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ON QUANTIZING T^*S^1

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In this paper we continue our study of Groenewold–Van Hove obstructions to quantization. We show that there exists such an obstruction to quantizing the cylinder T^*S^1. More precisely, we prove that there is no quantization of the Poisson algebra of T^*S^1 which is irreducible on a naturally defined $\mathrm{e}(2) \times \mathbb{R}$ Lie subalgebra. Furthermore, we determine the maximal "polynomial" Lie subalgebras that can be consistently quantized, and completely characterize the quantizations thereof. This example provides support for one of the conjectures in [1], but disproves part of another. Passing to coverings, we also derive a no-go result for $\mathbb{R}^2$ which is comparatively stronger than those originally found by Groenewold [2] and Van Hove [3].

1. Introduction

Let M be a symplectic manifold and $\mathcal{P}(M)$ its associated Poisson algebra. In [1] we conjectured that:

Let $\mathcal{B} \subset \mathcal{P}(M)$ be a "basic set" of observables, such that the Lie algebra it generates is finite-dimensional. Then there is no nontrivial strong quantization of $(\mathcal{P}(M), \mathcal{B})$.

To understand this, we recall the basic definitions; motivation for these can be found in [1].

DEFINITION 1. A *basic set* of observables $\mathcal{B}$ is a linear subspace of $\mathcal{P}(M)$ such that:
- $\mathcal{B}$ is finite-dimensional,
- the Hamiltonian vector fields X_f, $f \in \mathcal{B}$, are complete,

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- $\{X_f \mid f \in \mathcal{B}\}$ span the tangent spaces to M everywhere,
- $1 \in \mathcal{B}$,
- $\mathcal{B}$ is a minimal space satisfying these requirements.

A basic set typically consists of the components of the momentum map for a transitive Hamiltonian action of a finite-dimensional Lie group on M.

DEFINITION 2. Let $\mathcal{B}$ be a basic set, and let $\mathcal{O}$ be a Lie subalgebra of $\mathcal{P}(M)$ containing $\mathcal{B}$. Then a *quantization* of $(\mathcal{O}, \mathcal{B})$ is a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the algebra of symmetric operators which preserve some fixed domain D in some Hilbert space, such that for all $f, g \in \mathcal{O}$,

- $\mathcal{Q}(\{f, g\}) = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$,
- $\mathcal{Q}(1) = I$,
- if X_f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D,
- $\mathcal{Q}(\mathcal{B}) = \{\mathcal{Q}(f) \mid f \in \mathcal{B}\}$ is an irreducible set,
- D contains a dense set of separately analytic vectors for some basis of $\mathcal{Q}(\mathcal{B})$.

A quantization $\mathcal{Q}$ is *strong* if in addition D contains a dense set of separately analytic vectors for some Lie generating basis of $\mathcal{Q}(\mathcal{N}_{\mathcal{O}}(\mathcal{B}))$, where $\mathcal{N}_{\mathcal{O}}(\mathcal{B})$ denotes the normalizer in $\mathcal{O}$ of the Lie subalgebra generated by $\mathcal{B}$. Finally, since a quantization $\mathcal{Q}$ factors through a representation of $\mathcal{O}/\ker\mathcal{Q}$, we say that $\mathcal{Q}$ is *trivial* whenever $\operatorname{codim}\ker\mathcal{Q} \leq 1$.

All examples which have been analyzed to date validate the conjecture above; in particular, $\mathbb{R}^{2n}$ with the basic set

$$\mathcal{B} = \operatorname{span}\{1, q^i, p_i \mid i = 1, \ldots, n\}$$

[2, 3], and S^2 with

$$\mathcal{B} = \operatorname{span}\{1, S_1, S_2, S_3\},$$

where the S_i are the components of the spin vector [4]. In both cases the basic sets are already Lie algebras. On the other hand, there does exist a nontrivial strong quantization of T^2 with any of the basic sets

$$\mathcal{B}_k = \operatorname{span}\{1, \sin k\theta, \cos k\theta, \sin k\phi, \cos k\phi\}$$

for k a positive integer [5]. But the Lie algebras generated by the $\mathcal{B}_k$ are infinite-dimensional. In a sense, $\mathcal{B}_1$ is the toral analogue of the basic set for $\mathbb{R}^2$.

Given this dichotomy, a natural example with which to test the conjecture is the cylinder T^*S^1, since it is topologically "halfway" between $\mathbb{R}^2$ and T^2. Endow the cylinder with the canonical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \ell}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \ell},$$

where ℓ is the angular momentum conjugate to θ. While the symplectic self-action of T^*S^1 is not Hamiltonian (thinking of T^*S^1 as $T \times \mathbb{R}$, where T is the circle group), the cylinder can nonetheless be realized as a coadjoint orbit of the special Euclidean group SE(2).[1] The corresponding momentum map $T^*S^1 \to \mathrm{e}(2)^*$ has components $\{\ell, \sin\theta, \cos\theta\}$. Together with the constant function 1, these components span the basic set

$$\mathcal{B} = \mathrm{span}\{1, \ell, \sin\theta, \cos\theta\}.$$

Algebraically, $\mathcal{B}$ is the cylindrical analogue of those for $\mathbb{R}^2$ and T^2.

In this paper we show that the conjecture holds for the cylinder: there is no quantization of $\mathcal{P}(T^*S^1)$—strong or otherwise—which is irreducible when restricted to $\mathcal{B}$. However, there do exist quantizations of certain "polynomial" Lie subalgebras of $\mathcal{P}(T^*S^1)$; these are discussed and characterized in Section 3. Finally, we "lift" our results to $\mathbb{R}^2$, thereby producing a no-go result which is the strongest yet obtained for $\mathbb{R}^2$.

2. The obstruction

Our first task is to determine all possible quantizations of the basic set $\mathcal{B} \cong \mathrm{e}(2) \times \mathbb{R}$. According to the definition a quantization of $\mathcal{B}$ in this instance amounts to a Lie algebra representation $\mathcal{Q}$ by essentially self-adjoint operators on a common invariant dense domain in a Hilbert space which is both irreducible and integrable. Thus it suffices to compute the derived representations corresponding to the unitary irreducible representations ("UIRs") of the universal covering group of SE(2) $\times\, \mathbb{R}$.

Now, the universal covering group of SE(2) is the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ with the composition law

$$(t, x, y) \cdot (t', x', y') = (t + t', x' \cos t + y' \sin t + x, y' \cos t - x' \sin t + y).$$

Fortunately, it is straightforward to determine the UIRs of this group. From the theory of induced representations of semi-direct products [7] (see also [8, §5.8]), we compute that these representations are of two types:

(i) $\big(U(t, x, y)\psi\big)(\theta) = e^{i\lambda(x\cos\theta + y\sin\theta)} e^{i\nu t}\psi(\theta + t)$ on $L^2(S^1)$, and

(ii) $U(t, x, y)z = e^{i\mu t} z$ on $\mathbb{C}$.

Here λ, ν, μ are real parameters satisfying $\lambda > 0$ and $0 \le \nu < 1$. The corresponding derived representations are[2]

(i)′ $\mathcal{Q}(\ell) = -i\dfrac{d}{d\theta} + \nu I$, $\mathcal{Q}(\sin\theta) = \lambda\sin\theta$, $\mathcal{Q}(\cos\theta) = \lambda\cos\theta$ on $C^\infty(S^1, \mathbb{C}) \subset L^2(S^1)$, and

(ii)′ $\mathcal{Q}(\ell) = \mu$, $\mathcal{Q}(\sin\theta) = 0$, $\mathcal{Q}(\cos\theta) = 0$ on $\mathbb{C}$.

Thus the required representations of $\mathrm{e}(2) \times \mathbb{R}$ are given by (i)′ and (ii)′ supplemented by the condition $\mathcal{Q}(1) = I$. The parameter λ can be identified with the reciprocal of Planck's reduced constant,[3] which we take to be one.

[1] This fact is important in geometric optics, cf. [6, §17].

[2] We denote multiplication operators as functions.

[3] See [8, §4.6] for a discussion. There is an error here, however; λ should be identified with $\hbar^{-1}$, and not $\hbar^2$.

Rather than consider the entire Poisson algebra $\mathcal{P}(T^*S^1)$, we will focus on the Poisson subalgebra P of polynomials in elements of $\mathcal{B}$, i.e., sum of multiples of terms of the form

$$\ell^r \sin^m \theta \cos^n \theta$$

with r, m, n nonnegative integers. We remark in passing that P is dense in $C^\infty(T^*S^1)$, where the latter is given the topology of uniform convergence on compacta of a function and its derivatives. Let P^r be the subspace thereof consisting of polynomials which are at most degree r in ℓ, and P_r those which are homogeneous of degree r in ℓ. A short calculation shows that the (Lie) normalizer of $\mathcal{B}$ in both P and $\mathcal{P}(T^*S^1)$ is just itself, so that any quantization of $(P, \mathcal{B})$ or $\big(\mathcal{P}(T^*S^1), \mathcal{B}\big)$ is automatically strong.

Now suppose there existed a quantization $\mathcal{Q}$ of $(P, \mathcal{B})$ on some common invariant dense domain D in an infinite-dimensional Hilbert space $\mathcal{H}$. Arguing as in the proof of [1, Proposition 2], we may assume that $D = C^\infty(S^1, \mathbb{C})$ in $\mathcal{H} = L^2(S^1)$, so that $\mathcal{Q}$ restricted to $\mathcal{B}$ is given by (i)$'$. To begin, we generate some "von Neumann rules."

PROPOSITION 1. *$\mathcal{Q}(\ell^2) = \mathcal{Q}(\ell)^2 + b\mathcal{Q}(\ell) + cI$, where $b, c \in \mathbb{R}$ are arbitrary.*

Proof: First observe that $L^2(S^1)$ has an orthonormal basis $\{|n\rangle \mid n \in \mathbb{Z}\}$ of eigenvectors of $\mathcal{Q}(\ell)$, where $|n\rangle = \frac{1}{\sqrt{2\pi}} e^{in\theta}$. Thus $\mathcal{Q}(\ell)$ has spectrum $\{n + \nu \mid n \in \mathbb{Z}\}$, and the multiplicity of each eigenvalue is 1. Hence, if a normal operator commutes with $\mathcal{Q}(\ell)$ on $C^\infty(S^1, \mathbb{C})$, then it must be a function of $\mathcal{Q}(\ell)$.

Also observe that

$$\mathcal{Q}(\cos\theta)|n\rangle = \frac{1}{2}\big(|n+1\rangle + |n-1\rangle\big) \tag{1}$$

and

$$\mathcal{Q}(\sin\theta)|n\rangle = \frac{1}{2i}\big(|n+1\rangle - |n-1\rangle\big). \tag{2}$$

Now set $\Delta = \mathcal{Q}(\ell^2) - \mathcal{Q}(\ell)^2$. Then $[\Delta, \mathcal{Q}(\ell)] = 0$, so that Δ is a function of $\mathcal{Q}(\ell)$, say $\Delta = \xi\big(\mathcal{Q}(\ell)\big)$, and we want to compute ξ. Since

$$\xi\big(\mathcal{Q}(\ell)\big)|n\rangle = \xi(n + \nu)|n\rangle \tag{3}$$

and $\{|n\rangle\}$ span $L^2(S^1)$, it suffices to determine the sequence $\{\xi(n+\nu) \mid n \in \mathbb{Z}\}$.

Let us quantize the Poisson bracket identity

$$\big\{\{\ell^2, \sin\theta\}, \sin\theta\big\} + \big\{\{\ell^2, \cos\theta\}, \cos\theta\big\} = 2 \tag{4}$$

to get

$$\big[[\mathcal{Q}(\ell^2), \mathcal{Q}(\sin\theta)], \mathcal{Q}(\sin\theta)\big] + \big[[\mathcal{Q}(\ell^2), \mathcal{Q}(\cos\theta)], \mathcal{Q}(\cos\theta)\big] = -2,$$

whilst explicit calculation produces

$$\big[[\mathcal{Q}(\ell)^2, \mathcal{Q}(\sin\theta)], \mathcal{Q}(\sin\theta)\big] + \big[[\mathcal{Q}(\ell)^2, \mathcal{Q}(\cos\theta)], \mathcal{Q}(\cos\theta)\big] = -2.$$

Subtracting,

$$\big[[\Delta, \mathcal{Q}(\sin\theta)], \mathcal{Q}(\sin\theta)\big] + \big[[\Delta, \mathcal{Q}(\cos\theta)], \mathcal{Q}(\cos\theta)\big] = 0.$$

Denote the left-hand side of this equation by K. Now evaluate the matrix element $\langle n|K|n\rangle$ by substituting $\Delta = \xi(\mathcal{Q}(\ell))$. After a short computation using (1)–(3), we obtain the recursion relation

$$2\xi(n') - \xi(n'+1) - \xi(n'-1) = 0,$$

where $n' = n + \nu$. This has the solutions $\xi(n') = bn' + c$, where b, c are real as Δ is symmetric. Thus

$$\Delta = \xi(\mathcal{Q}(\ell)) = b\mathcal{Q}(\ell) + cI,$$

which yields the desired result. □

Next, we quantize the relations

$$\ell \sin\theta = -\tfrac{1}{2}\{\ell^2, \cos\theta\} \quad \text{and} \quad \ell\cos\theta = \tfrac{1}{2}\{\ell^2, \sin\theta\}$$

thereby obtaining

$$\mathcal{Q}(\ell\sin\theta) = \mathcal{Q}(\sin\theta)\mathcal{Q}(\ell) - \frac{i}{2}\mathcal{Q}(\cos\theta) + \frac{b}{2}\mathcal{Q}(\sin\theta) \tag{5}$$

and

$$\mathcal{Q}(\ell\cos\theta) = \mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + \frac{i}{2}\mathcal{Q}(\sin\theta) + \frac{b}{2}\mathcal{Q}(\cos\theta). \tag{6}$$

Then, quantizing the relation $\{\ell\cos\theta, \ell\sin\theta\} = \ell$, we conclude that $b = 0$ in the above. Finally, using (5), (6) and Proposition 1, we quantize

$$\ell^2\sin\theta = \tfrac{1}{2}\{\ell\cos\theta, \ell^2\} \quad \text{and} \quad \ell^2\cos\theta = -\tfrac{1}{2}\{\ell\sin\theta, \ell^2\}$$

to get

$$\mathcal{Q}(\ell^2\sin\theta) = \mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 - i\mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + \tfrac{1}{4}\mathcal{Q}(\sin\theta) \tag{7}$$

and

$$\mathcal{Q}(\ell^2\cos\theta) = \mathcal{Q}(\cos\theta)\mathcal{Q}(\ell)^2 + i\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell) + \tfrac{1}{4}\mathcal{Q}(\cos\theta). \tag{8}$$

Our main result is the following no-go theorem:

THEOREM 2. *There is no nontrivial quantization of* $(P, \mathcal{B})$.

Proof: We merely use (i)′ and the von Neumann rules (7) and (8) to quantize the bracket relation

$$2\left\{\{\ell^2\sin\theta, \ell^2\cos\theta\}, \cos\theta\right\} = 12\ell^2\sin\theta.$$

After simplifying, the left-hand side reduces to

$$12\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 - 12i\mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + 5\mathcal{Q}(\sin\theta),$$

whereas the right-hand side is

$$12\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 - 12i\mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + 3\mathcal{Q}(\sin\theta),$$

and the required contradiction is evident. □

This theorem holds for representations of type (i)$'$. But it is easy to see that there are no trivial quantizations of $(P, \mathcal{B})$ either, corresponding to representations of type (ii)$'$. Indeed, quantizing $\cos^2\theta = \frac{1}{2}\{\{\ell^2, \sin\theta\}, \sin\theta\}$ we obtain $\mathcal{Q}(\cos^2\theta) = 0$. Likewise, $\mathcal{Q}(\sin^2\theta) = 0$. But this is impossible:

$$I = \mathcal{Q}(1) = \mathcal{Q}(\cos^2\theta + \sin^2\theta) = \mathcal{Q}(\cos^2\theta) + \mathcal{Q}(\sin^2\theta) = 0.$$

Assembling the above results, we therefore have

COROLLARY 3. *There exists no quantization of* $\big(\mathcal{P}(T^*S^1), \mathcal{B}\big)$.

Thus $\big(\mathcal{P}(T^*S^1), \mathcal{B}\big)$ does indeed satisfy the conjecture of Section 1.

3. Quantizable subalgebras of observables

In view of the impossibility of quantizing $(P, \mathcal{B})$, one can ask for the maximal Lie subalgebras in P to which we can extend an irreducible representation of $\mathcal{B}$.

Such subalgebras certainly exist: For instance, there is a two-parameter family of quantizations of the pair $(P^1, \mathcal{B})$. They are the "position representations" on $C^\infty(S^1, \mathbb{C}) \subset L^2(S^1)$ given by

$$\mathcal{Q}_{\nu,\eta}\big(f(\theta)\ell + g(\theta)\big) = -if(\theta)\frac{d}{d\theta} + \left[\left(\eta - \frac{i}{2}\right) f'(\theta) + \nu f(\theta) + g(\theta)\right], \tag{9}$$

where ν labels the UIRs of the universal cover of SE(2) and η is real. (In this expression f, g are trigonometric polynomials. However, these quantizations can be extended to the case when f, g are arbitrary smooth functions on S^1.) Since P^1 is maximal (this is proven below), Corollary 3 implies that none of these quantizations can be extended beyond P^1 in P.

We now classify the maximal Lie subalgebras of P containing $\mathcal{B}$. First, we have

PROPOSITION 4. *P^1 is a proper maximal Lie subalgebra of P.*

We need a few preliminaries. Notice that we may equally well view P as consisting of sums of multiples of terms of types

$$\ell^r \sin m\theta, \quad \ell^r \cos n\theta, \quad \text{and} \quad \ell^r \sin m\theta \cos n\theta.$$

For each integer k define endomorphisms C_k and S_k of each P_r by

$$C_k(p) = \{\ell\cos k\theta, p\} \quad \text{and} \quad S_k(p) = \{\ell \sin k\theta, p\}.$$

LEMMA 1. *Each P_r is irreducible under the endomorphisms $\{C_k, S_k \mid k \in \mathbb{Z}\}$.*

Proof: Let S be an invariant subspace of P_r. Then its complexification $S_\mathbb{C} \subset (P_r)_\mathbb{C}$ is invariant under the ladder endomorphisms $L_k = C_k + iS_k$, $k \in \mathbb{Z}$. Set $e_m^r := \ell^r e^{im\theta}$; then

$$L_k(e_m^r) = i(m - kr)e_{m+k}^r. \tag{10}$$

We will show that $S_\mathbb{C} = (P_r)_\mathbb{C}$, whence $S = P_r$.

We assert that $S_{\mathbb{C}}$ contains a monomial. Indeed, given $p \in S_{\mathbb{C}}$ we may write

$$p = \sum_m a_m e_m^r.$$

Let M be any integer such that $a_M \neq 0$. Since $S_{\mathbb{C}}$ is invariant, it follows from (10) that if p is not already a monomial, then $p' = L_0(p) - iMp$ is nonvanishing and belongs to $S_{\mathbb{C}}$. But p' has one fewer term than p. Applying this procedure (which we refer to as the "elimination trick") to p' and continuing in this fashion, we eventually produce a monomial $e_N^r \in S_{\mathbb{C}}$.

Then, by successively applying the ladder endomorphisms L_k to e_N^r for various values of k, it is readily verified that we can obtain any other monomial $e_{N'}^r \in S_{\mathbb{C}}$. ▼

Proof of Proposition 4: Let $p \notin P^1$, and let R be the Lie algebra generated by p along with P^1. The degree of p in ℓ is $r > 1$. By bracketing p with $\cos\theta$ a total of $r-2$ times, we obtain an element of R which is quadratic in ℓ. Subtracting off the affine terms in ℓ—which belong to P^1—we obtain an element of $R \cap P_2$. Since both $\ell \sin k\theta$ and $\ell \cos k\theta$ belong to P^1, Lemma 1 implies that $P_2 \subset R$. Since $\{\ell^n, \ell^2 \cos\theta\} \in R \cap P_{n+1}$, Lemma 1 and induction yield $P_{n+1} \subset R$ for all $n > 1$, whence $P \subseteq R$. □

However, P^1 is not the only maximal Lie subalgebra of P containing $\mathcal{B}$. For each real number α, let W_α be the Lie subalgebra of $P_{\mathbb{C}}$ generated by

$$\left\{1, e_{\pm 1}^0, e_0^1, e_{2N+1}^2 + 2\alpha e_{2N+1}^1 \mid N \in \mathbb{Z}\right\},$$

where as before $e_m^r := \ell^r e^{im\theta}$. By construction W_α is totally real (i.e., $\overline{W_\alpha} = W_\alpha$), so it must be the complexification of a real subalgebra V_α. That each V_α is proper and maximal is established during the proof of Proposition 5 below.

First, we state a structural result concerning W_α, which follows from a consideration of Poisson brackets of elements of the form given above.

LEMMA 2. *For each r and N of opposite parity, there exists an element of W_α of the form*

$$e_N^r + r\alpha e_N^{r-1} + \text{l.d.t.},$$

where "l.d.t." *stands for lower degree terms in ℓ.*

With this observation, we can now prove

PROPOSITION 5. *P^1 and $V_\alpha, \alpha \in \mathbb{R}$, are the only proper maximal Lie subalgebras of P containing $\mathcal{B}$.*

Proof: Since all algebras under consideration here are real, we can manipulate their complexifications and then take real parts. It therefore suffices to prove that $P_{\mathbb{C}}^1$ and W_α are the only proper maximal totally real Lie subalgebras of $P_{\mathbb{C}}$ containing

$$\mathcal{B}_{\mathbb{C}} = \text{span}\left\{1, e_{\pm 1}^0, e_0^1\right\}.$$

The proof will proceed in several steps.

Step 1: Let $S \subset P$ be a proper maximal Lie subalgebra containing $\mathcal{B}$. If $S_{\mathbb{C}} \subset P^1_{\mathbb{C}}$ strictly, then $S_{\mathbb{C}}$ can't be maximal by Proposition 4. Thus either $S_{\mathbb{C}} = P^1_{\mathbb{C}}$ or $S_{\mathbb{C}} \not\subseteq P^1_{\mathbb{C}}$.

Suppose $S_{\mathbb{C}} \not\subseteq P^1_{\mathbb{C}}$, and consider any element of $S_{\mathbb{C}}$ which does not belong to $P^1_{\mathbb{C}}$. It must have degree at least 2 in ℓ. By repeatedly bracketing it with $e^0_{-1} \in \mathcal{B}_{\mathbb{C}}$, we obtain an element of $S_{\mathbb{C}} \cap P^2_{\mathbb{C}}$ of the form

$$\sum_m a_m e^2_m + \text{l.d.t.}$$

with $a_m \neq 0$ for at least one value of m. Since L_0 preserves $S_{\mathbb{C}}$ we may, by virtue of the elimination trick from the proof of Lemma 1, suppose that the coefficient of ℓ^2 in this expression is a trigonometric monomial. Thus we have

$$p := e^2_M + \text{l.d.t.} \in S_{\mathbb{C}}$$

for some fixed integer M. Since $S_{\mathbb{C}}$ is totally real, $\bar{p} \in S_{\mathbb{C}}$ as well.

Now compute

$$\{p, e^0_{\pm 1}\} = \pm 2ie^1_{M\pm 1} + \text{l.d.t.} \in S_{\mathbb{C}}.$$

By bracketing this expression with $\bar{p}$, we find that $p_{\pm 1} := e^2_{\pm 1} + \text{l.d.t.} \in S_{\mathbb{C}}$. Bracketing $p_{\pm 1}$ with $e^0_{\pm 1}$, we find that $e^1_{\pm 2} + \text{l.d.t.}$ belongs to $S_{\mathbb{C}}$. Further bracketing this last expression with $p_{\pm 1}$, we find that $p_{\pm 3} := e^2_{\pm 3} + \text{l.d.t.} \in S_{\mathbb{C}}$. Bracketing $p_{\pm 3}$ with $e^0_{\pm 1}$, we find that $e^1_{\pm 4} + \text{l.d.t.}$ belongs to $S_{\mathbb{C}}$. Continuing in this manner, we conclude that

$$e^2_{2N+1} + \text{l.d.t.} \in S_{\mathbb{C}} \tag{11}$$

and

$$e^1_{2N} + \text{l.d.t.} \in S_{\mathbb{C}} \tag{12}$$

for all integers N. Furthermore, by bracketing (12) with $e^0_{\pm 1}$, we have that

$$e^0_{2N+1} \in S_{\mathbb{C}} \tag{13}$$

for all integers N.

Step 2: We examine (11) and (12) more closely; we claim that these can be improved as follows: There exists a real number α such that

$$e^2_{2N+1} + 2\alpha e^1_{2N+1} \in S_{\mathbb{C}} \tag{14}$$

and

$$e^1_{2N} + \alpha e^0_{2N} \in S_{\mathbb{C}} \tag{15}$$

for each integer N, respectively.

First, we note the following useful result, which we refer to as the "bootstrap trick." Suppose that $e^0_{2K} \in S_{\mathbb{C}}$ for some $K \neq 0$. By bracketing e^0_{2K} with (12) we see that $e^0_{2K+2N} \in S_{\mathbb{C}}$ for all integers N. When combined with (13), this implies $P^0 \subset S_{\mathbb{C}}$, and then (12) yields $e^1_{2N} \in S_{\mathbb{C}}$ for all N, i.e., $(P_1)_{\mathbb{C}} \subset S_{\mathbb{C}}$. But altogether this implies

$P^1_{\mathbb{C}} \subset S_{\mathbb{C}}$, and Proposition 4 then forces $S_{\mathbb{C}} = P^1_{\mathbb{C}}$, contrary to assumption. Thus for no $N \neq 0$ can e^0_{2N} belong to $S_{\mathbb{C}}$.

If every $e^1_{2N} \in S_{\mathbb{C}}$, then (15) holds with $\alpha = 0$. So suppose that $e^1_{2L} \notin S_{\mathbb{C}}$ for some $L \neq 0$. Without loss of generality, we may assume that $L > 0$.[4] Then by virtue of (12), and taking into account (13), there must exist a polynomial

$$f^1_{2L} = e^1_{2L} + \sum_{K \neq 0} \alpha_{L,K} e^0_{2K} \in S_{\mathbb{C}}, \tag{16}$$

where $\alpha_{L,K} \neq 0$ for at least one value of K. Eliminating the top term in f^1_{2L}, we obtain:

$$f^0_{2L} := L_0\big(f^1_{2L}\big) - i2Lf^1_{2L} = 2i \sum_{K \neq 0} \alpha_{L,K}(K - L)e^0_{2K} \in S_{\mathbb{C}}.$$

If $\alpha_{L,K} \neq 0$ for some $K \neq L$, we may again use the elimination trick to remove every term in f^0_{2L} except the one corresponding to this value of K, thereby obtaining $e^0_{2K} \in S_{\mathbb{C}}$. The bootstrap trick then leads to a contradiction unless $\alpha_{L,K} = 0$ for all $K \neq L$, in which case (16) reduces to

$$f^1_{2L} = e^1_{2L} + \alpha_L e^0_{2L} \in S_{\mathbb{C}}, \tag{17}$$

where $\alpha_L := \alpha_{L,L} \neq 0$. We remark that α_L is uniquely determined by L. (Otherwise, upon subtracting two such f^1_{2L}, we would obtain $e^0_{2L} \in S_{\mathbb{C}}$, which would again lead to a contradiction.)

Now consider the quadratic term e^2_{2L+1}. It cannot belong to $S_{\mathbb{C}}$, since otherwise $\big\{e^2_{2L+1}, e^0_{-1}\big\} = -2ie^1_{2L}$ would also. From (11), then,

$$f^2_{2L+1} = e^2_{2L+1} + \sum_K \beta_{L,K} e^1_{2K+1} + \text{l.d.t.} \in S_{\mathbb{C}},$$

where we made use of (12). Now

$$\big\{f^2_{2L+1}, e^0_{-1}\big\} = -2i\Big(e^1_{2L} + \tfrac{1}{2}\sum_K \beta_{L,K} e^0_{2K}\Big) \in S_{\mathbb{C}}$$

and comparison with (17) along with the bootstrap trick gives $\beta_{L,K} = 2\alpha_L \delta_{K,L}$. Thus

$$f^2_{2L+1} = e^2_{2L+1} + 2\alpha_L e^1_{2L+1} + \text{l.d.t.}$$

But then

$$\big\{f^2_{2L+1}, e^0_1\big\} = 2i\big(e^1_{2L+2} + \alpha_L e^0_{2L+2}\big) \in S_{\mathbb{C}}.$$

If $e^1_{2L+2} \in S_{\mathbb{C}}$ then, to avoid a contradiction via the bootstrap trick, we must have $\alpha_L = 0$, which is impossible. Thus $e^1_{2L+2} \notin S_{\mathbb{C}}$, in which case comparison with (17) yields $\alpha_{L+1} = \alpha_L$.

[4] Since $S_{\mathbb{C}}$ is totally real, if $e^1_{2L} \notin S_{\mathbb{C}}$ then its conjugate $e^1_{-2L} \notin S_{\mathbb{C}}$ either.

A similar argument using e^2_{2L-1} in place of e^2_{2L+1} yields $e^1_{2L-2} \notin S_{\mathbb{C}}$ along with $\alpha_{L-1} = \alpha_L$, provided $L \neq 1$. Iterating, we obtain $e^1_{2N} + \alpha_L e^0_{2N} \in S_{\mathbb{C}}$ for all positive integers N. Analogously, starting with the conjugate e^1_{-2L} of e^1_{2L} (recall that $S_{\mathbb{C}}$ is totally real), we obtain $e^1_{2N} + \alpha_{-L} e^0_{2N} \in S_{\mathbb{C}}$ for all negative integers N. Comparing the bracket

$$\{f^2_{2L+1}, e^0_{-4L-1}\} = -2i(4L+1)(e^1_{-2L} + \alpha_L e^0_{-2L}) \in S_{\mathbb{C}}$$

with (17) and applying the bootstrap trick, we get $\alpha_{-L} = \alpha_L$. Thus (15) holds for all integers N with $\alpha := \alpha_L$. Finally, comparing the conjugate of f^1_{2L} with f^1_{-2L} gives $\overline{\alpha_L} = \alpha_{-L} = \alpha_L$, so α is real.

It is now a simple matter to prove (14). From the arguments above coupled with (13), we may write

$$f^2_{2N+1} = e^2_{2N+1} + 2\alpha e^1_{2N+1} + \sum_K \tau_{N,K} e^0_{2K} \in S_{\mathbb{C}}$$

for all integers N and some coefficients $\tau_{N,K}$. But now the elimination trick gives

$$i \sum_K (2K - 2N - 1)\tau_{N,K} e^0_{2K} \in S_{\mathbb{C}},$$

which leads to a contradiction unless $\tau_{N,K} = 0$ for all K.

Step 3: We first observe that according to (14)

$$W_\alpha \subseteq S_{\mathbb{C}} \tag{18}$$

for some $\alpha \in \mathbb{R}$. We will show that $S_{\mathbb{C}} \subseteq W_\alpha$.

Let $q \in S_{\mathbb{C}}$ be of degree r in ℓ, and suppose that $q \notin W_\alpha$. By Lemma 2, we know that $e^r_N + r\alpha e^{r-1}_N + \cdots \in W_\alpha$ for all N with opposite parity to r, and so by (18) we may eliminate all such combinations in q, thereby obtaining a polynomial $\tilde{q}$ which belongs to $S_{\mathbb{C}}$ but not W_α. Now either we can eliminate all terms of degree r in this manner, in which case $\tilde{q}$ has degree $\tilde{r} \leq r-1$, or else there is a term in $\tilde{q}$ of the form e^r_N where N and r have the same parity. In the latter instance, we may isolate this term using the elimination trick, and then bracket with e^0_{-1} r times to obtain $e^0_{N-r} \in S_{\mathbb{C}}$. Since N and r have the same parity, $N - r$ is even. If $N - r \neq 0$, then the bootstrap trick produces a contradiction. If $N = r$, then bracket instead with e^0_1 to obtain $e^0_{N+r} \in S_{\mathbb{C}}$ with $N + r$ even and nonzero.

We iterate this procedure, either encountering a contradiction at some point or finally ending up with a zeroth-degree polynomial $\tilde{q}' \in S_{\mathbb{C}}$ of the form $\sum_N \gamma_N e^0_{2N}$. Since $\tilde{q}' \notin W_\alpha$, $\gamma_M \neq 0$ for some $M \neq 0$. But the elimination trick can now be used to produce $e^0_{2M} \in S_{\mathbb{C}}$, which also yields a contradiction. Thus in all eventualities, the assumption that $q \notin W_\alpha$ produces a contradiction. It follows that $S_{\mathbb{C}} = W_\alpha$. □

In fact, other than P^1 and subalgebras thereof, this proof shows that the V_α are the *only* proper Lie subalgebras of P containing $\mathcal{B}$.

In contrast to P^1, we now show that there is *no* nontrivial quantization of V_α which represents $\mathcal{B}$ irreducibly. While the method of proof is the same as that of the no-go theorem for P in Section 2, we must make sure that all constructions take place in V_α. Upon replacing the identity (4) by

$$\big\{\{\ell^3 + 3\alpha\ell^2, \sin\theta\}, \sin\theta\big\} + \big\{\{\ell^3 + 3\alpha\ell^2, \cos\theta\}, \cos\theta\big\} = 6\ell + 6\alpha, \tag{19}$$

the proof of Proposition 1 can be immediately adapted to give

PROPOSITION 6. *$\mathcal{Q}(\ell^3 + 3\alpha\ell^2) = \mathcal{Q}(\ell)^3 + 3\alpha\mathcal{Q}(\ell)^2 + b'\mathcal{Q}(\ell) + c'I$, where $b', c' \in \mathbb{R}$ are arbitrary.*

This can be specialized further: Quantizing the bracket relations

$$(\ell^2 + 2\alpha\ell)\sin\theta = -\tfrac{1}{3}\{\ell^3 + 3\alpha\ell^2, \cos\theta\}$$

and

$$(\ell^2 + 2\alpha\ell)\cos\theta = \tfrac{1}{3}\{\ell^3 + 3\alpha\ell^2, \sin\theta\}$$

we obtain[5]

$$\begin{aligned}\mathcal{Q}\big((\ell^2 + 2\alpha\ell)\sin\theta\big) &= \mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 + \big(2\alpha\mathcal{Q}(\sin\theta) - i\mathcal{Q}(\cos\theta)\big)\mathcal{Q}(\ell)\\ &\quad + \frac{1+b'}{3}\mathcal{Q}(\sin\theta) - i\alpha\mathcal{Q}(\cos\theta)\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathcal{Q}\big((\ell^2 + 2\alpha\ell)\cos\theta\big) &= \mathcal{Q}(\cos\theta)\mathcal{Q}(\ell)^2 + \big(2\alpha\mathcal{Q}(\cos\theta) + i\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)\\ &\quad + \frac{1+b'}{3}\mathcal{Q}(\cos\theta) + i\alpha\mathcal{Q}(\sin\theta).\end{aligned} \tag{21}$$

Using these to quantize

$$\ell^3 + 3\alpha\ell^2 = \tfrac{1}{2}\big\{(\ell^2 + 2\alpha\ell)\cos\theta, (\ell^2 + 2\alpha\ell)\sin\theta\big\} - 2\alpha^2\ell,$$

we get

$$\mathcal{Q}(\ell^3 + 3\alpha\ell^2) = \mathcal{Q}(\ell)^3 + 3\alpha\mathcal{Q}(\ell)^2 + \frac{1+b'}{3}\mathcal{Q}(\ell) + \frac{1+b'}{3}\alpha I,$$

which is compatible with Proposition 6 iff $b' = \frac{1}{2}$ and $c' = \frac{\alpha}{2}$. Thus,

$$\mathcal{Q}(\ell^3 + 3\alpha\ell^2) = \mathcal{Q}(\ell)^3 + 3\alpha\mathcal{Q}(\ell)^2 + \tfrac{1}{2}\mathcal{Q}(\ell) + \tfrac{1}{2}\alpha I.$$

(As an aside, observe that fixing $b' = \frac{1}{2}$ here leads to an inconsistency with our calculations in Section 2, where $b = 0$. Indeed, comparing the expressions $(7) + 2\alpha \times (5)$ with (20) for $\mathcal{Q}\big((\ell^2 + 2\alpha\ell)\sin\theta\big)$, and $(8) + 2\alpha \times (6)$ with (21) for $\mathcal{Q}\big((\ell^2 + 2\alpha\ell)\cos\theta\big)$,

[5]These calculations were done using the *Mathematica* package *NCAlgebra* [9].

we see that they differ in the zeroth-degree terms in $\mathcal{Q}(\ell)$. We could equally well have used this discrepancy as a basis for the previous no-go result.)

Finally, using these von Neumann rules, we quantize

$$(\ell^4 + 4\alpha\ell^3 + 4\alpha^2\ell^2)\sin\theta = \tfrac{1}{3}\{(\ell^2 + 2\alpha\ell)\cos\theta, \ell^3 + 3\alpha\ell^2\}$$

and

$$(\ell^4 + 4\alpha\ell^3 + 4\alpha^2\ell^2)\cos\theta = -\tfrac{1}{3}\{(\ell^2 + 2\alpha\ell)\sin\theta, \ell^3 + 3\alpha\ell^2\}$$

to get

$$\begin{aligned}\mathcal{Q}\big((\ell^4 &+ 4\alpha\ell^3 + 4\alpha^2\ell^2)\sin\theta\big)\\ &= \mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^4\\ &\quad + \big(4\alpha\mathcal{Q}(\sin\theta) - 2i\mathcal{Q}(\cos\theta)\big)\mathcal{Q}(\ell)^3\\ &\quad + \big([4\alpha^2 + 2]\mathcal{Q}(\sin\theta) - 6i\alpha\mathcal{Q}(\cos\theta)\big)\mathcal{Q}(\ell)^2\\ &\quad + \big(4\alpha\mathcal{Q}(\sin\theta) - i[4\alpha^2 + 1]\mathcal{Q}(\cos\theta)\big)\mathcal{Q}(\ell)\\ &\quad + \big(\tfrac{1}{4} + \alpha^2\big)\mathcal{Q}(\sin\theta) - i\alpha\mathcal{Q}(\cos\theta)\end{aligned} \tag{22}$$

and

$$\begin{aligned}\mathcal{Q}\big((\ell^4 &+ 4\alpha\ell^3 + 4\alpha^2\ell^2)\cos\theta\big)\\ &= \mathcal{Q}(\cos\theta)\mathcal{Q}(\ell)^4\\ &\quad + \big(4\alpha\mathcal{Q}(\cos\theta) + 2i\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)^3\\ &\quad + \big([4\alpha^2 + 2]\mathcal{Q}(\cos\theta) + 6i\alpha\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)^2\\ &\quad + \big(4\alpha\mathcal{Q}(\cos\theta) + i[4\alpha^2 + 1]\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)\\ &\quad + \big(\tfrac{1}{4} + \alpha^2\big)\mathcal{Q}(\cos\theta) + i\alpha\mathcal{Q}(\sin\theta).\end{aligned} \tag{23}$$

We are now ready for

THEOREM 7. *There is no nontrivial quantization of* $(V_\alpha, \mathcal{B})$.

Proof: We consider representations of $\mathcal{B}$ of type (i)′, and use the von Neumann rules (20)–(23) to quantize the bracket relation

$$\begin{aligned}&\big\{\{(\ell^2 + 2\alpha\ell)\cos\theta, (\ell^4 + 4\alpha\ell^3 + 4\alpha^2\ell^2)\sin\theta\}, \cos\theta\big\}\\ &\quad + \big\{\{(\ell^4 + 4\alpha\ell^3 + 4\alpha^2\ell^2)\cos\theta, (\ell^2 + 2\alpha\ell)\sin\theta\}, \cos\theta\big\}\\ &\qquad = -30(\ell^4 + 4\alpha\ell^3 + 4\alpha^2\ell^2)\sin\theta - 24\alpha^2(\ell^2 + 2\alpha\ell)\sin\theta.\end{aligned}$$

After another computer calculation, the left-hand side reduces to

$$\begin{aligned}-30\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^4 &+ \big(60i\mathcal{Q}(\cos\theta) - 120\alpha\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)^3\\ &+ \big(180i\alpha\mathcal{Q}(\cos\theta) - [84 + 144\alpha^2]\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)^2\\ &+ \big(i[54 + 144\alpha^2]\mathcal{Q}(\cos\theta) - [168\alpha + 48\alpha^3]\mathcal{Q}(\sin\theta)\big)\mathcal{Q}(\ell)\\ &+ i[54\alpha + 24\alpha^3]\mathcal{Q}(\cos\theta) - \big[\tfrac{31}{2} + 66\alpha^2\big]\mathcal{Q}(\sin\theta),\end{aligned}$$

which is quite different from $-30 \times (22) - 24\alpha^2 \times (20)$:

$$\begin{aligned}-30\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^4 &+ \left(60i\mathcal{Q}(\cos\theta) - 120\alpha\mathcal{Q}(\sin\theta)\right)\mathcal{Q}(\ell)^3\\ &+ \left(180i\alpha\mathcal{Q}(\cos\theta) - [60 + 144\alpha^2]\mathcal{Q}(\sin\theta)\right)\mathcal{Q}(\ell)^2\\ &+ \left(i[30 + 144\alpha^2]\mathcal{Q}(\cos\theta) - [120\alpha + 48\alpha^3]\mathcal{Q}(\sin\theta)\right)\mathcal{Q}(\ell)\\ &+ i[30\alpha + 24\alpha^3]\mathcal{Q}(\cos\theta) - \left[\tfrac{15}{2} + 42\alpha^2\right]\mathcal{Q}(\sin\theta). \qquad \square\end{aligned}$$

On the other hand, there do exist trivial, but nonetheless nonzero, representations of type *(ii)'*, provided $\alpha \neq 0$. To see this, quantize (19) to obtain $\mathcal{Q}(\ell) = -\alpha I$. Moreover, since $\{e_{2N}^3 + 3\alpha e_{2N}^2 - 2\alpha^3 e_{2N}^0, e_1^0\} = 3i\left(e_{2N+1}^2 + 2\alpha e_{2N+1}^1\right)$, quantization yields $\mathcal{Q}\left(e_{2N+1}^2 + 2\alpha e_{2N+1}^1\right) = 0$ for all N. It follows from the definition of W_α that the only observables in V_α which have nonzero quantizations are of the form $b\ell + c$, with $\mathcal{Q}(b\ell + c) = (c - \alpha b)I$. This is reminiscent of the situation for S^2, cf. [4].

Thus the largest quantizable Lie subalgebra of P containing $\mathcal{B}$ is P^1. At the beginning of this section, we exhibited certain quantizations $\mathcal{Q}_{\nu,\eta}$ of $(P^1, \mathcal{B})$. In fact, as we now show, these are the only ones.

THEOREM 8. *If $\mathcal{Q}$ is a nontrivial quantization of $(P^1, \mathcal{B})$, then $\mathcal{Q} = \mathcal{Q}_{\nu,\eta}$ for some $\nu \in [0, 1)$ and $\eta \in \mathbb{R}$*

Proof: We may suppose that $\mathcal{Q}$ restricted to $\mathcal{B}$ is given by (i)' for some ν. In what follows it is convenient to use complex notation.

Because of the linearity of $\mathcal{Q}$, to establish (9) it suffices to prove that

$$\mathcal{Q}(e^{iN\theta}) = e^{iN\theta} \tag{24}$$

and

$$\mathcal{Q}\left(\ell e^{iN\theta}\right) = e^{iN\theta}\left(-i\frac{d}{d\theta} + \left[iN\eta + \frac{N}{2} + \nu\right]I\right). \tag{25}$$

We begin by quantizing the bracket relation $\{\ell, e^{iN\theta}\} = iNe^{iN\theta}$ to get $[\mathcal{Q}(\ell), \mathcal{Q}(e^{iN\theta})] = N\mathcal{Q}(e^{iN\theta})$. Writing $\mathcal{Q}(e^{iN\theta})|n\rangle = \sum_k D_{nk}^N|k\rangle$, we evaluate the matrix element

$$\langle m|[\mathcal{Q}(\ell), \mathcal{Q}(e^{iN\theta})]|n\rangle = N\langle m|\mathcal{Q}(e^{iN\theta})|n\rangle$$

to obtain $mD_{nm}^N - nD_{nm}^N = ND_{nm}^N$, i.e.,

$$(m - n - N)D_{nm}^N = 0.$$

Thus $D_{nm}^N = 0$ unless $m = n + N$, and hence

$$\mathcal{Q}(e^{iN\theta})|n\rangle = D_n^N|n + N\rangle, \tag{26}$$

where we have abbreviated $D_{n,n+N}^N =: D_n^N$. Note in particular that $D_n^0 = 1$ and $D_n^{\pm 1} = 1$, for all n, cf. (i)'. A similar analysis of the relation $\{\ell, \ell e^{iN\theta}\} = iN\ell e^{iN\theta}$

yields

$$\mathcal{Q}(\ell e^{iN\theta})|n\rangle = d_n^N|n + N\rangle, \tag{27}$$

where in particular $d_n^0 = n + \nu$ by virtue of (i)$'$.

Next quantize $\{e^{iN\theta}, e^{i\theta}\} = 0$ using (i)$'$ to obtain $[\mathcal{Q}(e^{iN\theta}), e^{i\theta}] = 0$. Applying this to a ket $|n\rangle$, (26) gives

$$(D_{n+1}^N - D_n^N)|n + N + 1\rangle = 0$$

for all integers n, from which we conclude that D_n^N depends only upon N, and so will be denoted D^N henceforth. Thus

$$\mathcal{Q}(e^{iN\theta}) = D^N e^{iN\theta}. \tag{28}$$

Now we quantize

$$\{\ell e^{iN\theta}, e^{iM\theta}\} = iMe^{i(N+M)\theta}$$

to obtain

$$[\mathcal{Q}(\ell e^{iN\theta}), \mathcal{Q}(e^{iM\theta})] = M\mathcal{Q}(e^{i(N+M)\theta}).$$

Applying this to a ket $|n\rangle$, (28) and (27) give

$$(d_{n+M}^N - d_n^N)D^M|n + M + N\rangle = MD^{N+M}|n + M + N\rangle,$$

from which we conclude that

$$(d_{n+M}^N - d_n^N)D^M = MD^{N+M}. \tag{29}$$

Setting $M = 1$ this reduces to $d_{n+1}^N - d_n^N = D^{N+1}$, which in turn implies that

$$d_n^N = d_0^N + nD^{N+1}. \tag{30}$$

On the other hand, when we take $M = -N$, (29) reduces to

$$(d_{n-N}^N - d_n^N)D^{-N} = -N.$$

Substituting (30) into this, we find that $D^{N+1}D^{-N} = 1$ for all integers N. Since $D^{\pm 1} = 1$, this implies that each $D^N = 1$. Thus (24) is proved.

Following the established pattern, upon quantizing the relation

$$\{\ell e^{iN\theta}, \ell e^{iM\theta}\} = i(M - N)\ell e^{i(M+N)\theta}$$

and using (30), we eventually produce

$$Md_0^M - Nd_0^N = (M - N)d_0^{M+N}. \tag{31}$$

Taking $M = -N$, this becomes

$$d_0^N + d_0^{-N} = 2\nu. \tag{32}$$

Taking $M = \pm 1$, (31) reduces to

$$d_0^1 - Nd_0^N = (1 - N)d_0^{N+1}$$

and

$$d_0^{-1} + Nd_0^N = (1+N)d_0^{N-1}.$$

Relabeling $N \mapsto N+1$ in this last equation, and then eliminating d_0^{N+1} between these two equations using (32) gives the relation

$$d_0^N = Nd_0^1 + (1-N)\nu. \tag{33}$$

Furthermore, we observe that by the definition of a quantization

$$\mathcal{Q}\big(\ell e^{-iN\theta}\big) \subset \mathcal{Q}\big(\ell e^{iN\theta}\big)^*,$$

which in particular forces

$$\overline{d_0^N} - d_0^{-N} = N. \tag{34}$$

Adding (32) and (34), we get $\Re\big(d_0^N\big) = \frac{N}{2} + \nu$. From (33) and its conjugate, we obtain $\Im\big(d_0^N\big) = N\Im\big(d_0^1\big) =: N\eta$. Substituting back into (27) and recalling (30), we end up with

$$\mathcal{Q}\big(\ell e^{iN\theta}\big)|n\rangle = \big(n + iN\eta + \tfrac{N}{2} + \nu\big)\,|n+N\rangle,$$

which is equivalent to (25). □

Thus within the subalgebra of polynomials, the quantizations $\mathcal{Q}_{\nu,\eta}$ of $(P^1, \mathcal{B})$ are the best one can do.

4. Discussion

Although for topological and algebraic reasons the quantization of T^*S^1 might be expected to share some of the features of both those of $\mathbb{R}^2$ and T^2, we see that in all essential respects it behaves like the plane. For both $\mathbb{R}^2$ and T^*S^1 there is an obstruction, and a maximal Lie subalgebra of polynomial observables that can be consistently quantized consists of those polynomials which are affine in the momentum. Most likely, the underlying reason is that in these examples the given basic sets are the generators of transitive (finite-dimensional) Lie group actions (the Heisenberg group H(2) for $\mathbb{R}^2$, and the special Euclidean group SE(2) for T^*S^1), whereas this is not true for the basic sets $\mathcal{B}_k$ on the torus.

There are some differences, however, which reflect the non-simple connectivity of T^*S^1. For instance, on $\mathbb{R}^2$, there are exactly three maximal polynomial Lie subalgebras containing the basic set $\mathrm{span}\{1, q, p\}$, whereas according to Proposition 5 there is an infinity of such containing $\mathcal{B}$ for the cylinder. Moreover, on $\mathbb{R}^2$ all three of these maximal subalgebras can be consistently quantized [1]. But on T^*S^1, in view of Theorem 7 and the discussion at the beginning of Section 3, only one of these can be nontrivially quantized (viz. P^1), leading to the position representations (9) on $L^2(S^1)$. Thus there is no analogue of the metaplectic representation for T^*S^1. Since θ is an angular variable, there is also no cylindrical counterpart of the momentum representation. Another key difference will be discussed below. Thus the possible polynomial quantizations of T^*S^1 are more limited than those of $\mathbb{R}^2$; this is surprising, given that T^2 admits a full quantization.

Although an obstruction exists for the cylinder, as predicted by the conjecture in Section 1, this example serves to disprove part of another conjecture in [1] concerning the maximal Lie subalgebras of observables that can be consistently quantized. In the present context, this conjecture can be stated:

Let $\mathcal{B}$ be a basic set, which is itself a Lie subalgebra of $\mathcal{P}(M)$. Then every integrable irreducible representation of $\mathcal{B}$ can be extended to a quantization of $(\mathcal{N}(\mathcal{B}), \mathcal{B})$, where $\mathcal{N}(\mathcal{B})$ is the Lie normalizer of $\mathcal{B}$ in $\mathcal{P}(M)$. Furthermore, no nontrivial quantization of $(\mathcal{N}(\mathcal{B}), \mathcal{B})$ can be extended beyond $\mathcal{N}(\mathcal{B})$.

For the cylinder, $\mathcal{N}(\mathcal{B}) = \mathcal{B}$. But from Section 3, we see that the representation (i)$'$ can be extended to a quantization of $(\mathcal{O}, \mathcal{B})$, where $\mathcal{O}$ is the Lie subalgebra of observables which are affine in the (angular) momentum ℓ. On $\mathbb{R}^2$, the basic set span$\{1, q, p\}$ is not self-normal, and it is this difference which is largely responsible for the absence of a "metaplectic-type" representation on T^*S^1.

The existence of consistent quantizations of $(\mathcal{O}, \mathcal{B})$ can be understood from the standpoint of geometric quantization theory; since $\mathcal{O}$ is the normalizer of the vertical polarization $\{f(\theta)\}$ on T^*S^1 [10]. In fact, the parameter $\nu \in [0, 1)$ in the quantizations $\mathcal{Q}_{\nu,\eta}$ labels the inequivalent connections on the prequantization line bundle $L = T^*S^1 \times \mathbb{C}$. (On the other hand, the parameter η in (9)—which also appears in the "quantum kinematics" of Doebner et al. [11]—seems to have no geometric significance.) While $\mathcal{O}$ thus finds a natural interpretation in the context of polarizations, it is not at all clear how (or even if) these quantizations could be predicted by considerations involving $\mathcal{B}$ alone. An important open problem is therefore to repair this conjecture.

It is interesting to observe that if we regard θ as a *real* variable, then $\mathcal{B} = \text{span}\{1, \sin\theta, \cos\theta, \ell\}$ forms a basic set on $\mathbb{R}^2$ (with coordinates θ, ℓ); indeed, $\mathbb{R}^2$ is a Hamiltonian homogeneous space for the universal cover of SE(2).[6] Thus, if we wish, we may regard $\mathcal{B}$ as an "exotic" basic set on the plane. A moment's reflection shows that the results of Section 2 carry through to this context (with one minor exception, noted below). Thus we obtain an exotic no-go result for $\mathbb{R}^2$:

THEOREM 9. *There exists no quantization of $(\mathcal{P}(\mathbb{R}^2), \mathcal{B})$.*

Comparatively, this result is stronger than Groenewold's original no-go theorem [2]. Indeed, the latter only states that there does not exist a quantization of the (standard) polynomial subalgebra of $\mathbb{R}^2$ which extends the metaplectic representation. This cannot be strengthened to a statement analogous to Theorem 9 without introducing ad hoc assumptions à la [3]. (A fuller discussion of this point can be found in [1, §4.1].) So in fact Theorem 9 is the optimal no-go result extant for $\mathbb{R}^2$.

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[6]Specifically, the action of $\mathbb{R} \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$ is

$$(t, x, y) \cdot (\theta, \ell) = (\theta + t, \ell + x\sin(\theta + t) - y\cos(\theta + t)).$$

We make three remarks. First, in $\mathcal{P}(\mathbb{R}^2)$, $\mathcal{B} = \mathrm{span}\{1, \sin\theta, \cos\theta, \ell\}$ is *not* self-normal; in particular, $\theta \in \mathcal{N}(\mathcal{B})$. It would be interesting to discover the ramifications of this, especially as regards the conjecture above. Second, this basic set separates points only locally on $\mathbb{R}^2$, not globally, unlike the Heisenberg basic set on $\mathbb{R}^2$ or even $\mathcal{B}$ on T^*S^1. Third, as described in Section 3, there do exist quantizations of $(P^1, \mathcal{B})$ on $L^2(S^1)$. Curiously, they do not seem to arise from geometric quantization theory (i.e., via a choice of polarization on $\mathbb{R}^2$, since S^1 cannot be the leaf space of any foliation of the plane).

Finally, one topic for future exploration would be to consider the higher-dimensional analogues of the cylinder, viz. T^*S^n with group $\mathrm{SE}(n)$.

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