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Compact parallelizable four-dimensional symplectic and complex manifolds.

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The main result of this paper is to give the first example of a compact symplectic manifold M with no complex structure. A. van de Ven gave the first examples of compact almost complex manifolds with no complex structures [Proc. Nat. Acad. Sci. U.S.A. 55 (1966), 1624–1627; MR 33#6651]. Then S.-T. Yau [Topology 15 (1976), no. 1, 51–53; MR 53#890] and N. Brotherton [Bull. London Math. Soc. 10 (1978), no. 3, 303–304; MR 80e:57022] gave examples of parallelizable manifolds that admit no complex structures. Much simpler than any of the examples of van de Ven, Yau or Brotherton is the example of the authors. It is a 3-step 4-dimensional nilmanifold Nil_3. The manifold Nil_3 also comes up in a paper by C. T. C. Wall [Topology 25 (1986), no. 2, 119–153; MR 88d:32038].

The manifold Nil_3 can be described explicitly in terms of matrices [see the reviewer and B. R. Moreiras, Boll. Un. Mat. Ital. A (7) 1 (1987), no. 3, 343–350; MR 89b:57015]. Since Nil_3 is parallelizable, it is obviously almost complex. Furthermore, the exterior differential d has constant coefficients with respect to the canonical parallelization of Nil_3; this fact allows one to write down symplectic forms in terms of the parallelization.

The proof that Nil_3 has no complex structure is a consequence of the following facts: (1) the main theorem of the paper of P. Deligne, P. Griffiths, J. Morgan and D. Sullivan [Invent. Math. 29 (1975), no. 3, 245–274; MR 52#3584], which implies that all Massey products on a compact Kahler manifold vanish; (2) K. Kodaira's Theorem 25 [Amer. J. Math. 86 (1964), 751–798; MR 32#4708], which states that a complex surface is a deformation of an algebraic surface if and only if its first Betti number is even; (3) the fact that Nil_3 has a first Betti number equal to 2.

The first Betti number $b_1(M)$ of a 4-dimensional compact nilmanifold M equals 2, 3 or 4. The case $b_1(M)=2$ has been discussed above. If $b_1(M)=4$, then M is a torus. The remaining case when $b_1(M)=3$ has been considered by Kodaira [op. cit., Theorem 19] and by W. P. Thurston [Proc. Amer. Math. Soc. 55 (1976), no. 2, 467–468; MR 53#6578]. Let Nil_2 be a 2-step 4-dimensional nilmanifold. Kodaira observed that Nil_2 is complex and symplectic. Thurston described Nil_2 more geometrically and observed that Nil_2 is symplectic but has no positive definite Kahler metric. The authors clarify the situation by proving that N_2 has indefinite Kahler metrics (and hence is complex).

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