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Coisotropic Imbeddings, Dirac Brackets and Quantization

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I. *Introduction*

It has long been recognized that the language of symplectic geometry is particularly suited to Hamiltonian and Lagrangian mechanics. The usefulness of this approach carries over even to degenerate and constrained dynamical systems, provided one generalizes to *presymplectic* geometries. This is a necessary generalization, for presymplectic phase spaces arise naturally in a variety of physical contexts (cf. [2,5,6] and references contained therein).

The presymplectic geometry of a classical system, along with the specification of a Hamiltonian, provides a complete self-contained description of the dynamics of the system. Nonetheless, it is often desirable to view the presymplectic phase space under consideration as being imbedded in a symplectic manifold. This is especially true, for instance, when considering the transition to quantum mechanics. Here the difficulty is that while one knows in principle how to quantize a symplectic system, it is not a priori clear how to quantize a *pre*symplectic one. An obvious way to proceed is to imbed the presymplectic manifold in a symplectic manifold and quantize the latter. It will be shown in §4 that for the quantization to be internally consistent this imbedding must be *coisotropic*.

Thus it is important to determine whether a given presymplectic manifold can be coisotropically imbedded in some symplectic manifold. This problem also appears in contexts other than that of quantization theory; for example, in connection with Dirac's theory of constrained dynamical systems [1,9,12]. Within this framework, the Poisson bracket associated to the symplectic structure on the imbedding space is known as the *Dirac bracket*. This structure, introduced by Dirac in order to simplify the canonical analysis of constrained systems, has important applications in field theory (a number of detailed examples may be found in [9]).

These remarks illustrate the significance of coisotropic imbeddings in both the classical and quantum physics of presymplectic systems. The purpose of this paper is to collect several recent results on coisotropic imbeddings which have a bearing on the existence and uniqueness of Dirac brackets as well as their application to quantization theory. In fact, from a mathematical standpoint the theory of coisotropic imbeddings *is* just that of Dirac brackets. In §2, the classical results on Dirac brackets are recalled, including some global refinements due to Śniatycki. In §3, the notion of Dirac bracket is reformulated in the more general setting of coisotropic imbeddings. A fundamental theorem to the effect that coisotropic imbeddings of a given presymplectic manifold always exist and are "locally" unique is stated and its ramifications discussed. In the last section, I briefly outline some joint work with J. Śniatycki [7] on applications of coisotropic imbeddings to the quantization of presymplectic dynamical systems.

II. *Dirac Brackets and Constraint Theory*

Let (P,ζ) be a symplectic manifold, and suppose that M is a closed submanifold of P. The submanifold M inherits a (generically) presymplectic structure Ω from (P,ζ); I assume that the characteristic distribution $ker\ \Omega$ has constant dimension.

The pair (P,ζ) may be taken to represent the phase space of a classical system, while M may be thought of as the "constraint set", that is, the submanifold of P consisting of all states which are admissible initial data for the equations of motion of the system. Given (P,ζ) and a Hamiltonian on P, one constructs M by the standard methods of constraint theory, cf. [1,2,6,9].

Submanifolds M of P are classified as follows: M is said to be

(i) *coisotropic* or *first class* if $TM^{\perp} \subseteq TM$,

(ii) *symplectic* or *second class* if $TM \cap TM^{\perp} = \{0\}$, and

(iii) *mixed* in all other cases.

Here, $TM^{\perp}$ denotes the "orthogonal" complement of TM in $TP|M$ with respect to ζ. From the point of view of the submanifold M, this classification reduces to a characterization of $\ker \Omega$.

In Dirac's terminology [1,2], a *constraint* is a smooth function on P which vanishes on M. A constraint is *first class* if its Poisson bracket with every other constraint is identically zero on M and *second class* otherwise. Thus, a coisotropic (resp. symplectic) submanifold is locally characterized by the vanishing of functionally independent first (resp. second) class constraints, whereas a mixed submanifold requires both first as well as second class constraints for its local description.

This classification of constraint submanifolds (or, equivalently, the partitioning of constraints into classes) has deep physical significance [1,2,6]. The existence of first class constraints, for example, signals the presence of 'gauge' degrees of freedom, that is, variables whose time-evolution is not uniquely determined by the equations of motion. The appearance of second class constraints, on the other hand, indicates that there are non-dynamic degrees of freedom in the theory.

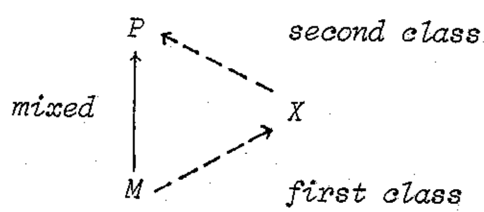
Dirac [1] pointed out that, since the degrees of freedom associated with the second class constraints do not evolve in time, they are physically irrelevant and so might as well be dropped from further consideration. More importantly, he also discovered that the presence of second class constraints precludes the possibility of consistently quantizing the system "as is" (cf. §4). Consequently, in order to simplify the canonical analysis and cast the theory into a form suitable for quantization, Dirac proposed eliminating the second class constraints entirely from the theory (and along with them the non-dynamic degrees of freedom). He then constructed a new canonical formalism in which only first class constraints appear and which leads to a consistent quantization. The crucial step in this process is the introduction of a "Dirac bracket".

Dirac's classical constructions go as follows. Locally describe the constraint set M in P by means of first and second class constraints f^i and g^α, chosen in such a way as to maximize the number of first class constraints [2]. The local submanifold V of P defined by $g^\alpha = 0$ is then second class, and the local submanifold $V \cap M$ is clearly first class *in* V as it is given by the vanishing of the f^i. One now removes P from the formalism altogether and in its place sets up a canonical apparatus on V; for this, it is only necessary to define a Poisson bracket on V. This new Poisson bracket $\{\cdot,\cdot\}^*$ is called a *Dirac bracket* and is given, in terms of the Poisson bracket $\{\cdot,\cdot\}$ on P, by

$$\{f,g\}^* = \{\tilde{f},\tilde{g}\} + \sum_{\alpha,\beta} \{\tilde{f},g^{\alpha}\}C_{\alpha\beta}\{g^{\beta},\tilde{g}\},$$

where f and g are smooth functions on V, $\tilde{f}$ and $\tilde{g}$ are any local extensions of f and g to P, and $C_{\alpha\beta}$ is the inverse of the matrix whose entries are the Poisson brackets $\{g^{\alpha},g^{\beta}\}$ of the second class constraints.

Geometrically, the transition from the Poisson bracket to the Dirac bracket consists of choosing a symplectic submanifold X of P which contains M as a coisotropic submanifold. This is neatly summarized by the diagram



The symplectic structure ω on X is that induced by ζ via the imbedding $X \to P$; Śniatycki [12] has shown that the ω-Poisson bracket is none other than the "mysterious" Dirac bracket $\{\cdot,\cdot\}^*$.

Since Dirac's constructions are local, it remains to determine whether a submanifold X of P with the desired properties always exists *globally*. Fortunately, Śniatycki [12] has settled this issue in the affirmative. On the other hand, it is clear that M and (P,ζ) cannot determine the global properties of (X,ω) to any significant extent, so that in general many different choices of X are possible. Locally, this nonuniqueness is reflected in the various possible choices of constraint functions and their decomposition into the first and second classes.

Dirac brackets play an important role in field theory, especially when used in conjunction with group theoretical techniques [4]. For example, in a large variety of physically interesting systems the first class "part" of the constraint submanifold M arises as the zero level set of the momentum mapping associated with the symplectic action of a gauge group on (P,ζ). Upon introducing Dirac brackets, one finds that this group action restricts to a Hamiltonian action on (X,ω) with corresponding momentum mapping J in such a way that, in X, M is given precisely by $J^{-1}(0)$. This realization of the constraint set is quite useful since, for a given gauge group, the structure of the set $J^{-1}(0)$ is well-understood [10]. Moreover, in such cases not only do Dirac brackets allow one to eliminate the second class constraints from the theory, they also enable one to write the field equations in adjoint form [4,10].

III. *Coisotropic Imbeddings*

In the preceding discussion the basic object of physical interest -- the presymplectic phase space (M,Ω) -- was given a priori as a submanifold of a symplectic manifold. In practice this is not always so; for example, it may happen that (M,Ω) arises as a submanifold of yet another presymplectic manifold (as is typical in Lagrangian mechanics [5]), or even that it exists independently of any other structure. In any case, as pointed out in the Introduction, one would like to be able to imbed (M,Ω) as a coisotropic submanifold of a symplectic manifold.

A *coisotropic imbedding* of a presymplectic manifold (M,Ω) in a symplectic manifold (X,ω) is a closed imbedding $j : M \to X$ such that $j(M)$ is a coisotropic submanifold of (X,ω) and $j^*\omega = \Omega$. By analogy with §2, one calls the Poisson bracket associated to ω a *Dirac bracket*.

Given a presymplectic manifold (M,Ω), there are two fundamental issues to be resolved: Do coisotropic imbeddings of (M,Ω) exist and, if so, to what extent are they unique? The latter question is important as one would like to ensure, in a physical context, that the notion of Dirac bracket is not too arbitrary to be useful. From the discussion in §2, it is clear that the best that one can hope for is to be able to classify symplectic neighborhoods of coisotropic imbeddings.

The following Theorem answers both of these questions. Let $E \to M$ be the *characteristic bundle* of (M,Ω), i.e., the bundle whose fiber over $m \in M$ is *ker* $\Omega(m)$; let E^* denote the dual bundle.

<u>Coisotropic Imbedding Theorem</u> [3]: *There exists a symplectic structure on a neighborhood of the zero-section of* E^* *such that the imbedding of* (M,Ω) *in this neighborhood as the zero-section is coisotropic. Moreover, given any two coisotropic imbeddings* $j_1 : (M,\Omega) \to (X_1,\omega_1)$ *and* $j_2 : (M,\Omega) \to (X_2,\omega_2)$, *there exists a symplectomorphism* ψ *from a neighborhood of* $j_1(M)$ *in* X_1 *onto a neighborhood of* $j_2(M)$ *in* X_2 *such that* $j_2 = \psi \circ j_1$.

The proof of these results (which in fact hold in infinite-dimensions) may be found in [3].

The last part of this Theorem implies that a neighborhood of a coisotropic submanifold M in a symplectic manifold (X,ω) is completely determined by (M,Ω) up to a symplectomorphism which reduces to the identity on M. In other words, the coisotropic imbedding $(M,\Omega) \to (X,\omega)$ is "locally unique". Combining this with the existence result, one has that *every symplectic manifold containing* (M,Ω) *as a coisotropic submanifold is, near* M*, symplectomorphic to a symplectic neighborhood of the zero-section in* E^*. This Theorem thus provides a complete local characterization of coisotropic imbeddings of presymplectic manifolds into symplectic manifolds.

These results are complementary to those obtained by Weinstein [16] regarding isotropic imbeddings. In particular, the isotropic and coisotropic cases coincide when $\Omega = 0$, in which case M is to be imbedded as a Lagrangian submanifold. Since in this instance $E^* = T^*M$, the Coisotropic Imbedding Theorem implies Weinstein's result that every symplectic manifold containing M as a Lagrangian submanifold is, near M, symplectomorphic to a neighborhood of the zero-section in T^*M.

Physically, this Theorem has the consequence that *Dirac brackets always exist and are locally unique*. This means that in a local sense it is possible to canonically associate to every presymplectic dynamical system a symplectic system. Furthermore, it implies that the classical symplectic dynamics obtained in this manner is effectively independent of the choice of Dirac bracket. This, of course, is to be expected since it is the presymplectic phase space (M,Ω) that is of primary physical significance rather than the auxiliary ambient space (X,ω).

IV. *Quantization*

One can effectively reduce the problem of quantizing a presymplectic phase space (M,Ω) to that of quantizing a constrained symplectic system by "extending" (M,Ω) to a symplectic manifold (X,ω). The quantization of the extended phase space (X,ω) produces a space H of quantum states and associates to some class of smooth functions f on X quantum operators $\mathcal{Q}f$ on H (those aspects of geometric quantization theory needed here may be found in [13]). Supposing without loss of generality that M is globally defined in X by the vanishing of constraints, one then postulates that the physically admissible quantum states of the system are those which belong to the subspace H_0 of H defined by

$$H_0 = \{\sigma \in H \mid \mathcal{Q}f[\sigma] = 0 \text{ for all quantizable constraints } f\} .$$

In essence, one enforces the constraints on the quantum rather than the classical level.

This approach, however, may lead to inconsistencies. Specifically, in the presence of quantizable second class constraints H_0 reduces to zero, i.e., there are no nontrivial eigenstates of the constraint operators $\mathcal{Q}f$. Indeed, if g is a second class constraint, then there exists a constraint f such that $\{f,g\}|M = 1$. Then for each wave function $\sigma \in H_0$,

$$\sigma = 1\sigma = \mathcal{Q}(\{f,g\})[\sigma] = \frac{1}{i\hbar}[\mathcal{Q}f,\mathcal{Q}g]\sigma = 0 .$$

Hence the method of "quantization via imbedding" may lead to meaningful results only if all constraints are first class. From a global standpoint, this means that in order to obtain consistent quantum dynamics it is necessary to place a restriction on the allowable types of imbeddings $(M,\Omega) \to (X,\omega)$: they must be coisotropic. According to the Coisotropic Imbedding Theorem, one may always assume that this is the case.

As remarked upon earlier, the fact that a neighborhood of the coisotropic submanifold M in the extended phase space (X,ω) is completely determined by (M,Ω) up to symplectomorphism has the consequence that the classical symplectic dynamics of the system is effectively independent of the choice of coisotropic imbedding $(M,\Omega) \to (X,\omega)$. One would like to determine the extent to which this classical property carries over to the quantal domain. That is, to what extent is the quantum dynamics of the system independent of the choice of coisotropic imbedding? To what extent does the coisotropic submanifold M determine the space of quantum states and the quantization of observables?

As X can be quite arbitrary globally and since the quantization process depends significantly upon the global topology and geometry of the imbedding space, it is clear that one cannot expect the quantization of the system to be independent of the choice of (X,ω) in general. Nevertheless, it is possible to eliminate a substantial portion of this dependence by restricting consideration to certain simple types of coistropic imbeddings. For instance, one may always choose the imbedding $M \to X$ in such a way that X is a tubular neighborhood of the zero-section of some vector bundle over M (e.g., E^*). Then the existence and uniqueness of the prequantization and metaplectic structures -- which a priori depend upon the cohomology of X -- can be characterized solely in terms of the topology of M, since such an X is homotopic to M. Recent work of Vaisman [15] shows that this is also the case regarding the existence of (certain types of) polarizations of (X,ω).

It remains to determine to what extent a similar result holds concerning the quantum representation space H. Since the structure of H is very sensitive to the choice of polarization, so also is the extent to which H can be reconstructed from the knowledge of the values of the wave functions on the submanifold M.

Let F be a real polarization of (X,ω) and denote by X/F the space of all integral manifolds of F and by π the canonical projection $X \to X/F$. The polarization F is *strongly admissible* provided X/F is a manifold and π is a submersion.

Since the following result does not require strong admissibility, it can be used to study "wild" polarizations.

Theorem 1: *Each wave function $\sigma \in H$ is uniquely determined by its restriction to M iff the Bohr-Sommerfeld variety S is contained in $\pi^{-1}(\pi(M))$.*

Whether or not $S \subseteq \pi^{-1}(\pi(M))$ is to some extent determined by the nature of the intersection of F with $TM^{\perp}$. I assume that $\ell = \dim\,(F \cap TM^{\perp})$ is constant on M.

Theorem 2: *Let M be compact and let F be a strongly admissible polarization of (X,ω) such that $F \cap TM^{\perp} = \{0\}$. Then each wave function $\sigma \in H$ is uniquely determined by its restriction to M.*

By an appropriate redefinition of the extended phase space, one may eliminate the compactness assumption.

When $\ell \neq 0$ it is not usually true that each wave function is uniquely determined by its restriction to M. If, however, this is the case, then one obtains a topological restriction on the polarization.

<u>Theorem 3</u>: *Let* F *be a complete strongly admissible polarization of* (X,ω) *such that* $\pi : X \to X/F$ *is a locally trivial fibration. If each wave function* $\sigma \in H$ *is uniquely determined by its restriction to* M*, then the integral manifolds of* F *are diffeomorphic to* $\mathbb{T}^k \times \mathbb{R}^{n-k}$ *with* $k \geq \ell$.

Now turn to the quantization of observables $f \in C^\infty(X)$. For a fixed polarization F, the operator $\mathcal{Q}f$ will in general depend upon the global properties of both f and X. However, the Coisotropic Imbedding Theorem implies that if the construction of $\mathcal{Q}f$ employs only arbitrarily small neighborhoods of M in X then $\mathcal{Q}f$ will be insensitive to both the large-scale behavior of f and the choice of imbedding $M \to X$.

<u>Theorem 4</u>: *Suppose that* M *is compact and let* F *be a strongly admissible polarization of* (X,ω) *with* $\ell = 0$. *Let* $f \in C^\infty(X)$ *be such that* $\mathcal{Q}f$ *exists and let* ϕ_f^t *denote the flow of* f. *If there exists an* $\varepsilon > 0$ *such that, for each* $t \in [0,\varepsilon)$, $F_t = F \cap T\phi_f^t(F)$ *is a distribution on* X *satisfying* $F_t \cap TM^\perp = \{0\}$ *then, for each* $\sigma \in H$, $\mathcal{Q}f[\sigma]$ *can be determined by operations in an arbitrarily small neighborhood of* M *in* X.

As a corollary, one has that if F is transverse to $TM^{\perp}$ and f preserves F in the sense that $T\phi_f^t(F) = F$, then Qf exists and is independent of the choice of coisotropic imbedding $(M,\Omega) \to (X,\omega)$.

The proofs of Theorems 1-4 may be found in [7]; the reader is referred to [7] and [14] for further discussion.

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The present work is but a first step in understanding the quantization of a presymplectic dynamical system. Here, emphasis has been placed on the method of "quantization via coisotropic imbedding", since this technique is perhaps the most natural and straightforward. Furthermore, according to the Coisotropic Imbedding Theorem, this quantization can in principle always be carried out without placing additional assumptions on (M,Ω). In addition to the quantization via imbedding technique, there are several other ways of quantizing a presymplectic manifold (M,Ω): quantize the reduced phase space $M/(ker\ \Omega)$, quantize subsequent to imposing a maximal gauge condition, and quantize (M,Ω) directly. These various methods and their interrelations have been studied by Günther [8], Simms [11], Śniatycki [14] and Woodhouse [17].

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