by

$$T^*\Phi(g,\beta) = \Phi_{g^{-1}}{}^*\beta ,$$

is also free and proper. There is a natural Ad*-equivariant momentum map for $T^*\Phi$ defined by

$$\langle J(\beta),\xi \rangle = \Theta(\xi_{T^*Q}(\beta)) = \beta(\xi_Q) . \tag{2.3}$$

We refer to the Hamiltonian G-space (T^*Q,ω,G,J) so defined, along with a fixed choice of $\mu \in \mathscr{g}^*$, as a *constrained cotangent system*.

Regarding reduction, one of the main advantages of our formalism is the following proposition.

Proposition (2.2): Every $\mu \in \mathscr{g}^*$ is a regular value of J.

Proof: Suppose that $T_\beta J$ was not surjective for some $\beta \in T^*Q$, in which case there exists $\xi \in \mathscr{g}$ such that $\langle T_\beta J(v),\xi \rangle = 0$ for all $v \in T_\beta(T^*Q)$. Then (2.1) yields $\omega(\xi_{T^*Q}(\beta),v) = 0$ for all v and nondegeneracy implies that $\xi_{T^*Q}(\beta) = 0$, contradicting the fact that $T^*\Phi$ is free. ■

Each level set $J^{-1}(\mu)$ is therefore an imbedded submanifold of T^*Q. Furthermore, since $T^*\Phi$ is free and proper and G_μ is closed in G, the action of G_μ on $J^{-1}(\mu)$ is also free and proper. This observation, combined with Proposition (2.2) and the Marsden–Weinstein Theorem, give the following proposition.

Proposition (2.3): $J^{-1}(\mu)$ is reducible for every $\mu \in \mathscr{g}^*$.

Insofar as the quantization of these systems is concerned, however, it is not necessary to consider general $\mu \in \mathscr{g}^*$. We will see in Sec. III B that only those μ which are "invariant" are relevant.

Definition: $\mu \in \mathscr{g}^*$ is *invariant* if $\mathrm{Ad}_g^*(\mu) = \mu$ for all $g \in G$.

Equivalently, μ is invariant iff $G_\mu = G$. For such μ the reduction of $J^{-1}(\mu)$ is particularly simple and elegant. We first reduce $J^{-1}(0)$ and then transform the case $\mu \neq 0$ to this; note that $\mu = 0$ is always invariant.

Let $\overline{\omega} = d\overline{\Theta}$ be the canonical symplectic structure on $T^*\overline{Q}$, where $\overline{Q} = Q/G$.

Proposition (2.4): The reduced phase space $(\overline{T^*Q}_0, \overline{\omega}_0)$ is symplectomorphic to $(T^*\overline{Q},\overline{\omega})$.

Proof: First note that the pullback bundle

$$\pi_Q^*(T^*\overline{Q}) = J^{-1}(0) , \tag{2.4}$$

where $\pi_Q: Q \to \overline{Q}$ is the canonical submersion. Indeed, since $T\pi_Q(\xi_Q) = 0$, a 1-form β on Q belongs to $\pi_Q^*(T^*\overline{Q})$ iff $\beta(\xi_Q) = 0$ iff $\beta \in J^{-1}(0)$ by (2.3). Quotienting by G in (2.4) then gives $T^*\overline{Q} \approx \overline{T^*Q}_0$.

It remains to show that the reduced symplectic form $\overline{\omega}_0$ on $\overline{T^*Q}_0$ can be identified with $\overline{\omega}$ on $T^*\overline{Q}$. Using (2.4) and the induced commutative diagram

$$\begin{array}{ccc} J^{-1}(0) & \xrightarrow{\;j_0\;} & T^*Q \\ {\scriptstyle \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Q \circ \tau_Q} \\ T^*\overline{Q} & \xrightarrow{\;\tau_{\overline{Q}}\;} & \overline{Q} \end{array} ,$$

a straightforward computation establishes $\pi_0^*\overline{\Theta} = j_0^*\Theta$ and consequently $\pi_0^*\overline{\omega} = j_0^*\omega$. The result now follows from the uniqueness of the reduced symplectic structure in the Marsden–Weinstein Theorem. ■

When μ is nonzero but invariant, we first choose a left connection α on the left principal G-bundle $\pi_Q: Q \to \overline{Q}$. Set $\alpha_\mu = \mu \circ \alpha$. Then α_μ is G-invariant and, viewed as a one-form on Q, takes values in $J^{-1}(\mu)$. Construct the invariant symplectic form $\Omega_\mu = \omega + \tau_Q^* \, d\alpha_\mu$ on T^*Q. The key step in the transition from $\mu = 0$ to $\mu \neq 0$ is the following proposition.

Proposition (2.5): There exists a G-equivariant presymplectomorphism of $(J^{-1}(0), j_0^*\Omega_\mu)$ with $(J^{-1}(\mu), j_\mu^*\omega)$.

Proof: Define a diffeomorphism δ_μ of T^*Q by

$$\delta_\mu(\beta) = \beta + \alpha_\mu(\tau_Q(\beta)) . \tag{2.5}$$

Since α_μ is invariant δ_μ is equivariant and, as $J(\alpha_\mu) = \mu$, δ_μ induces a diffeomorphism $J^{-1}(0) \to J^{-1}(\mu)$, which we also denote by δ_μ.

Now δ_μ is just translation along the fibers, so

$$\delta_\mu^*\Theta = \Theta + \tau_Q^*\alpha_\mu$$

and hence $\delta_\mu^*\omega = \Omega_\mu$. But this and the relation $j_\mu \circ \delta_\mu = \delta_\mu \circ j_0$ imply that δ_μ is a presymplectomorphism. ■

Propositions (2.4) and (2.5) enable us to identify the reduced manifolds $\overline{T^*Q}_\mu$ for μ invariant with $T^*\overline{Q}$. To complete the reduction we have only to compute the reduced symplectic forms $\overline{\omega}_\mu$.

Lemma (2.6): There exists a closed two-form F_μ on $\overline{Q}$ such that $\pi_Q^* F_\mu = d\alpha_\mu$.

Proof: We first claim that $d\alpha_\mu = \mu \circ D\alpha$, where $D\alpha$ is the curvature of the connection α. To prove this, take the μ component of the Cartan structure equation

$$d\alpha(u,v) = [\alpha(u),\alpha(v)] + D\alpha(u,v)$$

and observe that

$$\mu \circ [\alpha(u),\alpha(v)] = \mu(\mathrm{ad}_{\alpha(u)}\alpha(v))$$

$$= (\mathrm{ad}_{\alpha(u)}^*\mu)\,(\alpha(v))$$

vanishes as μ is invariant. Thus $d\alpha_\mu$ is horizontal. Since in addition $d\alpha_\mu$ is invariant and π_Q is a submersion, $d\alpha_\mu$ projects to a two-form F_μ on $\overline{Q}$ with the required properties. ■

The reduction of $(J^{-1}(0), j_0^*\Omega_\mu)$ is clearly $(T^*\overline{Q}, \overline{\Omega}_\mu)$, where

$$\overline{\Omega}_\mu = \overline{\omega} + \tau_Q^* F_\mu \, . \tag{2.6}$$

Taking Proposition (2.5) into account and noting that, by equivariance, δ_μ passes to the quotient, we have proven the following theorem.

Theorem (Kummer–Marsden–Satzer reduction): Consider the constrained cotangent system (T^*Q,ω,G,J,μ). If $\mu\in\mathscr{g}^*$ is invariant then each choice of connection on Q defines a symplectomorphism between the reduced phase space $(\overline{T^*Q}_\mu,\overline{\omega}_\mu)$ and $(T^*\overline{Q},\overline{\Omega}_\mu)$, where $\overline{Q}=Q/G$ and $\overline{\Omega}_\mu$ is defined by (2.6).

In essence, the reduction of a cotangent bundle is again a cotangent bundle. This fact will be of paramount importance in the sequel.

Remarks: (1) It is crucial here that μ be invariant. When $G_\mu \subset G$, $\overline{T^*Q}_\mu$ can only be identified with a symplectic *subbundle* of $T^*(Q/G_\mu)$ (cf. Refs. 13 and 15). In other words, the invariance of μ is *necessary as well as sufficient* for the reduced manifold to be a cotangent bundle.

(2) In general the symplectic structure $\overline{\Omega}_\mu$ on $T^*\overline{Q}$ will *not* be canonical due to the presence of the curvature term F_μ. This "extra" term has been used as a means of introducing Yang–Mills-type interactions (see, e.g., Refs. 18 and 19 and the references contained therein for the details of this; we shall encounter an instance of this phenomenon in our study of the Kaluza–Klein theory in Sec. V C). However, when Q carries a flat connection we may take $\overline{\Omega}_\mu$ to be exact.

(3) Although the proof of the Kummer-Marsden–Satzer Theorem required choosing a connection, this choice is irrelevant. Other such choices simply lead to different, but nonetheless symplectomorphic, realizations of $(\overline{T^*Q}_\mu, \overline{\omega}_\mu)$. It is also possible to derive this theorem using a Riemannian metric to obtain the required invariant $J^{-1}(\mu)$-valued one-form α_μ (cf. Refs. 14 and 15). This approach seems more cumbersome and less "physical" than the one employed here, which is due to Kummer.[13]

(4) Montgomery[20] has recently shown, subject to certain additional assumptions, that the Kummer–Marsden–Satzer reduction procedure may be extended to the case when Φ is not free.

We close this section by noting that we may also reduce observables: if $f\in C^\infty(T^*Q)$ is invariant, then it projects to $\overline{T^*Q}_\mu$. To describe this function on $T^*\overline{Q}$, set $f_\mu = f\circ\delta_\mu$ and define $\overline{f}_\mu\in C^\infty(T^*\overline{Q})$ by $\overline{f}_\mu\circ\pi_0 = f_\mu$. Since $f_\mu\circ j_0 = f\circ j_\mu$, it follows that $\overline{f}_\mu$ represents the reduced observable. In particular, if h is a Hamiltonian on (T^*Q,ω) then $\overline{h}_\mu$ is the "amended" Hamiltonian on $(T^*\overline{Q},\overline{\Omega}_\mu)$ (cf. Refs. 14–16).

III. QUANTIZATION

To properly address the subtleties and complexities of the transition from the classical to the quantal domain it is essential to use a well-defined quantization technique. We choose the geometric quantization framework of Kostant and Souriau because it is formulated in terms of symplectic geometry. In Sec. III A we briefly outline those elements of this theory that are needed here, referring the reader to Sniatycki[21] and Woodhouse[12] for comprehensive expositions.

A. Quantization structures

Let (X,ω) be a $2n$-dimensional symplectic manifold. The supplementary structures needed for the geometric quantization of (X,ω) are a polarization, a prequantization line bundle, and a metalinear frame bundle.

A (real) *polarization* of (X,ω) is an involutive n-dimensional distribution P on X such that $P^\perp = P$.

A *prequantization* of (X,ω) consists of a complex line bundle $l: L\to X$ with a connection ∇ such that

$$\text{curvature } \nabla = -(1/h)\, l^*\omega \, , \tag{3.1}$$

where h is Planck's constant. A prequantization of (X,ω) exists iff the de Rham class of ω/h in $H^2(X,\mathbb{R})$ is integral and, if nonempty, the set of all prequantizations is parametrized up to equivalence by a principal homogeneous space for the character group of $\pi_1(X)$.

Remark: We make no distinction between L and its associated principal $\mathbb{C}^*$-bundle.

Fix a polarization P of (X,ω) and let FP be the linear frame bundle of P. It is a right principal $GL(n,\mathbb{R})$-bundle over X. Let $ML(n,\mathbb{R})$ be the $n\times n$ metalinear group, that is, the set of all matrices of the form

$$\widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & z \end{pmatrix},$$

where $M\in GL(n,\mathbb{R})$ and $z^2 = \det M$. A *metalinear frame bundle* for P is a right principal $ML(n,\mathbb{R})$-bundle $\widetilde{FP}$ over X along with a 2:1 projection $\rho: \widetilde{FP}\to FP$ such that the diagram

$$\begin{array}{ccc} \widetilde{FP}\times ML(n,\mathbb{R}) & \longrightarrow & \widetilde{FP} \\ \Big\downarrow \rho\times\sigma & & \Big\downarrow \rho \\ FP\times GL(n,\mathbb{R}) & \longrightarrow & FP \end{array}$$

commutes, where the horizontal arrows are the group actions and $\sigma: ML(n,\mathbb{R})\to GL(n,\mathbb{R})$ is the twofold projection $\widetilde{M}\mapsto M$.

The existence of a metalinear frame bundle is equivalent to the vanishing of a class in $H^2(X,\mathbb{Z}_2)$ characteristic of FP and, if nonempty, the set of all such is parametrized up to equivalence by $H^1(X,\mathbb{Z}_2)$.

Remark: We need not consider the more general metaplectic structures here since we will not be moving polarizations.

Let $\Delta: ML(n,\mathbb{R})\to\mathbb{C}$ be the unique holomorphic square root of the determinant function on $GL(n,\mathbb{R})$ such that $\Delta(\widetilde{I}) = 1$, where $\widetilde{I}$ is the identity. The bundle $\sqrt{\wedge^n P}$ of *half-forms* relative to P is the bundle associated to $\widetilde{FP}$ with typical fiber $\mathbb{C}$ on which $ML(n,\mathbb{R})$ acts by multiplication by Δ. This bundle has a canonically defined partial flat connection covering P. Denote by $\Gamma(\sqrt{\wedge^n P})$ the space of all smooth sections of $\sqrt{\wedge^n P}$. Each $\nu\in\Gamma(\sqrt{\wedge^n P})$ can be identified with a function $\nu^\#: \widetilde{FP}\to\mathbb{C}$ satisfying

$$\nu^\#(\widetilde{f}\widetilde{M}) = \Delta(\widetilde{M})^{-1}\cdot\nu^\#(\widetilde{f}) \tag{3.2}$$

for all metaframes $\widetilde{f}\in\widetilde{FP}$ and $\widetilde{M}\in ML(n,\mathbb{R})$.

Consider the bundle $L\otimes\sqrt{\wedge^n P}$. It carries a partial flat connection covering P induced from those on L and $\sqrt{\wedge^n P}$. A section $\Psi\in\Gamma(L\otimes\sqrt{\wedge^n P})$ is said to be *polarized* if it is covariantly constant along P. Let $\mathscr{H}$ be the subspace of $\Gamma(L\otimes\sqrt{\wedge^n P})$ consisting of polarized sections. Elements of

$\mathscr{H}$ are interpreted as smooth quantum wave functions, i.e., $\mathscr{H}$ is the *smooth* quantum state space associated to (X,ω) by the geometric quantization procedure in the representation defined by the polarization P.

We now turn to the quantization of classical observables $f \in C^\infty(X)$. Suppose f preserves P in the sense that $T\phi^t(P) = P$ for all $t \in \mathbf{R}$, where ϕ^t is the flow of (the Hamiltonian vector field of) f, which we assume is complete. Then f is quantizable as a first-order linear differential operator $\mathscr{Q}f$ on $\mathscr{H}$. The mechanics of this are as follows. The flow ϕ^t has a natural lift to L consisting of connection-preserving automorphisms. On the other hand, ϕ^t operates on FP by push forward of frames—this is well defined since f is polarization preserving—and this flow automatically lifts to $\widetilde{F}P$ because ρ is a 2:1 submersion. Assembling these, we obtain a one-parameter group of automorphisms of $L \otimes \sqrt{\wedge^n P}$ that in turn induces a one-parameter group of linear isomorphisms of $\mathscr{H}$, which we also denote by ϕ^t. Setting $\hbar = h/2\pi$, the quantum observable $\mathscr{Q}f$ is then defined by

$$\mathscr{Q}f[\Psi] = i\hbar \frac{d}{dt}(\phi^t\Psi)|_{t=0}, \qquad (3.3)$$

for all $\Psi \in \mathscr{H}$.

Remark: This technique is not applicable if the observables to be quantized do not preserve the polarization. One must use the Blattner–Kostant–Sternberg kernels to quantize such functions and the corresponding quantum operators—if they exist—will generally be more complicated.

Of principal interest is when X is a cotangent bundle T^*Q with the canonical symplectic structure $\omega = d\Theta$. We study this case in detail and present several formulas which will be useful later.

Let (q^i,p_i), $i = 1,\dots,n$, be a canonical bundle chart on $U \subset T^*Q$. Then

$$\Theta|U = \sum_{i=1}^{n} p_i\, dq^i \qquad (3.4)$$

and

$$\omega|U = \sum_{i=1}^{n} dp_i \wedge dq^i . \qquad (3.5)$$

A cotangent bundle carries a naturally defined polarization: the *vertical polarization* $V = \ker T\tau_Q$. Locally,

$$V = \text{span}\left\{\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_n}\right\}.$$

Since ω is exact, T^*Q is always prequantizable. Relative to a locally trivializing section λ: $U \to L$, the covariant differential is given by

$$\nabla\lambda = (1/i\hbar)\,\Theta \otimes \lambda . \qquad (3.6)$$

A *metalinear structure* on a cotangent bundle will always mean a metalinear frame bundle for the vertical polarization. In this case the existence criterion is quite simple: a metalinear structure exists on T^*Q iff $w_1(Q)^2 = 0$, where $w_1(Q)$ is the first Stiefel–Whitney class of Q (see Ref. 22).

Let $\underline{f} = (\partial/\partial p_1,\dots,\partial/\partial p_n)$ be a local frame field for V and $\tilde{\underline{f}}$ a lift of $\underline{f}$ to $\widetilde{F}V$. Define $\nu_{\tilde{\underline{f}}} \in \Gamma(\sqrt{\wedge^n V})$ according to

$$\nu_{\tilde{\underline{f}}}^{\#} \circ \tilde{\underline{f}} = 1 . \qquad (3.7)$$

Both λ and $\nu_{\tilde{\underline{f}}}$ are covariantly constant along V and every section Ψ of $L \otimes \sqrt{\wedge^n V}$ may be written

$$\Psi|U = \psi(q,p)\lambda \otimes \nu_{\tilde{\underline{f}}}, \qquad (3.8)$$

for some smooth function ψ on U. Such a Ψ is polarized iff $\psi = \psi(q)$ only. In particular, when L and $\sqrt{\wedge^n V}$ are trivial with global sections λ and ν, respectively, the association $\psi(q,p)\lambda \otimes \nu \mapsto \psi(q,p)$ defines an isomorphism $\Gamma(L \otimes \sqrt{\wedge^n V}) \approx C^\infty(T^*Q,\mathbf{C})$. The space $\mathscr{H}$ of polarized sections may similarly be identified with $C^\infty(Q,\mathbf{C})$.

Now suppose g is an observable which preserves V. Then for Ψ given locally by (3.8), (3.3) reduces to

$$\mathscr{Q}g[\Psi]|U = [\{-i\hbar\nabla_{X_g} + g - \tfrac{1}{2}i\hbar\,\text{tr}\,A_{\underline{f}}(X_g)\}\psi\lambda] \otimes \nu_{\tilde{\underline{f}}}, \qquad (3.9)$$

where X_g is the Hamiltonian vector field of g and the components a^i_j of the matrix $A_{\underline{f}}(X_g)$ are found from

$$\left[X_g, \frac{\partial}{\partial p_i}\right] = \sum_{j=1}^{n} a^i_j \frac{\partial}{\partial p_j}.$$

B. The quantization of constrained systems

We wish to study the equivalence of the geometric quantizations of the extended and reduced phase spaces of a constrained classical system with symmetry (X,ω,G,J,μ). In this section we outline our strategy and delineate general criteria which must be met before we can proceed with the more technical aspects of the theory (which occupy the remainder of Secs. III and IV).

Our main concerns here are obtaining a natural quantization of the extended phase space and constructing a compatible quantization of the reduced phase space.

First of all, the extended phase space quantization must be "natural" in the sense that the classical symmetry algebra $\mathscr{g}$ also appears as a symmetry algebra on the quantum level. Hence the functions J_ζ must all be quantizable and the association $J_\zeta \mapsto \mathscr{Q}J_\zeta$ must be a Lie algebra homomorphism:

$$[\mathscr{Q}J_\zeta, \mathscr{Q}J_\eta] = i\hbar\mathscr{Q}J_{[\zeta,\eta]}, \qquad (3.10)$$

for all $\zeta,\eta \in \mathscr{g}$. This enables us to express the constraints $J = \mu$ as conditions

$$\mathscr{Q}J_\zeta[\Psi] = \langle\mu,\zeta\rangle\Psi \qquad (3.11)$$

on the quantum wave functions $\Psi \in \mathscr{H}$.

Unfortunately, such a quantization will usually be inconsistent: the constraint operators $\mathscr{Q}J_\zeta$ will have no nonzero eigenstates corresponding to the eigenvalues $\langle\mu,\zeta\rangle$. For suppose Ψ satisfied (3.11) so that

$$[\mathscr{Q}J_\zeta, \mathscr{Q}J_\eta]\Psi = (\langle\mu,\eta\rangle\langle\mu,\zeta\rangle - \langle\mu,\zeta\rangle\langle\mu,\eta\rangle)\Psi$$

vanishes. But then (3.10) yields

$$\mathscr{Q}J_{[\zeta,\eta]}[\Psi] = \langle\mu,[\zeta,\eta]\rangle\Psi \not\equiv 0,$$

which forces $\Psi = 0$. Thus the space $\mathscr{H}_\mu$ of physically admissible wave functions will be trivial.

To obtain meaningful results the offending term

$$\langle\mu,[\zeta,\eta]\rangle = \langle \text{ad}^*_\zeta\mu,\eta\rangle$$

in the last equation must vanish for all ζ and η, and this happens iff μ is invariant. *One can therefore consistently quantize (X,ω,G,J,μ) only if μ is invariant.*

This invariance condition can be expressed geometrically. From Proposition (2.1) we have that μ is invariant iff

$G_\mu = G$ iff $J^{-1}(\mu)$ is a coisotropic submanifold of (X,ω), i.e.,

$$TJ^{-1}(\mu)^\perp \subseteq TJ^{-1}(\mu) .$$

The invariance of μ thus plays two key roles in our formalism: it is the primary obstruction to obtaining a consistent natural quantization of the system, and it guarantees that the reduction of a cotangent bundle is again a cotangent bundle. *Henceforth we assume that μ is invariant.*

We now return to the naturality question, viz., under what conditions will quantization produce a representation of $\mathscr{g}$ on $\mathscr{H}$? Once the invariance of μ ensures that no outright inconsistencies will occur, this reduces to a problem of making suitable choices of the geometric quantization structures discussed in the previous section. We must choose these so that the J_ζ are all quantizable and moreover that the quantum operators $\mathscr{Q}J_\zeta$ thus obtained satisfy (3.10). In general, this will be possible iff the polarization P is G-invariant for then the J_ζ are all polarization-preserving functions (see Ref. 21, §6.2). However, if P is not invariant the $\mathscr{Q}J_\zeta$ need not exist and, even if they are defined, (3.10) will not necessarily follow.

Having obtained a natural quantization of the extended phase space we now turn to the quantization of the reduced phase space. Our task is to correlate these two quantizations. We first observe that, by construction, the quantization of $(\overline{X}_\mu,\overline{\omega}_\mu)$ is completely determined by the structure of the constraint set. Consequently, if there is to be any hope for an equivalence of the extended and reduced phase space quantizations, we must ensure that the extended phase space quantization has this same property. This translates into the requirement that the extended wave functions be uniquely determined by their restrictions to $J^{-1}(\mu)$, and effectively places a further restriction on the choice of polarization.[7]

It remains only to construct quantization structures on $(\overline{X}_\mu,\overline{\omega}_\mu)$ that are compatible with those we already have on (X,ω). The basic idea is to project the quantization structures on the latter down to the former. An invariant polarization P on (X,ω) will project to a polarization on $(\overline{X}_\mu,\overline{\omega}_\mu)$ if, for example, P is transverse to $TJ^{-1}(\mu)^\perp$. For the prequantization and metalinear structures, we accomplish this by first lifting the action of G on X to $L\,|J^{-1}(\mu)$ and $\widetilde{F}P\,|J^{-1}(\mu)$ in a suitable manner. This is always possible infinitesimally, and the obstruction to extending from $\mathscr{g}$ to G is purely topological. In particular, there is no problem if G is simply connected; otherwise, one must choose L and $\widetilde{F}P$ appropriately—provided, of course, such "invariant" structures exist. We then quotient by these G-actions, producing bundles which are the required quantization structures on $(\overline{X}_\mu,\overline{\omega}_\mu)$.

We have now laid the foundation for comparable quantizations of the extended and reduced phase spaces. The next step is to check the above criteria and to explicitly construct the appropriate quantization structures for constrained cotangent systems (T^*Q,ω,G,J,μ), where μ is invariant, $\dim Q = n$, $\dim G = r$, and $\dim \overline{Q} = \bar{n} = n - r$. In the remainder of this section we work out the details for the three geometric quantization structures. Then, in Sec. IV, we show that the conditions we have set forth here are sufficient to guarantee the equivalence of the extended and reduced phase space quantizations.

C. Polarization

Let $V = \ker T\tau_Q$ and $\overline{V} = \ker T\tau_{\overline{Q}}$ be the vertical polarizations on T^*Q and $T^*\overline{Q}$, respectively.

Lemma (3.1): $V \cap TJ^{-1}(\mu)^\perp = \{0\}$.

Proof: Let $v \in V_\beta \cap T_\beta J^{-1}(\mu)^\perp$. Since $G_\mu = G$, it follows from Proposition (2.1) that $v \in T_\beta(G\cdot\beta)$, i.e., $v = \zeta_{T^*Q}(\beta)$ for some $\zeta \in \mathscr{g}$. But then $T\tau_Q(v) = \zeta_Q(\tau_Q(\beta)) = 0$, which implies that $v = 0$ because Φ is free. ■

Taking the symplectic orthogonal complement of Lemma (3.1) gives

$$V + TJ^{-1}(\mu) = TT^*Q$$

over $J^{-1}(\mu)$. Counting fiber dimensions, we have

$$2n = \dim(V + TJ^{-1}(\mu))$$
$$= \dim V + \dim TJ^{-1}(\mu) - \dim(V \cap TJ^{-1}(\mu)) .$$

By Proposition (2.2) μ is a regular value of J, so $\dim J^{-1}(\mu) = 2n - r$. It follows that $V \cap TJ^{-1}(\mu)$ is an involutive $\bar{n}$-dimensional distribution on $J^{-1}(\mu)$. Furthermore, since $\tau_{\overline{Q}} \circ \pi_\mu = \pi_Q \circ \tau_Q$,

$$T\tau_{\overline{Q}}(T\pi_\mu(V \cap TJ^{-1}(\mu)))$$
$$= T\pi_Q(T\tau_Q(V \cap TJ^{-1}(\mu))) = \{0\} ,$$

so that $T\pi_\mu(V \cap TJ^{-1}(\mu))$ is vertical on $T^*\overline{Q}$. We have therefore proven the following proposition.

Proposition (3.2): $\overline{V} = T\pi_\mu(V \cap TJ^{-1}(\mu))$.

Clearly the action $T^*\Phi$ leaves V invariant. It follows automatically—regardless of the choices of the prequantization and metalinear structures—that the quantization of (T^*Q,ω,G,J,μ) in the Schrödinger representation will be natural.

Also note that every leaf of V intersects $J^{-1}(\mu)$. Indeed, since $J^{-1}(0)$ contains the zero section of T^*Q this is certainly true for $J^{-1}(0)$. But $J^{-1}(\mu)$ is obtained from $J^{-1}(0)$ by translation along the leaves of V (cf. Sec. II B), so this holds for $J^{-1}(\mu)$ as well. As V-wave functions are covariantly constant along V, they will be uniquely determined by their restrictions to $J^{-1}(\mu)$.

These results, coupled with the fact that the vertical polarization on T^*Q projects to the vertical polarization on $T^*\overline{Q}$, imply that we may consistently and compatibly quantize both the extended and reduced phase spaces in the Schrödinger representation (which is the physicists' "canonical" quantization). Moreover, since geometric quantization is very sensitive to the choice of polarization, it is a definite advantage to have at our disposal concrete examples of polarizations that satisfy all the criteria of Sec. III B.

D. Prequantization

Let L be a prequantization line bundle for (T^*Q,ω) with connection form γ. We must construct a compatible prequantization line bundle for $(T^*\overline{Q},\overline{\Omega}_\mu)$. Preliminary results along these lines have been obtained by Puta.[9]

The first step is to lift the action of $\mathscr{g}$ on T^*Q to $L_\mu = L\,|J^{-1}(\mu)$. For each $\zeta \in \mathscr{g}$ let $\zeta_L = \zeta^*_{T^*Q}$ be the horizontal lift of ζ_{T^*Q} to L_μ.

Proposition (3.3): $\zeta \mapsto \zeta_L$ is a Lie algebra antihomomorphism.

Proof: We have to verify that $[\zeta,\eta]_L = -[\zeta_L,\eta_L]$ and for this it suffices to prove that $[\zeta_L,\eta_L]$ is horizontal. The prequantization condition (3.1) gives

$$\gamma([\zeta_L,\eta_L]) = -d\gamma(\zeta_L,\eta_L)$$
$$= (1/h)\, l^*[\omega(\zeta_{T^*Q},\eta_{T^*Q})] ,$$

which vanishes by virtue of Proposition (2.1) and the fact that $J^{-1}(\mu)$ is coisotropic, since both ζ_{T^*Q} and η_{T^*Q} belong to $TJ^{-1}(\mu)^\perp \subseteq TJ^{-1}(\mu)$. ■

Remarks: (1) The association $\zeta \mapsto \zeta_L$ is an *antihomomorphism* as the G-action is on the *left*.

(2) Proposition (3.3) will fail if μ is not invariant, so that we cannot lift the $\mathscr{g}$-action to L_μ for arbitrary μ. In particular, the action will generally not be defined on all of L (unless, e.g., G is Abelian), but, insofar as reduction is concerned, we need only obtain an action on L_μ.

To extend this $\mathscr{g}$-action to a G-action is more difficult. There are two possible obstructions: the incompleteness of some of the vector fields ζ_L and the nonsimple connectivity of G. The first of these presents no problem: since L is a *line* bundle and the ζ_{T^*Q} are complete the ζ_L will be also. When $\pi_1(G) \neq 0$, however, some L_μ may admit G-actions while others will not.

Fix an orbit $G\cdot\beta \subseteq J^{-1}(\mu)$. Since orbits are isotropic in T^*Q [cf. Proposition (2.1)], the prequantization condition implies that $L\,|\,(G\cdot\beta)$ is flat. As the ζ_L are horizontal the $\mathscr{g}$-action on $L\,|\,(G\cdot\beta)$ will integrate to a G-action iff the holonomy of $L\,|\,(G\cdot\beta)$ is trivial. The crucial observation is that the holonomy of $L\,|\,(G\cdot\beta)$ is the *same* for all orbits in a *given* level set $J^{-1}(\mu)$.

To show this let $c(t)$, $0 \leqslant t \leqslant T$, be a loop in G based at the identity and let $c_\beta(t) = T^*\Phi_{c(t)}(\beta)$ be the corresponding loop in $G\cdot\beta$ based at β. From Ref. 12, §5.5.2, we find that the element in the holonomy group of $l^{-1}(\beta)$ determined by c_β is

$$\exp\left(\frac{i}{\hbar}\int_{c_\beta}\Theta\right). \tag{3.12}$$

Let ζ_t be the curve in $\mathscr{g}$ defined by

$$\zeta_t = TL^{-1}_{c(t)}(c_*(t)) ,$$

where $L_{c(t)}$ is left translation by $c(t)$. Then a short calculation using (2.3) yields

$$\int_{c_\beta}\Theta = \int_0^T \langle J(c_\beta(t)),\ \zeta_t\rangle dt = \int_0^T \langle\mu,\zeta_t\rangle dt , \tag{3.13}$$

which depends only upon μ and the homotopy class of c and not the particular orbit $G\cdot\beta \subseteq J^{-1}(\mu)$. It therefore makes sense to speak of "the holonomy" of L_μ.

Proposition (3.4): The action of $\mathscr{g}$ on L_μ can be extended to a G-action iff L_μ has trivial holonomy.

This proposition is essentially a "quantization condition"; we will see it in operation in Sec. V.

Assuming that L_μ has trivial holonomy, we are now able to construct the reduced prequantization line bundle. Since the G-action on L_μ is necessarily free and proper we may form $\overline{L}_\mu = L_\mu/G$, which is clearly a complex line bundle over $T^*\overline{Q}$. Denote the projections $L_\mu \to \overline{L}_\mu$ and $\overline{L}_\mu \to T^*\overline{Q}$ by π_μ and $\overline{l}_\mu$, respectively.

Set $\gamma_\mu = j_\mu^*\gamma$, where $j_\mu: L_\mu \to L$ is the inclusion. By (3.1) and (2.1)

$$\mathscr{L}_{\zeta_L}\gamma = -(1/h)l^*(\zeta_{T^*Q} \lrcorner\, \omega)$$
$$= (1/h)l^*\, dJ_\zeta$$

and so

$$\mathscr{L}_{\zeta_L}\gamma_\mu = (1/h)l^*d\langle\mu,\zeta\rangle = 0 .$$

Consequently γ_μ projects to a complex-valued one-form $\overline{\gamma}_\mu$ on $\overline{L}_\mu$ such that

$$\gamma_\mu = \pi_\mu^*\,\overline{\gamma}_\mu . \tag{3.14}$$

Theorem (3.5): $(\overline{L}_\mu,\overline{\gamma}_\mu)$ is a prequantization line bundle for $(T^*\overline{Q},\overline{\Omega}_\mu)$.

Proof: It is straightforward to check that $\overline{\gamma}_\mu$ is indeed a connection form on $\overline{L}_\mu$.

To prove the Theorem we must verify the prequantization condition

$$d\overline{\gamma}_\mu = -(1/h)\,\overline{l}_\mu^*\overline{\Omega}_\mu .$$

Now $d\overline{\gamma}_\mu = \overline{l}_\mu^*\rho$ for some two-form ρ on $T^*\overline{Q}$. By (3.14), (3.1), and the Kummer–Marsden–Satzer Theorem,

$$(\overline{l}_\mu\circ\pi_\mu)^*\rho = d\gamma_\mu$$
$$= -(1/h)j_\mu^*l^*\omega$$
$$= -(1/h)l^*j_\mu^*\omega$$
$$= -(1/h)l^*\pi_\mu^*\overline{\Omega}_\mu$$
$$= -(1/h)(\overline{l}_\mu\circ\pi_\mu)^*\overline{\Omega}_\mu .$$

Since $\overline{l}_\mu\circ\pi_\mu$ is a submersion, $\rho = -(1/h)\overline{\Omega}_\mu$. ■

Thus if L_μ has trivial holonomy, $(T^*\overline{Q},\overline{\Omega}_\mu)$ is prequantizable. In particular, the de Rham class $(1/h)[\overline{\Omega}_\mu]_{T^*\overline{Q}} = (1/h)[F_\mu]_{\overline{Q}}$ must be integral. We will have a nice physical interpretation of this result in Sec. V C.

E. Metalinear structures

Let $\widetilde{F}V$ be a metalinear frame bundle for the vertical polarization V. Since V is invariant, G acts on FV by push forward of frames. Following the general technique of Ref. 10, we will relate FV on T^*Q to $F\overline{V}$ on $T^*\overline{Q}$ and use this relation, along with a lift of the G-action on FV to $\widetilde{F}V$, to induce a metalinear structure on $T^*\overline{Q}$ from that on T^*Q.

We can substantially simplify matters by working on configuration spaces rather than cotangent bundles. To this end, let FQ and F^*Q be the linear frame and coframe bundles of Q, respectively. There exist natural G-actions $F\Phi$ on FQ and $F^*\Phi$ on F^*Q again given by push forward of frames and coframes. Let $Z: Q \to T^*Q$ be the zero section.

Proposition (3.6): There exist canonical G-equivariant isomorphisms $FQ \approx Z^*(FV)$ and $FV \approx \tau_Q^*(FQ)$.

Proof: There is a canonical equivariant identification of T^*Q with Z^*V and hence of V with $\tau_Q^*(T^*Q)$. These induce similar identifications of F^*Q with $Z^*(FV)$ and FV with $\tau_Q^*(FQ)$. Moreover, associating to each basis its dual basis gives rise to a canonical isomorphism $FQ \approx F^*Q$ which is equivariant with respect to the actions $F\Phi$ and $F^*\Phi$. Combining these isomorphisms establishes the Proposition. ■

This result allows us to transform back and forth from T^*Q to Q. Similarly, there are canonical isomorphisms $F\overline{Q} \approx \overline{Z}^*(F\overline{V})$ and $F\overline{V} = \tau_{\overline{Q}}^*(F\overline{Q})$.

Now consider the subbundle B of FQ consisting of frames of the form $\underline{b} = (\underline{v}, \underline{\xi}_Q)$, where $\underline{\xi}$ is a positively oriented orthonormal frame for $\mathscr{g}$ with respect to the given bi-invariant metric on G. The space B is a right principal H-bundle over Q, where

$$H = \left\{ \begin{pmatrix} N & O \\ P & R \end{pmatrix} \middle| \, N \in \mathrm{GL}(\bar{n},\mathbb{R}), \; R \in \mathrm{SO}(r) \right\}$$

is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ that stabilizes B.

Let K be the subgroup of H consisting of those matrices which leave invariant the projection of $\underline{v}$ to $F\overline{Q}$; explicitly,

$$K = \left\{ \begin{pmatrix} I & O \\ P & R \end{pmatrix} \middle| \, R \in \mathrm{SO}(r) \right\}.$$

It is a normal subgroup of H and $H/K \approx \mathrm{GL}(\bar{n},\mathbb{R})$. We may therefore identify

$$B/K \approx \pi_{\overline{Q}}^*(F\overline{Q}). \tag{3.15}$$

Since $F\Phi_g(\xi_Q) = -(\mathrm{Ad}_{g^{-1}}\xi)_Q$ and the metric on G is bi-invariant, $\mathrm{Ad}_{g^{-1}}\underline{\xi}$ is also a positively oriented orthonormal frame for $\mathscr{g}$. It follows that the left action $F\Phi$ on FQ induces an action on B that commutes with the right action of H. This allows us to quotient by G in (3.15), thereby obtaining a natural isomorphism

$$G \setminus (B/K) \approx F\overline{Q}. \tag{3.16}$$

We next lift these constructions to the metabundles. Let ρ: $\widetilde{FQ} \rightarrow FQ$ be a metalinear frame bundle for Q and set $\widetilde{H} = \sigma^{-1}(H)$, where σ: $\mathrm{ML}(n,\mathbb{R}) \rightarrow \mathrm{GL}(n,\mathbb{R})$ is the 2:1 projection. Then $\widetilde{B} = \rho^{-1}(B)$ is a right principal $\widetilde{H}$-bundle over Q. Since the determinant of any matrix in K is unity, $\sigma^{-1}(K)$ has two connected components. We identify K with the component of the identity in $\sigma^{-1}(K)$. Then K is a normal subgroup of $\widetilde{H}$ and $\widetilde{H}/K \approx \mathrm{ML}(\bar{n},\mathbb{R})$. The bundle $\widetilde{B}/K$ is therefore a right principal $\mathrm{ML}(\bar{n},\mathbb{R})$-bundle on Q such that the diagram

$$\begin{array}{ccc} \widetilde{B}/K \times \mathrm{ML}(\bar{n},\mathbb{R}) & \longrightarrow & \widetilde{B}/K \\ \downarrow & & \downarrow \\ B/K \times \mathrm{GL}(\bar{n},\mathbb{R}) & \longrightarrow & B/K \end{array} \tag{3.17}$$

commutes, where the horizontal arrows are the right group actions and the vertical arrows are twofold projections.

Suppose for the moment that the G-action $F\Phi$ on FQ lifts to an action $\widetilde{F\Phi}$ on $\widetilde{FQ}$. Then $\widetilde{F\Phi}$ will preserve $\widetilde{B}$ and commute with the right $\widetilde{H}$-action on $\widetilde{B}$, and thus give rise to a left G-action on $\widetilde{B}/K$ that commutes with the right action of $\mathrm{ML}(\bar{n},\mathbb{R})$. The space $G \setminus (\widetilde{B}/K)$ of G-orbits in $\widetilde{B}/K$ will therefore inherit the structure of a right principal $\mathrm{ML}(\bar{n},\mathbb{R})$-bundle over $\overline{Q}$ such that the projection $G \setminus (\widetilde{B}/K) \rightarrow G \setminus (B/K)$ is 2:1. Applying (3.17) and (3.16) it follows that $G \setminus (\widetilde{B}/K)$ will define a metalinear frame bundle $\widetilde{F\overline{Q}}$ for $\overline{Q}$.

In summary, if Q is metalinear and the action of G on FQ lifts to $\widetilde{FQ}$, then $\overline{Q}$ is metalinear. To complete the analysis we must determine whether in fact the group action lifts.

Since ρ is a 2:1 submersion, the action of $\mathscr{g}$ on FQ obtained by differentiating $F\Phi$ lifts to an action on $\widetilde{FQ}$ by complete vector fields. Thus the only possible obstruction to lifting $F\Phi$ is the nonsimple connectivity of G. Now this $\mathscr{g}$-action defines an involutive distribution on $\widetilde{FQ}$, and we may extend to a G-action iff this distribution has trivial holonomy when restricted to each orbit in FQ.

To measure the holonomy, introduce the characteristic homomorphism $\pi_1(FQ,\underline{f}) \rightarrow \mathbb{Z}_2$ of $\widetilde{FQ}$ (see Ref. 23, §13) and consider the map $\pi_1(G) \rightarrow \pi_1(FQ,\underline{f})$ defined by $[c(t)] \mapsto [F\Phi_{c(t)}(\underline{f})]$. The above distribution has no holonomy over $G\cdot\underline{f}$ iff the composite homomorphism χ_G: $\pi_1(G) \rightarrow \mathbb{Z}_2$ is trivial. Furthermore, since $\mathbb{Z}_2$ has no nontrivial automorphisms, χ_G is independent of the choice of $\underline{f} \in FQ$. Thus if the $\mathscr{g}$-action extends over just one orbit, it extends over all of them.

Proposition (3.7): The action of G on FQ lifts to $\widetilde{FQ}$ iff the natural homomorphism $\chi_G: \pi_1(G) \rightarrow \mathbb{Z}_2$ is trivial.

There is another version of this result which is often useful. Note that the restriction of FQ to any orbit $G\cdot q$ in Q is trivial: each $\underline{f} \in F_q Q$ defines a global section $\Phi_g(q) \mapsto F\Phi_g(\underline{f})$ of $FQ\,|\,(G\cdot q)$. Proposition (3.7) then implies that we have lifting iff the restriction of $\widetilde{FQ}$ to any (and hence every) orbit in Q is trivial.

It remains to pull our results back to T^*Q and $T^*\overline{Q}$. We first observe that Proposition (3.6) holds on the metalinear level, i.e., if $\widetilde{FV}$ is a metalinear frame bundle for V then $\widetilde{FQ} = Z^*(\widetilde{FV})$ is one for Q and, conversely, every metalinear frame bundle $\widetilde{FV}$ is $\tau_Q^*(\widetilde{FQ})$ for some metalinear structure on Q. Similar resetuls are true for $\widetilde{F\overline{V}}$ and $\widetilde{F\overline{Q}}$. Now, since the mechanics are the same in both cases, it is clear that the G-action on FQ lifts to $\widetilde{FQ} = Z^*(\widetilde{FV})$ iff that on FV lifts to $\widetilde{FV} = \tau_Q^*(\widetilde{FQ})$. It follows that these metalinear identifications are G-equivariant. Denote by the same letter $\widetilde{B}$ the pullback bundle $\tau_Q^*(\widetilde{B}) \subset \widetilde{FV}$, and set $\widetilde{B}_\mu = \widetilde{B}\,|J^{-1}(\mu)$. We have proven the following theorem.

Theorem (3.8): If the action of G on FV lifts to $\widetilde{FV}$, then there exists a compatible metalinear structure

$$\widetilde{F\overline{V}} = G \setminus (\widetilde{B}_\mu/K)$$

on $T^*\overline{Q}$.

Remark: Similar results hold for metaplectic structures.[10]

IV. EQUIVALENCE OF COMPATIBLE QUANTIZATIONS

The stage is now set to prove the equivalence of the quantizations of the extended and reduced phase spaces. We have shown that quantization data on (T^*Q,ω,G,J,μ) consisting of the vertical polarization V, a prequantization line bundle (L,γ) and a metalinear frame bundle $\widetilde{FV}$ induce quantization data $\overline{V}$, $(\overline{L}_\mu,\overline{\gamma}_\mu)$ and $\widetilde{F\overline{V}}$ on $(T^*\overline{Q},\overline{\Omega}_\mu)$, provided both L_μ and $\widetilde{FV}$ have trivial holonomy. The corresponding quantizations of (T^*Q,ω) and $(T^*\overline{Q},\overline{\Omega}_\mu)$ are said to be *compatible*. Our main result is that *compatible quantizations are equivalent*. We will make this precise after disposing of some preliminaries.

A. Preliminaries

Let $\sqrt{\wedge^n V}$ and $\sqrt{\wedge^{\bar{n}} \overline{V}}$ be the bundles of half-forms relative to V and $\overline{V}$, respectively. Set $\sqrt{\wedge^n V_\mu} = \sqrt{\wedge^n V}\,|J^{-1}(\mu)$.

Proposition (4.1): There exists a canonical isomorphism $\sqrt{\wedge^n V_\mu} \approx \pi_\mu^* \sqrt{\wedge^{\bar n} \bar V}$.

Proof: Fix $\beta \in J^{-1}(\mu)$ and consider $\nu_\beta \in \sqrt{\wedge^n V_\beta}$. As $\Delta(\widetilde M) = 1$ for all $\widetilde M \in K$, (3.2) implies that $\nu_\beta^\#(\underline{\tilde b}\widetilde M) = \nu_\beta^\#(\underline{\tilde b})$. Since by Theorem (3.8) $\pi_\mu^* \bar F \bar V = \widetilde B_\mu / K$, it follows that for $\underline{\tilde b} \in \widetilde B_\beta$ the equation

$$\bar\nu_\beta^\#([\underline{\tilde b}]) = \nu_\beta^\#(\underline{\tilde b}) \tag{4.1}$$

defines an element $\bar\nu_\beta$ of $(\pi_\mu^* \sqrt{\wedge^{\bar n} \bar V})_\beta$, where the brackets denote K-equivalence classes. Conversely, given $\bar\nu_\beta \in (\pi_\mu^* \sqrt{\wedge^{\bar n}\bar V})_\beta$, (4.1) defines an element $\nu_\beta \in \sqrt{\wedge^n V_\beta}$ since, according to (3.2), any half-form is completely determined by its restriction to $\widetilde B_\beta$. The association (4.1) is thus the desired complex line bundle isomorphism. ■

The action of G on $\widetilde F V$ gives rise to a left action of G on $\sqrt{\wedge^n V}$ which commutes with the right action of $\mathrm{ML}(n,\mathbb{R})$ by multiplication by Δ. Since all our constructions are G-equivariant, Proposition (4.1) yields $G \backslash \sqrt{\wedge^n V_\mu} \approx \sqrt{\wedge^{\bar n} \bar V}$. Combining this with the results of Sec. III D, we have the following corollary.

Corollary (4.2): $G \backslash (L \otimes \sqrt{\wedge^n V})_\mu \approx \bar L_\mu \otimes \sqrt{\wedge^{\bar n} \bar V}$.

Our next task is to relate the *two* induced actions of $\mathscr{g}$ on $\Gamma(L \otimes \sqrt{\wedge^n V})_\mu$, which we must be careful to distinguish. The first is that provided by the naturality of the extended phase space quantization (cf. Sec. III B) and is used to quantize the momentum map. The second is needed to construct compatible quantization structures on the reduced phase space (cf. Secs. III D and III E and the preceding discussion). These two actions agree on $\Gamma(\sqrt{\wedge^n V_\mu})$ but differ on $\Gamma(L_\mu)$. We derive expressions for the generators of both actions.

The first action on $\Gamma(L \otimes \sqrt{\wedge^n V})_\mu$ is generated by the quantum constraint operators $\mathscr{Q}J_\xi$. According to (3.9) these are given by

$$\mathscr{Q}J_\xi[\Psi] = \{[-i\hbar\nabla_{\xi_{T^*Q}} + J_\xi - \tfrac{1}{2}i\hbar \operatorname{tr} A_{\underline{f}}(\xi_{T^*Q})]\psi\lambda\} \otimes \nu_{\underline{f}}, \tag{4.2}$$

for each local section Ψ of the form (3.8). Tracing through the derivation of this formula (cf. Ref. 21, §6.1), we find that the last term on the right-hand side of (4.2) arises from the action of $\mathscr{g}$ on $\sqrt{\wedge^n V_\mu}$ while the first two terms are due to the action of $\mathscr{g}$ on L_μ by *connection-preserving* vector fields

$$\xi^*_{T^*Q} - \zeta_{J_\xi/h}, \tag{4.3}$$

where $\zeta_{J_\xi/h}$ is the fundamental vector field on L defined by the function J_ξ/h. On the other hand, the second action of $\mathscr{g}$ on L_μ is generated by just the *horizontal* vector fields $\xi^*_{T^*Q}$. It is easy to see that removing the last term from (4.3) eliminates the second term from (4.2). Since both actions agree on $\sqrt{\wedge^n V_\mu}$, it follows from (4.2) that the generators $\mathscr{G}_\xi$ of the second action are related to the $\mathscr{Q}J_\xi$ by

$$\mathscr{Q}J_\xi[\Psi] = \mathscr{G}_\xi[\Psi] + J_\xi\Psi, \tag{4.4}$$

for all $\Psi \in \Gamma(L \otimes \sqrt{\wedge^n V})_\mu$.

B. Smooth equivalence

We are finally ready to compare the extended and reduced phase space quantizations. They are correlated by the following theorem.

Smooth Equivalence Theorem: If the quantizations of the extended phase space (T^*Q,ω,G,J,μ) and the reduced phase space $(T^*\bar Q,\bar\Omega_\mu)$ are compatible, then there exists a canonical isomorphism $\mathscr{H}_\mu \approx \bar{\mathscr{H}}_\mu$.

Proof: Let $\Psi \in \mathscr{H}_\mu$. Equation (4.4) implies that $\mathscr{G}_\xi[\Psi]|J^{-1}(\mu) = 0$, so $\Psi|J^{-1}(\mu)$ is G-invariant. By Corollary (4.2), Ψ projects to a smooth section $\bar\Psi$ of $\bar L_\mu \otimes \sqrt{\wedge^{\bar n}\bar V}$. Since Ψ is polarized, Proposition (3.2) shows that $\bar\Psi$ is also. Thus $\bar\Psi \in \bar{\mathscr{H}}_\mu$.

For the converse, suppose $\bar\Psi \in \bar{\mathscr{H}}_\mu$. Corollary (4.2), Proposition (3.2), and Eq. (4.4) imply that $\bar\Psi$ pulls back to a unique G-invariant section Ψ_μ of $(L \otimes \sqrt{\wedge^n V})_\mu$ which is covariantly constant along $V \cap TJ^{-1}(\mu)$ and satisfies

$$\mathscr{Q}J_\xi[\Psi_\mu] = \langle \mu,\xi\rangle \Psi_\mu. \tag{4.5}$$

Since every leaf of V is simply connected and intersects $J^{-1}(\mu)$, parallel transport along V produces a globally defined polarized section Ψ of $L \otimes \sqrt{\wedge^n V}$ which agrees with Ψ_μ on $J^{-1}(\mu)$. Now consider the polarized sections

$$\Psi_\xi = \mathscr{Q}J_\xi[\Psi] - \langle\mu,\xi\rangle\Psi,$$

for each $\xi \in \mathscr{g}$. Every Ψ_ξ is uniquely determined by its restriction to $J^{-1}(\mu)$. But $\Psi_\xi|J^{-1}(\mu) = 0$ by virtue of (4.5), so $\Psi_\xi \equiv 0$ and hence $\Psi \in \mathscr{H}_\mu$.

This establishes the existence of the required isomorphism. ■

Remarks: (1) We emphasize that this isomorphism is entirely canonical since our constructions of the reduced quantization data are.

(2) When $\mu = 0$ both of the $\mathscr{g}$-actions on $\mathscr{H}_0$ coincide. We may then restate the conclusion of this Theorem as follows: There exists a canonical isomorphism between the space of gauge-invariant smooth polarized sections of $L \otimes \sqrt{\wedge^n V}$ and the space of smooth polarized sections of $\bar L_0 \otimes \sqrt{\wedge^{\bar n}\bar V}$. This special case is due to Sniatycki.[10]

We now derive a local expression for the isomorphism $\mathscr{H}_\mu \approx \bar{\mathscr{H}}_\mu$ which will be useful later. Let

$$(q^1,\dots,q^n) = (\bar q^1,\dots,\bar q^{\bar n}, g^1,\dots,g^r)$$

be a chart on $\pi_Q^{-1}(\bar U) \subset Q$ induced by a local trivialization $\pi_Q^{-1}(\bar U) \approx \bar U \times G$, and let (q^i,p_i), $i = 1,\dots,n$, and $(\bar q^i,\bar p_i)$, $i = 1,\dots,\bar n$, be the corresponding canonical charts on T^*Q and $T^*\bar Q$, respectively. Set $\underline{f} = (\partial/\partial p_1,\dots,\partial/\partial p_n)$ and define $\nu_{\underline{f}} \in \Gamma(\sqrt{\wedge^n V})$ by (3.7). It is convenient to construct another half-form $\nu_{\underline{b}}$ on T^*Q as follows (cf. Sec. III E). Using the given bi-invariant metric g on G, fix a positively oriented orthonormal frame $\underline{\xi} = (\xi_1,\dots,\xi_r)$ for $\mathscr{g}$ and set

$$\underline{b} = \left(\frac{\partial}{\partial \bar q^1},\dots,\frac{\partial}{\partial \bar q^{\bar n}},\xi_1,\dots,\xi_r\right).$$

Then

$$\underline{b} = \left(\frac{\partial}{\partial \bar q^1},\dots,\frac{\partial}{\partial \bar q^{\bar n}},\frac{\partial}{\partial g^1},\dots,\frac{\partial}{\partial g^r}\right)\begin{pmatrix} I & O \\ O & C\end{pmatrix},$$

for some matrix C with $\det C = (\det g)^{-1/2}$. Applying Proposition (3.6) we find that under the isomorphism $FV \approx \tau_Q^*(FQ)$ the frame $(\partial/\partial\bar q^1,\dots,\partial/\partial\bar q^{\bar n},\partial/\partial g^1,\dots,\partial/\partial g^r)$ maps onto $\underline{f}$ while $\underline{b}$ maps onto a frame in B which we also denote by $\underline{b}$. Define $\nu_{\underline{b}}$ by $\nu_{\underline{b}}^\# \circ \underline{b} = 1$; from the above and (3.2) it follows that

$$\nu_{\underline{b}} = (\det g)^{1/4}\,\nu_{\underline{f}}\,. \tag{4.6}$$

Then each $\Psi\in\mathscr{H}$ may be locally written as either

$$\Psi = \check{\psi}(q)\lambda\otimes\nu_{\underline{b}} \tag{4.7}$$

or

$$\Psi = \check{\psi}(q)\,(\det g)^{1/4}\lambda\otimes\nu_{\underline{f}}\,. \tag{4.8}$$

Now suppose $\Psi\in\mathscr{H}_\mu$ so that Ψ satisfies (3.11). Then using (4.7), (4.6), (4.2), (3.6), (2.3), and the fact that the components of ξ_Q are constant in this chart, we compute

$$\Psi = k(\bar{q})\exp\left(\frac{i}{\hbar}\sum_{a=1}^{r}\mu_a g^a\right)\lambda\otimes\nu_{\underline{b}}\,, \tag{4.9}$$

where k is arbitrary.

On the other hand, both $\underline{b}$ and $\underline{f}$ project to $\underline{\bar{f}} = (\partial/\partial\bar{p}_1,...,\partial/\partial\bar{p}_n)\in F\bar{V}$. Defining $\bar{\nu}_{\underline{f}}\in\Gamma(\sqrt{\ }\wedge^n\bar{V})$ by (3.7), it follows from (4.1) that $\nu_{\underline{b}}$ projects to $\bar{\nu}_{\underline{f}}$. Similarly, from Sec. III D we find that λ projects to the section $\bar{\lambda}_\mu$ of $\bar{L}_\mu$ defined by $\bar{\lambda}_\mu\circ\pi_\mu = \bar{l}_\mu\circ\lambda$. Since locally every $\bar{\Psi}\in\bar{\mathscr{H}}_\mu$ takes the form

$$\bar{\Psi} = \bar{\psi}(\bar{q})\bar{\lambda}_\mu\otimes\bar{\nu}_{\underline{f}}\,, \tag{4.10}$$

we have upon comparing (4.9) and (4.10) that the isomorphism $\mathscr{H}_\mu\to\bar{\mathscr{H}}_\mu$ is given by

$$k(\bar{q})\exp\left(\frac{i}{\hbar}\sum_{a=1}^{r}\mu_a g^a\right)\mapsto k(\bar{q})\,. \tag{4.11}$$

Compatible quantizations thus have canonically isomorphic spaces of physically admissible wave functions. But compatibility should ensure more than this: it should also intertwine the quantizations of G-invariant observables. More precisely, let $f\in C^\infty(T^*Q)$ be G-invariant in which case it reduces to $\bar{f}_\mu\in C^\infty(T^*\bar{Q})$ as indicated in Sec. II B. Then the quantum operators $\mathscr{Q}f$ on $\mathscr{H}$ and $\bar{\mathscr{Q}}\bar{f}_\mu$ on $\bar{\mathscr{H}}_\mu$ should be such that

$$\begin{array}{ccc} \mathscr{H}_\mu & \xrightarrow{\ \mathscr{Q}f\ } & \mathscr{H}_\mu \\ \downarrow & \bar{\mathscr{Q}}\bar{f}_\mu & \downarrow \\ \bar{\mathscr{H}}_\mu & \longrightarrow & \bar{\mathscr{H}}_\mu \end{array} \tag{4.12}$$

commutes, where the vertical arrows are the isomorphisms provided by the Smooth Equivalence Theorem. This is actually so, at least if f is polarization preserving.

Theorem (4.3): Let f be a G-invariant polarization-preserving observable. Then diagram (4.12) commutes.

Proof: First note that because f preserves V, $\bar{f}_\mu$ preserves $\bar{V}$ by Proposition (3.2). Consequently $\bar{f}_\mu$ is quantizable.

Let ϕ^t and $\bar{\phi}^t_\mu$ be the flows of f and $\bar{f}_\mu$ on T^*Q and $T^*\bar{Q}$, respectively, with

$$\pi_\mu\circ\phi^t = \bar{\phi}^t_\mu\circ\pi_\mu\,.$$

Since both f and $\bar{f}_\mu$ are polarization preserving, these flows induce one-parameter groups of bundle automorphisms of $L\otimes\sqrt{\ }\wedge^n V$ and $\bar{L}_\mu\otimes\sqrt{\ }\wedge^n\bar{V}$, which we denote by the same symbols (cf. Sec. III A). Since f is invariant, a straightforward calculation using the techniques of Sec. III E along with the fact that $F\phi^t(\xi_Q(q)) = \xi_Q(\phi^t(q))$ shows that ϕ^t preserves $\tilde{B}_\mu$. Thus ϕ^t is equivariant and it follows from Corollary (4.2) that

$$\begin{array}{ccc} (L\otimes\sqrt{\ }\wedge^n V)_\mu & \xrightarrow{\ \phi^t\ } & (L\otimes\sqrt{\ }\wedge^n V)_\mu \\ \downarrow & \bar{\phi}^t_\mu & \downarrow \\ \bar{L}_\mu\otimes\sqrt{\ }\wedge^n\bar{V} & \longrightarrow & \bar{L}_\mu\otimes\sqrt{\ }\wedge^n\bar{V} \end{array} \tag{4.13}$$

commutes, where the vertical arrows are projections.

Now consider the corresponding one-parameter groups of linear isomorphisms ϕ^t of $\mathscr{H}$ and $\bar{\phi}^t_\mu$ of $\bar{\mathscr{H}}_\mu$. From the definition of $\mathscr{Q}J_\xi$ applied to $\phi^t\Psi$ we have $\mathscr{Q}J_\xi[\phi^t\Psi] = \phi^t\,\mathscr{Q}J_\xi[\Psi]$ as ϕ^t is equivariant. In particular, if $\Psi\in\mathscr{H}_\mu$, then so is $\phi^t\Psi$. Thus (4.13) and the Smooth Equivalence Theorem imply that the induced diagram

$$\begin{array}{ccc} \mathscr{H}_\mu & \xrightarrow{\ \phi^t\ } & \mathscr{H}_\mu \\ \downarrow & \bar{\phi}^t_\mu & \downarrow \\ \bar{\mathscr{H}}_\mu & \longrightarrow & \bar{\mathscr{H}}_\mu \end{array} \tag{4.14}$$

commutes.

The quantum operators $\mathscr{Q}f$ and $\bar{\mathscr{Q}}\bar{f}_\mu$ are defined by

$$\mathscr{Q}f[\Psi] = i\hbar\frac{d}{dt}(\phi^t\Psi)|_{t=0}$$

and

$$\bar{\mathscr{Q}}\bar{f}_\mu[\bar{\Psi}] = i\hbar\frac{d}{dt}(\bar{\phi}^t_\mu\bar{\Psi})|_{t=0}$$

[cf. (3.3)]. If $\Psi\in\mathscr{H}_\mu$ then $\phi^t\Psi\in\mathscr{H}_\mu$ and consequently $\mathscr{Q}f[\Psi]\in\mathscr{H}_\mu$. Thus diagram (4.12) is well defined and its commutativity now follows immediately from the above definitions and (4.14). ■

Roughly, this result states that one may quantize invariant observables in either formalism with equivalent results. However, the Theorem does not apply when f is not polarization preserving. In such cases diagram (4.12) may not exist and, when it does, it will generally not commute.

C. Unitary equivalence

We now discuss the one facet of the equivalence problem that we have overlooked thus far—the Hilbert space structure. The Theorems of Sec. IV B pertain only to *smooth* quantizations, i.e., the linear spaces $\mathscr{H}_\mu$ and $\bar{\mathscr{H}}_\mu$ of C^∞ wave functions. Do our results still apply when the quantum inner products are introduced? More precisely, does the linear isomorphism $\mathscr{H}_\mu\approx\bar{\mathscr{H}}_\mu$ of the Smooth Equivalence Theorem extend to a unitary isomorphism of the corresponding quantum Hilbert spaces?

For constrained cotangent systems, the spaces $\mathscr{H}$ and $\bar{\mathscr{H}}_\mu$ of polarized states carry canonically defined inner products.[12,21] Using the setup of the previous section, we may describe these as follows. The inner product of two wave functions $\Psi,\Phi\in\mathscr{H}$ of the form (4.7) with supports in $\pi_Q^{-1}(\bar{U})$ is

$$(\Psi,\Phi)_Q = \int_{\pi_Q^{-1}(\bar{U})}\psi(q)\phi^*(q)\sqrt{\det g}\,d^nq\,, \tag{4.15}$$

where the star denotes complex conjugation. Similarly, for $\bar{\Psi},\bar{\Phi}\in\bar{\mathscr{H}}_\mu$ of the form (4.10) with supports in $\bar{U}$,

$$(\bar{\Psi},\bar{\Phi})_{\bar{Q}} = \int_{\bar{U}}\bar{\psi}(\bar{q})\bar{\phi}^*(\bar{q})d^n\bar{q}\,. \tag{4.16}$$

We complete these spaces with respect to these inner products thereby obtaining the quantum Hilbert spaces $\mathscr{h}$ and $\overline{\mathscr{h}}_\mu$, respectively.

Although this procedure is in itself straightforward, a difficulty arises when considering the space

$$\mathscr{h}_\mu = \{\Psi \in \mathscr{h} \mid \mathscr{Q} J_\xi [\Psi] = \langle \mu, \xi \rangle \Psi\}$$

of physically admissible states. It may happen that $\mathscr{h}_\mu$ will consist only of distributional wave functions. For instance, if G is noncompact some of the eigenvalues $\langle \mu, \xi \rangle$ will lie in the continuous spectra of the corresponding constraint operators $\mathscr{Q} J_\xi$. In such cases the inner product on $\mathscr{h}$ will not induce one on $\mathscr{h}_\mu$ so that, in general, $\mathscr{h}_\mu$ and $\overline{\mathscr{h}}_\mu$ can only be compared as linear spaces. However, one may use the Smooth Equivalence Theorem and the inner product on $\overline{\mathscr{h}}_\mu$ to induce one on $\mathscr{h}_\mu$ in such a way that $\mathscr{h}_\mu$ and $\overline{\mathscr{h}}_\mu$ will then be unitarily related. We will see an example of this phenomenon in Sec. V A.

This problem cannot occur if G is compact, in which case $\mathscr{h}_\mu$ is a genuine subspace of $\mathscr{h}$.

Unitary Equivalence Theorem: If G is compact then $\mathscr{h}_\mu$ and $\overline{\mathscr{h}}_\mu$ are unitarily equivalent.

Proof: Let $\Psi, \Phi \in \mathscr{h}_\mu$. Substituting (4.9) into (4.15) we obtain the induced inner product

$$(\Psi,\Phi)_\mu = \int_{\pi_{\overline{Q}}^{-1}(\overline{U})} k(\overline{q}) h^*(\overline{q}) \sqrt{\det g}\, d^n q ,$$

where h is to Φ as k is to Ψ. Writing $d^n q = d^r g\, d^{\bar{n}} \overline{q}$, this reduces to

$$(\Psi,\Phi)_\mu = \text{vol}(G) \int_{\overline{U}} k(\overline{q}) h^*(\overline{q}) d^{\bar{n}} \overline{q} , \tag{4.17}$$

where

$$\text{vol}(G) = \int_G \sqrt{\det g}\, d^r g$$

is finite since G is compact.

The isomorphism $\mathscr{H}_\mu \rightarrow \overline{\mathscr{H}}_\mu$ of the Smooth Equivalence Theorem clearly extends to $\mathscr{h}_\mu$ thereby enabling us to project Ψ and Φ on T^*Q to wave functions $\overline{\Psi}$ and $\overline{\Phi}$ on $T^*\overline{Q}$. Using the explicit form (4.11) of this projection in (4.16) yields

$$(\overline{\Psi},\overline{\Phi})_{\overline{Q}} = \int_{\overline{U}} k(\overline{q}) h^*(\overline{q}) d^{\bar{n}} \overline{q} . \tag{4.18}$$

It follows from (4.17) that $\overline{\Psi}, \overline{\Phi} \in \overline{\mathscr{h}}_\mu$. The mapping $\mathscr{U}$: $\mathscr{h}_\mu \rightarrow \overline{\mathscr{h}}_\mu$ defined locally by

$$k(\overline{q}) \exp\left(\frac{i}{\hbar} \sum_{a=1}^{r} \mu_a g^a\right) \lambda \otimes \nu_{\underline{b}} \mapsto \sqrt{\text{vol}(G)}\, k(\overline{q}) \overline{\lambda}_\mu \otimes \overline{\nu}_{\underline{\overline{b}}} \tag{4.19}$$

is therefore a vector space isomorphism, and a comparison of (4.17) with (4.18) shows that it is unitary. ■

Similarly, Theorem (4.3) carries over to the Hilbert space case when G is compact. Namely, if f is a G-invariant polarization-preserving observable, then the unitary isomorphism $\mathscr{U}$ intertwines the quantum operators corresponding to f and its reduction $\overline{f}_\mu$:

$$\overline{\mathscr{Q}}\, \overline{f}_\mu = \mathscr{U}^{-1} (\mathscr{Q} f) \mathscr{U} .$$

To summarize, if G is compact, then a smooth equivalence of the extended and reduced phase space quantizations naturally extends to a unitary equivalence which intertwines the quantizations of invariant observables. When G is noncompact no such natural unitary equivalence exists *a priori*, but the Smooth Equivalence Theorem may be used to induce one.

V. EXAMPLES

We present several examples which illustrate our techniques and theorems. In most cases we will explicitly verify our results by direct computation.

A. Center of mass reduction in the N-body problem

Our presentation follows that in §10.4 of Abraham and Marsden[15]; see also Robinson.[24]

Consider N masses $m_1,...,m_N$ moving in $\mathbb{R}^k$. Upon removing collisions we have

$$Q = (\mathbb{R}^k)^N - \Delta^N ,$$

where

$$\Delta^N = \bigcup_{1<i,j<N} \Delta^N_{ij}$$

and

$$\Delta^N_{ij} = \{\mathbf{q} = (\underline{q}^1,...,\underline{q}^N) \in (\mathbb{R}^k)^N \mid \underline{q}^i = \underline{q}^j\} .$$

The translation group $\mathbb{R}^k$ acts freely and properly on Q according to

$$\Phi(\underline{g},\mathbf{q}) = (\underline{q}^1 + \underline{g},...,\underline{q}^N + \underline{g}) .$$

To construct the orbit space introduce the diffeomorphism

$$C\colon Q \rightarrow \{(\dot{\mathbb{R}}^k)^{N-1} - \Delta^{N-1}\} \times \mathbb{R}^k$$

given by

$$(\underline{q}^1,...,\underline{q}^N) \mapsto \left(m_1(\underline{q}^1 - \underline{q}^N),...,m_{N-1}(\underline{q}^{N-1} - \underline{q}^N), \sum_{i=1}^{N} m_i \underline{q}^i\right), \tag{5.1}$$

where $\dot{\mathbb{R}}^k = \mathbb{R}^k - \{\underline{0}\}$. The corresponding $\mathbb{R}^k$-action $C \circ \Phi_{\underline{g}} \circ C^{-1}$ is just translation in the last factor by $M\underline{g}$, where $M = \Sigma_{i=1}^N m_i$ is the total mass of the system, so that

$$\overline{Q} \approx (\dot{\mathbb{R}}^k)^{N-1} - \Delta^{N-1} .$$

Thus $Q = \overline{Q} \times \mathbb{R}^k$ is trivial as a principal $\mathbb{R}^k$-bundle.

This result is useful for understanding the structure of $\overline{Q}$ which, in general, is quite complicated. More meaningful physically, however, is the representation of $\overline{Q}$ as the $N(k-1)$-dimensional submanifold $C^{-1}(\overline{Q} \times \{\underline{0}\})$ of Q obtained by fixing the center of mass of the system at the origin. Thus we view

$$\overline{Q} = \left\{\mathbf{q} \in Q \,\middle|\, \sum_{i=1}^{N} m_i \underline{q}^i = \underline{0}\right\} . \tag{5.2}$$

The extended phase space is $T^*Q = Q \times (\mathbb{R}^k)^N$ with symplectic form $\omega = d\Theta$, where $\Theta = \Sigma_{i=1}^N \underline{p}_i \cdot d\underline{q}^i$. The cotangent action is $T^*\Phi(\underline{g},(\mathbf{q},\mathbf{p})) = (\Phi_{\underline{g}}(\mathbf{q}),\mathbf{p})$ with momentum map

$$J(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{N} \underline{p}_i . \tag{5.3}$$

Now $\mathbb{R}^k$ is Abelian so every $\mu \in \mathscr{g}^* \approx \mathbb{R}^k$ is invariant; for simplicity we consider only $\mu = 0$. Taking (5.2) and (2.4) into account, we can identify $T^*\overline{Q}$ with $\tau_Q^{-1}(\overline{Q}) \cap J^{-1}(\underline{0})$, i.e.,

$$T^*\overline{Q} = \left\{ (\mathbf{q},\mathbf{p}) \in T^*Q \,\middle|\, \sum_{i=1}^{N} m_i \underline{q}^i = \underline{0}, \quad \sum_{i=1}^{N} \underline{p}_i = \underline{0} \right\}. \tag{5.4}$$

Although there may be various prequantizations of (T^*Q,ω) depending upon the topology of Q, we can always take $L = T^*Q \times \mathbb{C}$ with trivializing section $\lambda(\mathbf{q},\mathbf{p}) = (\mathbf{q},\mathbf{p},1)$. Since $\mathbb{R}^k$ is simply connected, the action $T^*\Phi$ lifts horizontally to all of L; $\overline{L}_0$ therefore exists. Using (5.4) to explicitly identify $\overline{L}_0$ with $L\,|T^*\overline{Q}$, it follows that $\overline{L}_0 = T^*\overline{Q} \times \mathbb{C}$.

Since T^*Q is parallelizable one possibility for $\widetilde{F}V$ is simply $T^*Q \times \mathrm{ML}(Nk,\mathbb{R})$. The induced action of $\mathbb{R}^k$ on FV is trivial on the fibers and consequently lifts to $\widetilde{F}V$. Thus the corresponding metalinear frame bundle $\widetilde{F\overline{V}}$ for

$$\overline{V} = \mathrm{span}\left\{ \underline{v}_1 \cdot \frac{\partial}{\partial \underline{p}_1} + \dots + \underline{v}_N \cdot \frac{\partial}{\partial \underline{p}_N} \,\middle|\, \sum_{i=1}^{N} \underline{v}_i = \underline{0} \right\}$$

is also a product.

We now quantize the extended phase space. Setting $\underline{f} = (\partial/\partial \underline{p}_1,\dots,\partial/\partial \underline{p}_N)$, we have from Sec. III A that every polarized $\Psi \in \Gamma(L \otimes \surd \wedge^{Nk} V)$ can be written

$$\Psi = \psi(\mathbf{q})\lambda \otimes \nu_{\underline{f}}. \tag{5.5}$$

From (4.2), (5.3), and (5.5) the quantum constraint operators are

$$\mathscr{Q}J[\Psi] = -i\hbar\{\nabla_1 + \dots + \nabla_N)\psi(\mathbf{q})\}\lambda \otimes \nu_{\underline{f}},$$

where ∇_i is the ordinary gradient with respect to $\underline{q}^i$. Thus the physically admissible quantum states are those that satisfy

$$(\nabla_1 + \dots + \nabla_N)\psi(\mathbf{q}) = 0.$$

It follows that $\mathscr{H}_0$ can be identified with the set of all $\psi \in C^\infty(Q,\mathbb{C})$ of the form, say,

$$\psi = \psi(\underline{q}^1 - \underline{q}^N,\dots,\underline{q}^{N-1} - \underline{q}^N). \tag{5.6}$$

Similarly, quantizing the reduced phase space gives $\overline{\mathscr{H}}_0 \approx C^\infty(\overline{Q},\mathbb{C})$. We have from (5.1) that every $\bar{\psi} \in C^\infty(\overline{Q},\mathbb{C})$ is of the form

$$\bar{\psi} = \bar{\psi}(m_1(\underline{q}^1 - \underline{q}^N),\dots,m_{N-1}(\underline{q}^{N-1} - \underline{q}^N)). \tag{5.7}$$

A comparison of (5.6) with (5.7) yields the isomorphism $\mathscr{H}_0 \rightarrow \overline{\mathscr{H}}_0$ predicted by the Smooth Equivalence Theorem.

From (4.15) and (4.16) it follows that the Hilbert spaces $\mathscr{k}$ and $\bar{\mathscr{k}}_0$ are $L^2((\mathbb{R}^k)^N - \Delta^N)$ and $L^2((\mathbb{R}^k)^{N-1} - \Delta^{N-1})$, respectively. Now $\underline{0}$ is in the continuous spectrum of $\mathscr{Q}J$ and from (5.6) it is clear that none of the translationally invariant wave functions are square integrable. Thus $\mathscr{k}_0$ and $\bar{\mathscr{k}}_0$ can only be compared as linear spaces.

Remark: For nonzero $\mu \in \mathbb{R}^k$, reduction fixes the center of mass as moving with velocity μ/M.

B. Angular momentum

We study the system consisting of a single particle moving in $\dot{\mathbb{R}}^n = \mathbb{R}^n - \{\underline{0}\}$ with constant angular momentum. This example is interesting for two reasons. First, when $n > 2$, the action

$$\Phi(A,\underline{q}) = A\underline{q} \tag{5.8}$$

of $\mathrm{SO}(n)$ on $\dot{\mathbb{R}}^n$ is not free so that the fundamental assumption underlying all our results is violated. Nonetheless, as we shall see, the conclusions of our theorems still hold. Second, the case $n = 2$ illustrates how the obstructions to lifting the group action to the various quantization structures give rise to "quantization conditions."

Since the case $n = 2$ is exceptional, we first consider only $n > 2$.

The action (5.8) is always proper and effective. The orbits are concentric spheres and thus

$$\dot{\mathbb{R}}^n/\mathrm{SO}(n) \approx \mathbb{R}^+. \tag{5.9}$$

Viewing $\underline{q}$ and $\underline{p}$ as column vectors, the cotangent action on $T^*\dot{\mathbb{R}}^n = \dot{\mathbb{R}}^n \times \mathbb{R}^n$ becomes

$$T^*\Phi(A,(\underline{q},\underline{p})) = (A\underline{q},A\underline{p}) \tag{5.10}$$

and, upon identifying $\mathrm{so}(n)^*$ and $\mathrm{so}(n)$ with $\wedge^2(\mathbb{R}^n)$, the angular momentum map can be written

$$J(\underline{q},\underline{p}) = \underline{q} \wedge \underline{p}. \tag{5.11}$$

The coadjoint action of $\mathrm{SO}(n)$ on $\wedge^2(\mathbb{R}^n)$ is

$$\mathrm{Ad}^*_A(\underline{q} \wedge \underline{p}) = A^{-1}\underline{q} \wedge A^{-1}\underline{p},$$

from which it follows that

$$\mathrm{SO}(n)_\mu \approx \mathrm{SO}(2) \times \mathrm{SO}(n-2),$$

for $\mu \neq 0$. Consequently 0 is the *only* invariant element of $\mathrm{so}(n)^*$ for $n > 2$.

Remark: With reference to the discussion in Sec. III B, it is not surprising that this system cannot be consistently quantized when $\mu \neq 0$. Indeed, (3.11) would correspond to simultaneously specifying *all* the components of the angular momentum—a well-known quantum mechanical impossibility.

Now 0 is not a regular value of J, but it is weakly regular. Actually J has rank $n-1$ on $J^{-1}(0)$, so that $J^{-1}(0)$ is an $(n+1)$-dimensional submanifold of $\dot{\mathbb{R}}^n \times \mathbb{R}^n$. From (5.11) we have

$$J^{-1}(0) = \{(\underline{q},s\underline{q})\,|\,\underline{q} \neq 0,\ s \in \mathbb{R}\}, \tag{5.12}$$

which shows that $J^{-1}(0)$ is a real line bundle over $\dot{\mathbb{R}}^n$. This bundle has a global nonvanishing section $\underline{q} \mapsto (\underline{q},\underline{q})$ so that in fact

$$J^{-1}(0) = \dot{\mathbb{R}}^n \times \mathbb{R}. \tag{5.13}$$

To reduce $J^{-1}(0)$, first note that the action of $\mathrm{SO}(n)$ on $\dot{\mathbb{R}}^n \times \mathbb{R}$ induced by (5.13), (5.12), and (5.10) is just $(\underline{q},s) \mapsto (A\underline{q},s)$. Then (5.13) and (5.9) imply that

$$J^{-1}(0)/\mathrm{SO}(n) \approx \mathbb{R}^+ \times \mathbb{R},$$

with projection $\pi_0(\underline{q},s\underline{q}) = (\|\underline{q}\|,s\|\underline{q}\|)$. Now fix $\check{\underline{q}} \in \dot{\mathbb{R}}^n$ with $\|\check{\underline{q}}\| = 1$. The map i_0: $\mathbb{R}^+ \times \mathbb{R} \rightarrow J^{-1}(0) \subset \dot{\mathbb{R}}^n \times \mathbb{R}^n$, defined by

$$i_0(r,s) = (r\check{\underline{q}},s\check{\underline{q}}), \tag{5.14}$$

is a section of π_0. Then from (3.5)

$$i_0^*\omega = i_0^*\left(\sum_{i=1}^{n} dp_i \wedge dq^i\right)$$

$$= \sum_{i=1}^{n} d(r\check{q}^i) \wedge d(s\check{q}^i)$$

$$= (\check{q}\cdot\check{q})\, dr\wedge ds$$
$$= dr\wedge ds\,,$$

which is the canonical symplectic structure on $T^*\mathbb{R}^+ = \mathbb{R}^+\times\mathbb{R}$. We have therefore shown that the reduced phase space $(J^{-1}(0)/\mathrm{SO}(n),\bar{\omega}_0)$ is just $(T^*(\dot{\mathbb{R}}^n/\mathrm{SO}(n)),\bar{\omega})$, i.e., the conclusions of the Kummer–Marsden–Satzer Theorem hold despite the fact that Φ is not free (see also Ref. 20).

Turn now to prequantization. Since $\dot{\mathbb{R}}^n$ is simply connected for $n>2$, the prequantization line bundle is unique and trivial. But $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n>2$, so we must check the holonomy of L_0. Let $(q,p)\in J^{-1}(0)$ and consider the orbit $\mathrm{SO}(n)\cdot(q,p)$. As $\mathrm{SO}(n)\cdot(q,p)\approx S^{n-1}$ is simply connected for $n>2$, it follows from Proposition (3.4) that $T^*\Phi$ lifts horizontally to L_0. Thus $\bar{L}_0$ exists and, since the reduced phase space is contractible, $\bar{L}_0$ is also trivial.

For the metalinear structure, the facts that $\dot{\mathbb{R}}^n$ is orientable and simply connected for $n>2$ imply that $\widetilde{FV}$ exists and is unique. Since FV is trivial so is $\widetilde{FV}$. By the remarks following Proposition (3.7) and Theorem (3.8), $T^*\bar{Q}$ is metalinear. Again, since $\mathbb{R}^+\times\mathbb{R}$ is contractible, $\widetilde{\bar{F}\bar{V}}$ is trivial.

Quantizing this system, we have from Sec. III A that the extended wave functions are

$$\Psi = \psi(q)\lambda\otimes\nu_{\tilde{I}} \tag{5.15}$$

and from Sec. IV C that $\mathscr{h} = L^2(\dot{\mathbb{R}}^n)$. Using (4.2) and (5.11) we compute

$$\mathscr{Q}J_\xi[\Psi] = -i\hbar\left(\sum_{i,j=1}^{n}\xi_{ij}\left(q^i\frac{\partial}{\partial q^j} - q^j\frac{\partial}{\partial q^i}\right)\psi(q)\right)\lambda\otimes\nu_{\tilde{I}}\,, \tag{5.16}$$

for $\xi\in\Lambda^2(\mathbb{R}^n)$. Thus the rotationally invariant states look like

$$\Psi = \psi(\|q\|)\lambda\otimes\nu_{\tilde{I}} \tag{5.17}$$

and, in hyperspherical coordinates, the induced inner product on $\mathscr{h}_0$ is

$$(\Psi,\Phi)_0 = \mathrm{vol}(S^{n-1})\int_0^\infty \psi(r)\phi^*(r)r^{n-1}\,dr \tag{5.18}$$

[compare (4.17)]. Hence $\mathscr{h}_0 = L^2(\mathbb{R}^+,r^{n-1})$.

Similarly, the space $\bar{\mathscr{h}}_0 = L^2(\mathbb{R}^+)$ consists of states

$$\bar{\Psi} = \bar{\psi}(r)\bar{\lambda}_0\otimes\bar{\nu}_{\partial/\partial s}\,. \tag{5.19}$$

Since the group action is not free, we have no set technique as in Sec. IV for constructing an isomorphism $\mathscr{h}_0\to\bar{\mathscr{h}}_0$. Nonetheless, it is clear from (5.17)–(5.19) that

$$\psi(r)\lambda\otimes\nu_{\tilde{I}}\mapsto\sqrt{\mathrm{vol}(S^{n-1})}\; r^{(n-1)/2}\;\psi(r)\bar{\lambda}_0\otimes\bar{\nu}_{\partial/\partial s} \tag{5.20}$$

defines a unitary isomorphism $\mathscr{U}$ of $L^2(\mathbb{R}^+,r^{n-1})$ with $L^2(\mathbb{R}^+)$. Thus we have unitary equivalence.

Now consider, for instance, the radial momentum $p_r = (q\cdot p)/\|q\|$. It is $\mathrm{SO}(n)$-invariant and the reduced observable is $i_0^* p_r = s$. Since p_r preserves V both p_r and s are quantizable. From (3.9) and (3.4)–(3.6) we compute

$$\mathscr{Q}p_r[\Psi] = -\left(i\hbar\left\{\frac{1}{\|q\|}(q\cdot\nabla) + \frac{n-1}{2\|q\|}\right\}\psi(q)\right)\lambda\otimes\nu_{\tilde{I}}\,,$$

for Ψ given by (5.15). On $\mathscr{h}_0 = L^2(\mathbb{R}^+,r^{n-1})$ in hyperspherical coordinates, $\mathscr{Q}p_r$ takes the form

$$\mathscr{Q}p_r[\psi(r)] = -i\hbar\left(\frac{d}{dr} + \frac{n-1}{2r}\right)\psi(r)\,.$$

Likewise, on $\bar{\mathscr{h}}_0 = L^2(\mathbb{R}^+)$,

$$\bar{\mathscr{Q}}s[\bar{\psi}(r)] = -i\hbar\frac{d}{dr}\bar{\psi}(r)\,.$$

It is routine to verify that the unitary map (5.20) intertwines $\mathscr{Q}p_r$ and $\bar{\mathscr{Q}}s$ according to

$$-i\hbar\left(\frac{d}{dr} + \frac{n-1}{2r}\right) = \mathscr{U}^{-1}\left(-i\hbar\frac{d}{dr}\right)\mathscr{U}\,.$$

Theorem (4.3) therefore holds when $n>2$, at least for the radial momentum observable.

When $n=2$ the action Φ is free and we may apply all our previous results. Other than this, the main difference between the cases $n=2$ and $n>2$ is that, since SO(2) is Abelian, *every* $\mu\in\mathrm{so}(2)^*\approx\mathbb{R}$ is invariant.

Consider the standard connection

$$\alpha = (1/\|q\|^2)\,(q\wedge dq)\,, \tag{5.21}$$

on $\dot{\mathbb{R}}^2 = \mathrm{SO}(2)\times\mathbb{R}^+$. Since α is flat, Kummer–Marsden–Satzer reduction implies that the reduced phase space is $(\mathbb{R}^+\times\mathbb{R},\bar{\omega})$ as before. Composing $i_0\colon\mathbb{R}^+\times\mathbb{R}\to J^{-1}(0)$ given by (5.14) with the SO(2)-equivariant diffeomorphism $\delta_\mu\colon J^{-1}(0)\to J^{-1}(\mu)$ given by (2.5) defines a global section

$$i_\mu(r,s) = (r\check{q},s\check{q} + \mu\alpha(r\check{q})) \tag{5.22}$$

of $\pi_\mu = \pi_0\circ\delta_\mu^{-1}$. Hence

$$J^{-1}(\mu) = \mathrm{SO}(2)\times\mathbb{R}^+\times\mathbb{R}$$

is trivial as a principal SO(2)-bundle.

Now, $H^2(\dot{\mathbb{R}}^2\times\mathbb{R}^2,\mathbb{Z}) = 0$ so that again the prequantization line bundle is unique and trivial. Letting $[c(t)]$ be the generator of $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ and using (3.12) and (3.13), we find that the holonomy of L_μ is $\exp((2\pi i/\hbar)\mu)$. Proposition (3.4) then gives rise to a *quantization condition*: L_μ is reducible iff $\mu = m\hbar$ for some integer m. When $\mu = m\hbar$, $\bar{L}_\mu$ is trivial as before.

We must also be careful with the metalinear structures. Since $H^1(\dot{\mathbb{R}}^2\times\mathbb{R}^2,\mathbb{Z}_2) = \mathbb{Z}_2$ there are *two* metalinear frame bundles for V. On the other hand, there is exactly *one* (necessarily trivial) metalinear structure on the reduced phase space. This indicates that one of the metalinear frame bundles on $\dot{\mathbb{R}}^2\times\mathbb{R}^2$ will not project to $\mathbb{R}^+\times\mathbb{R}$ and hence will lead to a spurious quantization.

To construct these metalinear structures, introduce polar coordinates (r,θ) on $\dot{\mathbb{R}}^2$ and set

$$U_+ = \{(r,\theta)\,|\,0<\theta<2\pi\},\quad U_- = \{(r,\theta)\,|\,-\pi<\theta<\pi\}$$

and

$$W_+ = \{(r,\theta)\,|\,0<\theta<\pi\},\quad W_- = \{(r,\theta)\,|\,-\pi<\theta<0\}\,.$$

Since FV is trivial, the transition functions $\tilde{M}_\pm\colon W_\pm\times\mathbb{R}^2\to\mathrm{ML}(2,\mathbb{R})$ for the two $\widetilde{FV}$ are

$$\tilde{M}_+ = \tilde{I},\quad \tilde{M}_- = \tilde{I}\,, \tag{5.23}$$

corresponding to the identity of $H^1(\dot{\mathbb{R}}\times\mathbb{R}^2,\mathbb{Z}_2)$, and

$$\tilde{M}_+ = \tilde{I},\quad \tilde{M}_- = \epsilon\,, \tag{5.24}$$

corresponding to its generator, where

$$\epsilon = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}.$$

The metalinear frame bundle defined by (5.23) is trivial and the SO(2)-action on FV lifts to this $\widetilde{F}V$ just as when $n > 2$. It is this metalinear structure which projects to the reduced phase space. For the second metalinear frame bundle, it is clear from (5.24) that both $\widetilde{F}V$ and the natural homomorphism $\chi_{SO(2)}: \mathbb{Z} \to \mathbb{Z}_2$ are nontrivial. It follows from Proposition (3.7) that the SO(2)-action does *not* lift to this $\widetilde{F}V$ which therefore does *not* project to $\mathbb{R}^+ \times \mathbb{R}$.

Let $\underline{b} = (\partial/\partial p_r, \partial/\partial p_\theta)$ be a global frame field for V and fix a lift $\bar{\underline{b}}$ of $\underline{b}$ to the trivial metalinear frame bundle. From (4.7) every polarized section Ψ of $L \otimes \sqrt{\wedge^2 V}$ can be written

$$\Psi = \check{\psi}(r,\theta)\lambda \otimes \nu_{\bar{b}} . \qquad (5.25)$$

Using (5.16) with $\xi = 1$, the quantum constraint $\mathscr{Q}J[\Psi] = m\hbar\Psi$ for $\mu = m\hbar$ becomes

$$-i\hbar \frac{\partial}{\partial \theta} \check{\psi}(r,\theta) = m\hbar\check{\psi}(r,\theta) . \qquad (5.26)$$

Thus the physically admissible states take the form $\check{\psi}(r,\theta) = k(r)e^{im\theta}$ consistent with (4.9).

The reduced phase space quantization proceeds exactly as before. The reduced wave functions are given by (5.19) and the isomorphism (4.11) becomes $k(r)e^{im\theta} \mapsto k(r)$.

When $\mu = 0$ these results correlate exactly with those obtained earlier for $n > 2$. The only difference is that here we have used the half-form $\nu_{\bar{b}}$ rather than $\nu_{\bar{l}}$ which, according to (3.2), satisfies $\nu_{\bar{l}} = \sqrt{r}\,\nu_{\bar{b}}$. Writing Ψ given by (5.25) in the form (5.15) we have that $\psi(r,\theta) = \sqrt{r}\check{\psi}(r,\theta)$. With this change of notation, (5.20) is just (4.19) and all our previous results immediately carry over to the case $n = 2$.

Remarks: (1) It is interesting to see what happens when we quantize the extended phase space using the *nontrivial* metalinear structure. Let $\tilde{\underline{b}}_\pm$ be lifts of $\underline{b}$ to this $\widetilde{F}V$ over $U_\pm \times \mathbb{R}^2$; then from (5.24),

$$\tilde{\underline{b}}_- = \begin{cases} \tilde{\underline{b}}_+, & \text{on } W_+ \times \mathbb{R}^2, \\ \tilde{\underline{b}}_+\epsilon, & \text{on } W_- \times \mathbb{R}^2. \end{cases} \qquad (5.27)$$

Defining local sections $\nu_\pm$ of $\sqrt{\wedge^2 V}$ over $U_\pm \times \mathbb{R}^2$ by $\nu_\pm^\# (\tilde{\underline{b}}_\pm) = 1$, it follows from (5.27) and (3.2) that $\nu_- = \pm \nu_+$ on $W_\pm \times \mathbb{R}^2$, respectively. The quantum wave functions are now

$$\Psi | (U_\pm \times \mathbb{R}^2) = \check{\psi}_\pm (r,\theta)\lambda \otimes \nu_\pm ,$$

where

$$\check{\psi}_-(r,\theta) = \pm \check{\psi}_+(r,\theta) \qquad (5.28)$$

on $W_\pm$. Such a Ψ satisfies the quantum constraint (5.26) iff

$$\check{\psi}_\pm (r,\theta) = k_\pm (r) e^{im\theta}$$

and (5.28) then implies that $k_-(r) = \pm k_+(r)$ on $W_\pm$, which forces $k_\pm (r) \equiv 0$. Thus, when the nontrivial metalinear structure is used, $\mathscr{H}_\mu = \{0\}$ and we have a spurious quantization.

(2) Earlier we showed that the extended and reduced phase space quantizations of the radial momentum p_r were unitarily related. Of course, this is not really surprising and was in fact guaranteed by our theorems when $n = 2$. It is therefore very curious that the same is *not* true for a rotationally invariant Hamiltonian *except* when $n = 3$.

Set $\mu = 0$ and let the Hamiltonian be

$$h(\underline{q},\underline{p}) = \tfrac{1}{2} \|\underline{p}\|^2 + V(\|\underline{q}\|) .$$

The reduced Hamiltonian on $\mathbb{R}^+ \times \mathbb{R}$ is

$$\bar{h}_0(r,s) = \tfrac{1}{2} s^2 + V(r) .$$

Neither of these is polarization preserving but may nonetheless be quantized using Blattner–Kostant–Sternberg kernels (cf. Chap. 6 of Ref. 21).

On $L^2(\dot{\mathbb{R}}^n)$ we compute

$$\mathscr{Q}h [\Psi] = \{[-(\hbar^2/2)\Delta + V(r)]\psi(\underline{q})\}\lambda \otimes \nu_{\bar{l}} .$$

In hyperspherical coordinates this reduces to

$$\mathscr{Q}h [\Psi] = \left\{\left[-\frac{\hbar^2}{2}\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}\right) + V(r)\right]\psi(r)\right\}\lambda \otimes \nu_{\bar{l}} ,$$

for $\Psi \in \mathscr{k}_0$. Similarly,

$$\bar{\mathscr{Q}}\bar{h}_0[\bar{\Psi}] = \left\{\left[-\frac{\hbar^2}{2}\frac{d^2}{dr^2} + V(r)\right]\bar{\psi}(r)\right\}\bar{\lambda}_0 \otimes \nu_{\bar{\partial}/\partial s}$$

on $L^2(\mathbb{R}^+)$. Using (5.20) it follows that

$$\mathscr{U}^{-1}(\bar{\mathscr{Q}}\bar{h}_0)\mathscr{U} = \mathscr{Q}h - \hbar^2(n-1)(n-3)/8r^2 .$$

Consequently, these two quantizations do not intertwine $\bar{\mathscr{Q}}\bar{h}_0$ and $\mathscr{Q}h$ unless $n = 3$. It would be nice to understand the underlying geometric reason for this.

When $n = 2$ and $\mu = m\hbar$, (5.22) and (5.21) imply that the amended Hamiltonian $\bar{h}_\mu = i_\mu^* h$ is

$$\bar{h}_\mu = \bar{h}_0 + \mu^2/2r^2 .$$

Since now $\Psi = k(r)e^{im\theta}\lambda \otimes \nu_{\bar{l}}$, the above expressions for the quantum Hamiltonians must be modified by replacing $V(r)$ by $V(r) + m^2\hbar^2/2r^2$. But $\mathscr{Q}h$ and $\bar{\mathscr{Q}}\bar{h}_\mu$ are still not unitarily related.

(3) We have punctured $\mathbb{R}^n$ in order to avoid pathologies. If the origin is not excluded Φ is no longer even effective, $\mathbb{R}^n/SO(n)$ is not a manifold and $J^{-1}(0)$ will be singular. Our entire formalism then fails to apply. For a discussion of this case, see Refs. 25 and 26.

C. Kaluza–Klein electrodynamics

The Kaluza–Klein theory of a relativistic charged particle provides another nice illustration of our formalism. We present only a brief account here; for more details on this and related topics see Refs. 16, 18, 19, 21, and 27.

Let $\bar{Q}$ represent four-dimensional space-time. The configuration space for our charged particle is a left principal T-bundle $\pi_Q: Q \to \bar{Q}$, T being the multiplicative group of complex numbers of modulus one. We identify the Lie algebra of T with $\mathbb{R}$ by associating to each $e \in \mathbb{R}$ the one-parameter group

$$z \mapsto \exp\{i(e_0/\hbar)et\}z , \qquad (5.29)$$

where e_0 is a parameter which we interpret as the "elementary" charge.

Suppose Q carries a T-invariant metric g of signature $(+ + + - -)$. Define a connection form α on Q by

$$\alpha(v) = g(1_Q, v) ,$$

for all $v \in TQ$, where 1_Q is the fundamental vector field on Q

corresponding to $1 \in \mathbb{R}$. By Lemma (2.6) there exists a closed two-form F on $\overline{Q}$ such that

$$\pi_Q^* F = d\alpha .$$

We construe F as the electromagnetic field. Since F is the curvature of a circle bundle, the de Rham class $(e_0/h)\ [F]_{\overline{Q}}$ must be integral. This condition may be viewed as a restriction on the allowable interactions of a particle of charge e_0 with the electromagnetic field F in the Kaluza–Klein formalism. Finally we define the space-time metric $\bar{g}$ via

$$\operatorname{hor} g = \pi_Q^* \bar{g} .$$

The group action is by construction free and proper and every $e \in \mathbb{R}$ is invariant. We fix the charge of our particle by imposing the *charge constraint* $J = e$ on T^*Q. Kummer–Marsden–Satzer reduction then identifies the reduced phase space $(J^{-1}(e)/T, \overline{\omega}_e)$ with $(T^*\overline{Q}, \overline{\Omega}_e)$; here

$$\overline{\Omega}_e = \overline{\omega} + e\tau_{\overline{Q}}^* F$$

is just the *charged symplectic structure* on $T^*\overline{Q}$.

Let the prequantization line bundle be $L = T^*Q \times \mathbb{C}$. Mimicking the calculation in the previous example while taking the precise form of (5.29) into account, we find that the holonomy of L_e is $\exp(2\pi i(e/e_0))$. In the Kaluza–Klein framework, then, the lifting criteria become *superselection rules*: L_e is reducible iff the particle's charge e is an integral multiple of the elementary charge e_0. When $e = ne_0$, the induced line bundle $\overline{L}_e$ is also trivial. As an aside, notice how the integrality condition on $(e_0/h)\ [F]_{\overline{Q}}$ and the superselection rule $e = ne_0$ combine to guarantee the integrality of $(1/h)[\overline{\Omega}_e]_{T^*\overline{Q}} = (e/h)[F]_{\overline{Q}}$ as required for the quantizability of the reduced phase space.

Now assume that $\overline{Q}$, and hence Q, is orientable. Using Proposition (3.6) and the metric g, we reduce the structure group $GL(5,\mathbb{R})$ of FV to $SO(3,2)$. Now $SO(3,2)$ is isomorphic to the intersection of $\sigma^{-1}(SO(3,2))$ with the component of the identity in $ML(5,\mathbb{R})$. Thus the transition functions for FV, valued in $SO(3,2)$, can be lifted to $\sigma^{-1}\ (SO(3,2)) \subset ML(5,\mathbb{R})$ thereby defining a metalinear frame bundle $\widetilde{F}V$. The characteristic homomorphism of $\widetilde{F}V$ so defined is obviously trivial. This and the orientability of Q imply that the associated bundle $\sqrt{\ }\wedge^5 V$ of half-forms is trivial.

Proposition (3.7) and Theorem (3.8) guarantee that $\widetilde{F}V$ projects to a metalinear frame bundle $\widetilde{F}\overline{V}$ on $T^*\overline{Q}$, which is exactly that constructed in a similar fashion to $\widetilde{F}\overline{V}$ by reducing the structure group $GL(4,\mathbb{R})$ of $F\overline{V}$ to $SO(3,1)$ using the space-time metric $\bar{g}$. The half-form bundle $\sqrt{\ }\wedge^4\overline{V}$ is likewise trivial.

Set $\Psi = \psi\lambda \otimes \nu_g$, where ν_g is defined as follows. Fix a positively oriented orthonormal frame $\underline{b}$ for FQ, where b_5 is tangent to the fibers of π_Q, and denote also by $\underline{b}$ the corresponding frame in FV (cf. Sec. III E). Then let ν_g be such that $\nu_g^\#(\underline{\tilde{b}}) = 1$. It follows from Sec. IV C and the triviality of both L and $\sqrt{\ }\wedge^5 V$ that $\mathscr{H} = L^2(Q, \sqrt{\det g})$. Similarly, we have $\overline{\Psi} = \overline{\psi}\overline{\lambda}_e \otimes \overline{\nu}_{\bar{g}}$ for $\overline{\Psi} \in \mathscr{H}_e = L^2(\overline{Q}, \sqrt{\det \bar{g}})$.

In a chart (q,z) on Q, where the q are space-time coordinates and z is the T-coordinate, the quantum constraint

$$\mathscr{Q}J[\Psi] = ne_0\Psi$$

becomes

$$-i\hbar \frac{\partial}{\partial z}\psi(q,z) = ne_0\psi(q,z) .$$

Thus the Kaluza–Klein quantum state space for a particle with charge $e = ne_0$ consists of wave functions

$$\Psi = k(q)\ e^{(i/\hbar)ne_0 z}\lambda \otimes \nu_g .$$

Since T is compact, the Unitary Equivalence Theorem asserts that the correspondence

$$k(q)e^{(i/\hbar)ne_0 z}\lambda \otimes \nu_g \mapsto k(q)\overline{\lambda}_e \otimes \overline{\nu}_{\bar{g}}$$

defines a unitary isomorphism of $L^2(Q, \sqrt{\det g})$ with $L^2(\overline{Q}, \sqrt{\det \bar{g}})$.

Our theorems therefore guarantee that the quantizations of a relativistic charged particle with $e = ne_0$ in both the Kaluza–Klein formalism and the conventional space-time-based approach are unitarily equivalent. Since all ordinary polarization-preserving observables—viz., the position, linear and angular momenta—are T-invariant, Theorem (4.3) shows that they may be equally well quantized in either formalism.

VI. DISCUSSION

We have proven theorems to the effect that one can quantize either the extended or reduced phase space of a constrained cotangent system with unitarily equivalent results. The examples in the previous section demonstrate the utility of our formalism. Here we briefly overview our constructions and conclusions with an eye to possible generalizations and improvements.

We begin by reexamining the conditions under which our formalism operates. These are (1) G must admit a bi-invariant metric and the action of G on Q must be free and proper, (2) $\mu \in \mathscr{g}^*$ must be invariant, and (3) the geometric quantization structures must be G-invariant.

Regarding (1), the only really severe restriction is that the action be free. In fact, virtually all our results are predicated upon this assumption although, as the $n > 2$ angular momentum example shows, our theorems *may* be valid without it. One might try to weaken this hypothesis as in Montgomery,[20] but it is not clear to what extent this is workable.

As noted earlier, condition (2) serves a dual purpose. It guarantees classically that the reduction of a cotangent bundle is again a cotangent bundle and quantum mechanically that one obtains a representation of $\mathscr{g}$ on $\mathscr{H}$. The possibility that the reduced phase space is not a cotangent bundle is not a problem in principle, although one then of course loses much of the structure that so simplified our formalism. On the quantum level the noninvariance of μ would not necessarily be a disaster either, since one can *always* find another extended phase space in which the constraint set is imbedded coisotropically.[7] One can then consistently quantize this new constrained system, but the price is that one will lose both the group-theoretical and cotangent bundle structures in the process.

Our last condition (3) on the invariance of the quantization structures is vital. As the examples show, one either cannot quantize or obtains spurious quantizations if the G-action does not lift appropriately to both the prequantization line and metalinear frame bundles. It would be interesting to know if every such structure on the reduced phase space arises by projection from an invariant one on the extended phase space and, conversely, whether every invariant such structure on T^*Q is the pullback of a compatible one on $T^*\bar{Q}$.

To what extent can our results be expected to carry over to more general settings? The simplest modification to our framework is to allow for other types of polarizations. This should not cause too much difficulty provided P is G-invariant, has simply connected leaves and intersects $J^{-1}(\mu)$ sufficiently regularly. Two distinguished possibilities are polarizations P which satisfy either

$$P \cap TJ^{-1}(\mu) = \{0\}$$

or

$$P|J^{-1}(\mu) \subseteq TJ^{-1}(\mu) .$$

One could also consider nonreal polarizations.

Of critical importance, however, is that the polarization be chosen in such a way that every quantum wave function is uniquely determined by its restriction to the constraint set. In essence, this means that the extended phase space quantization must be totally insensitive to what happens "off" $J^{-1}(\mu)$. This requirement seems reasonable, since in principle only those classical states contained in the constraint set are physically permissible and/or relevant. For further discussion of these matters, see Ref. 7. In any case, this condition played a key role in the proof of the Smooth Equivalence Theorem. Without it there is no effective way to properly correlate the extended and reduced phase space quantizations which will then, in general, be wildly incompatible. An interesting—and physically meaningful—illustration of the consequences of violating this condition is given by Ashtekar and Horowitz.[5] An even more bizarre example is studied in Gotay.[28]

The next step is to consider arbitrary constrained systems with or even without symmetry. The problem is now much more difficult since we cannot explicitly construct anything and no longer necessarily have at our disposal well-behaved polarizations. What is known in this general case is summarized in Refs. 7, 8, and 10–12.

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