Presymplectic manifolds and the Dirac–Bergmann theory of constraints

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We present an algorithm which enables us to state necessary and sufficient conditions for the solvability of generalized Hamilton-type equations of the form $\iota(X)\omega = \alpha$ on a presymplectic manifold (M,ω) where α is a closed 1-form. The algorithm is phrased in the context of global infinite-dimensional symplectic geometry, and generalizes as well as improves upon the local Dirac–Bergmann theory of constraints. The relation between our algorithm and the geometric constraint theory of Śniatycki, Tulczyjew, and Lichnerowicz is discussed.

I. INTRODUCTION

It is generally recognized[1–4] that classically, a physical system can be described in terms of a symplectic manifold, that is, a manifold M together with a nondegenerate closed 2-form ω. Physically, M is the phase space of the system while ω is essentially a generalization of the Poisson bracket.

The manifold M and the symplectic form ω are kinematical in nature; the dynamics of the system is determined by specifying a real-valued function H on phase space, the Hamiltonian. One then solves the Hamilton equations

$$\iota(X)\omega = dH, \tag{1.1}$$

thereby obtaining the dynamical trajectories of the system in phase space (i.e., the integral curves of the vector field X). The fact that ω is nondegenerate assures us that Eq. (1.1) has a unique solution; indeed, the nondegeneracy of ω means that the linear map $\flat: TM \to T^*M$ defined by $\flat(X) := \iota(X)\omega$ is an isomorphism. Thus for any H we can solve (1.1) uniquely: $X = \flat^{-1}(dH)$. Once X has been determined, one appeals to the standard results of differential equation theory in order to integrate X.

We want to consider in detail the case when ω is degenerate, in which case (M,ω) is said to be a *presymplectic* manifold. This situation usually arises when the system is constrained in some manner, and often when M is infinite-dimensional. When (M,ω) is degenerate, the Hamilton equations (1.1) may or may not possess solutions (and, in general, even if solutions exist they will not be unique) depending on whether or not dH is in the range of $\flat$. In the former case, the equations (1.1) possess nonunique solutions, the nonuniqueness being characterized by $\ker\omega$. It is the latter case which is the most interesting, for then (1.1) as it stands possesses no globally defined solutions. In order to "solve" the Hamilton equations, then, one must "modify" M, the equations (1.1), or both. We have developed an algorithm which enables us to produce and solve such a "modified" problem in both the finite- and infinite-dimensional cases. More precisely, we find whether or not there exists a submanifold N of M along which the equations (1.1) hold; if such a submanifold exists, we give a *constructive* method for finding it. Moreover, we show that this submanifold is *unique* in the sense that it contains any other submanifold along which (1.1) is satisfied (Sec. IV).

This work grew out of an attempt to globalize the Dirac–Bergmann theory of constraints,[5,6] first published circa 1950. In these papers, an algorithm was developed by Dirac, Bergmann, and his collaborators for dealing with Lagrangian systems which could not be put into canonical form in the usual manner owing to the fact that the momenta are not all independent functions of the velocities. This algorithm was nicely summarized by Dirac,[7] who showed that such systems could be put into a modified canonical form with the motion restricted to a "constraint" submanifold. Requiring the equations of motion to be consistent on this submanifold led to a sequence of further constraint submanifolds which either terminated or restricted the system to such an extent that no solution of the original variational problem could be found. He showed further that a modified Poisson bracket could be defined in such a way that certain constraints could be effectively eliminated, the remaining variables falling (in principle) into two classes: (i) those whose time development from given initial conditions is completely arbitrary, and (ii) those whose evolution is well defined by canonical equations of motion.

The point of developing this algorithm was not pedagogical, for several classical systems exist which display the above-mentioned feature; notably electromagnetism and gravity. Insofar as it is felt to be necessary to cast these theories into canonical form for the purpose of quantization, the Dirac–Bergmann algorithm provides, in principle, a method for doing this and, at the same time, for identifying the "physical observables" or "true degrees of freedom." In fact, Dirac applied his technique to general relativity[8] and electromagnetism[7] and showed that it was effective in isolating an

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appropriate set of variables with which to describe the motion.

While our algorithm is related to the Dirac–Bergmann method, there are several important differences in both the method and the results.

First, although the Dirac–Bergmann algorithm is clear in an algebraic sense, it is hard to gain an adequate geometric picture of what is taking place. Thus, we have chosen to phrase our discussion in *global* terms using the language of infinite-dimensional symplectic geometry. This manifestly coordinate-invariant language is eminently suited to both the algebraic and geometric aspects of the problem. To this end, much work has been done in recent years,[9–11] but the bulk of this has been mostly concerned with translating Dirac's concepts into the modern mathematical idiom and with symplectically reinterpreting the results of his algorithm. No one seems to have successfully globalized the *algorithm* itself. In Sec. III, we give a brief overview of this "geometric theory of constraints."

Secondly, as Dirac himself noticed,[12] his algorithm is ambiguous in the following sense (to be elaborated upon later): One is not certain whether or not the first-class secondary constraints should be included in the Hamiltonian. Put another way, Dirac is unable to show that the motions generated by the first-class secondary constraints are physically irrelevant (gauge) and hence cannot identify those observables which correspond to "true" degrees of freedom. Actually, this is not so much a problem with the Dirac–Bergmann *algorithm* per se as it is with its physical interpretation. The physical interpretation, in turn, is obscured by Dirac's nongeometric formulation of the constraint algorithm. In Sec. V, we show that our geometric algorithm not only globalizes (and thus substantiates) Dirac's results, but moreover that, strictly speaking, the Hamiltonian should *not* in general contain all the first-class secondary constraints.[13] This uncertainty concerning the first-class secondary constraints is fairly subtle, and we shall not consider it in depth in this paper. This question, and the related issue of the physical intepretation of our geometric algorithm will be discussed from another, more fundamental point of view in a companion paper.[14]

Lastly, our algorithm is applicable in situations considerably more general than those considered by Dirac. Specifically, the Dirac–Bergmann algorithm can only be applied when the degenerate manifold M is actually a "primary constraint submanifold" of some symplectic manifold W. The algorithm we propose does not require the *a priori* existence of such a nondegenerate manifold W. Physically, this may be of considerable importance in the case of an infinite number of degrees of freedom where ω may be degenerate even if there are no constraints.[15,16] The *a priori* presymplectic case is also of physical interest from the point of view of the quantization problem. Normally, when one quantizes a constrained system, one relies upon Śniatycki's theorem[9,11] to eliminate the second-class constraints from the theory. However, Śniatycki's theorem fails in the presymplectic case,[17] leading one to question whether or not such systems are actually quantizable.

After Dirac, a number of people approached the constraint problem from various viewpoints,[18–20] but no completely satisfactory analysis of the three above-mentioned aspects of the theory was forthcoming. (This paper amends an attempt made by one of us several years ago.[21]) In fact, there have been a number of papers[19,20] which challenge the validity of the Dirac–Bergmann algorithm on theoretical grounds. As our algorithm generalizes the Dirac–Bergmann theory, this approach would seem to verify the correctness of the latter, since our derivation is from a completely different (viz., geometrically rigorous) point of view. Moreover, it is not difficult to show that although several of the issues raised by these authors are of importance for the elucidation of the theory, their objections are without content (Sec. V).

Section II provides a very brief introduction to symplectic geometry and its application to Hamiltonian systems in an infinite-dimensional setting.[22] A more comprehensive treatment of these topics is given in the texts by Abraham and Marsden,[1] Souriau,[2] Chernoff and Marsden,[16] and Godbillon.[4] For some of the more advanced notions and applications, one should consult the lecture notes of both Woodhouse[3] and Weinstein.[23] The infinite-dimensional techniques used throughout this paper are clearly and comprehensively explained in the books by Marsden,[15] Chernoff and Marsden,[16] Lang.[24] In general, we shall try to keep our notation and terminology[25] consistent with that of Refs. 1, 16, 23, and 24.

Section III reviews the basic notions and tools of geometric constraint theory which are necessary for the presentation of the algorithm in Sec. IV and the correspondence with the Dirac–Bergmann theory detailed in Sec. V. Finally, we apply the algorithm to electromagnetism in Sec. VI as an example of the calculational techniques involved in the theory.

II. SYMPLECTIC GEOMETRY AND HAMILTONIAN MECHANICS[1–4,15,16,23,26]

Let M be a manifold modelled on a Banach space E, and suppose that ω is a closed 2-form on M. Then (M,ω) is said to be a *strong symplectic* manifold if the linear map $\flat: TM \to T^*M$ defined by $\flat(X) \equiv X^\flat := \iota(X)\omega$ is an isomorphism. However, it may happen that ω will be injective but not surjective, in which case (M,ω) is called a *weak symplectic* manifold, ω being *weakly* nondegenerate. Generically, $\flat$ will be neither injective nor surjective and ω is then *degenerate*. When E is finite-dimensional, there is of course no distinction between weak and strong symplectic forms. For brevity, strongly symplectic manifolds will often be referred to simply as *symplectic*, while weakly nondegenerate and degenerate forms will be dubbed *presymplectic*.

The simplest example of a weak symplectic manifold is the cotangent bundle T^*Q of any Banach manifold Q. In fact, on T^*Q there exists a canonical 1-form θ defined by

$$\langle v|\theta\rangle = \langle \pi_* v|\tau v\rangle$$

where $v \in TT^*Q$, and $\pi: T^*Q \to Q$, $\tau: TT^*Q \to T^*Q$ are the bundle projections. This 1-form defines the weak symplectic structure as follows: $\Omega = -d\theta$.

Locally, we can find a chart $U \subset F$, where F is the model space for Q, such that on U,[25]

$$\theta(x,\sigma)\cdot(a \oplus \pi) = \langle a|\sigma\rangle$$

and

$$\Omega(x,\sigma)\cdot(a \oplus \pi, b \oplus \tau) = \langle a|\tau\rangle - \langle b|\pi\rangle. \tag{2.1a}$$

If F is finite-dimensional, this is the same as saying that there exist coordinates (q^i,p_i) on U such that

$$\theta|U = p_i\, dq^i$$

and

$$\Omega \,|\, U = dq^i \wedge dp_i. \tag{2.1b}$$

The weak nondegeneracy of Ω follows from the above formulas after a simple calculation. In fact, when F is reflexive, Ω is strongly nondegenerate.[16]

However, not every strongly symplectic manifold (M,ω) is a cotangent bundle nor is ω always exact [e.g., (S^2,ω) where ω is a volume on S^2. Then ω cannot be exact, and S^2 is of course not a cotangent bundle] although locally both statements are true. That a strong symplectic manifold is locally a cotangent bundle follows from a normal form theorem, called Darboux's theorem,[27] which states that a chart always exists in which ω is constant. In such a chart ω must always have the form (2.1a) or (2.1b). However, this result need not hold in the presymplectic case.[28] This normal form theorem shows that strongly symplectic geometries are "flat"—this should be compared with the corresponding theorem in Riemannian geometry.

Another contrast with Riemannian geometry can be obtained by examining the infinitesimal automorphisms of a strong symplectic structure (i.e., the *locally Hamiltonian* vector fields). These are vector fields X such that

$$L_X\omega = 0. \tag{2.2}$$

As ω is closed, it is clear that X will be a locally Hamiltonian vector field iff $d\iota(X)\omega = 0$. Since ω is strongly nondegenerate, the map $\flat$ will have an inverse $\#$ and consequently we see that if α is a closed 1-form then $\alpha^\#$ will be a locally Hamiltonian vector field. As there are many closed forms on any manifold, there will exist many infinitesimal symplectic automorphisms. By way of contrast, in Riemannian geometry the existence of Killing vector fields is the exception rather than the rule.

Physically, the weak and strong symplectic manifolds one almost always encounters are cotangent bundles. This comes about as follows: One describes a physical system by specifying a manifold Q called configuration space and a function L, the Lagrangian, on velocity phase space TQ. One then casts the theory into canonical form by "changing variables" from (q^i,v^i) to (q^i,p_i) and replacing L by the Hamiltonian H via $H(q,p) = p_i v^i - L(q,v)$. Mathematically, this transition takes the form of a map $FL: TQ \to T^*Q$ which is called the *Legendre transformation* or the *fiber derivative*[1] and is defined by

$$\langle z \,|\, FL(w)\rangle := \frac{d}{dt} L(w+tz)\Big|_{t=0}, \tag{2.3}$$

where $z,w \in TQ$. The Hamiltonian is defined via

$$H \circ FL(w) := \langle w|FL(w)\rangle - L(w). \tag{2.4}$$

[This is provided that (2.4) does in fact define a single-valued function H on $FL(TQ)$. For further discussion regarding this point, see Ref. 43.] Additionally, in the finite-dimensional case, the canonical momenta are "defined" by

$$p_i \circ FL(w) = \partial L/\partial v^i(w). \tag{2.5}$$

One major advantage of changing a theory into Hamiltonian form is that T^*Q canonically carries a (weak) symplectic structure whereas TQ does not.

It is the weak symplectic structure on T^*Q which gives rise to the elegant simplicity of the Hamiltonian formalism. For example, the Hamilton equations (1.1), when written in terms of local Darboux coordinates [i.e., canonical coordinates for which (2.1b) holds] are simply

$$\frac{dq^i}{dt} \equiv X[q^i] = \frac{\partial H}{\partial p_i},$$

$$\frac{dp_i}{dt} \equiv X[p_i] = -\frac{\partial H}{\partial q^i}.$$

Similarly, one can use Ω (provided Ω is strongly symplectic)[26] to define the Poisson bracket of two functions f,g as follows

$$\{f,g\} := \Omega(\xi_f, \xi_g) \tag{2.6}$$

where $\xi_f := df^\#$. In a Darboux chart, $\{f,g\}$ reduces to the usual expression. The symplectic analog of a canonical transformation is a diffeomorphism $\zeta: T^*Q \to T^*Q$ such that $\zeta^*\Omega = \Omega$.

There do, however, exist physically interesting systems whose phase spaces are not cotangent bundles and whose symplectic forms are not exact. An example of such a system was given by Souriau,[2] who investigated the dynamics of a freely spinning massive particle in Minkowski spacetime from a symplectic viewpoint (in this example, $M = R^6 \times S^2$). Systems of this type do not possess configuration manifolds and consequently do not admit Lagrangian formulations (at least in the usual sense).

With this in mind, it is apparent that from a geometric viewpoint the Hamiltonian formulation of classical physics is of primary importance while the Lagrangian formalism is an alternative construction applicable only in special cases.

Turning now to the presymplectic case, we recall that a presymplectic manifold is obtained by relaxing the assumption that $\flat$ be bijective. Presymplectic manifolds arise quite frequently in physics, in particular when the Legendre transformation (2.3) is degenerate. This means that FL is no longer a local diffeomorphism, but merely an into map, the range of which defines a submanifold M of T^*Q. In more familiar terms, FL will fail to be a local diffeomorphism when the matrix

$$\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right)$$

is not invertible.

This is the starting point of the Dirac–Bergmann constraint theory, in which M is called the *primary constraint* submanifold. The *primary constraints* are a collection of functions on T^*Q which locally define M as a submanifold of T^*Q. One particular set of primary constraints are those relations (2.5) (or combinations thereof) which do not define the momenta p_i as independent functions of the velocities v^i. Geometrically, M will inherit a presymplectic structure from T^*Q by pulling Ω back to M via the inclusion $j{:}M\to T^*Q$. We are thus faced with the problem of determining the dynamics of a physical system on the presymplectic phase space $(M, j^*\Omega)$ where the Hamiltonian H is given by (2.4).

The presymplectic phase spaces of the above discussion are rather special in that they are *naturally* submanifolds of weakly nondegenerate manifolds. But this is not always the case, even in physics, as was shown by Künzle[29] who obtained genuinely *pre*symplectic phase spaces for spinning particles in curved spacetimes.

Thus, from both a mathematical and physical standpoint, there is considerable justification in considering presymplectic geometry in its own right. The physical issue that one is then confronted with is the following: A system is described by a presymplectic phase space (M,ω) and a Hamiltonian H on M. What does one mean by "consistent equations of motion" on M, and how does one obtain and solve such equations? The algorithm we propose will select a certain submanifold N of M upon which we can define and solve "consistent equations of motion." Before we can proceed to discuss the algorithm however, we must first examine the properties of such submanifolds.

III. GEOMETRIC CONSTRAINT THEORY[3,6,7,9–11,23]

We would like to have a classification scheme for submanifolds of presymplectic manifolds which is at the same time mathematically convenient and physically meaningful. Dirac[7] first developed a local classification of submanifolds of strongly symplectic manifolds which Śniatycki and Tulczyjew later globalized as the "geometric theory of constraints."[9,10] This classification is of the utmost importance insofar as the physical interpretation of the algorithm is concerned.[14] We briefly review this classification (generalized to the presymplectic case) following Śniatycki, Tulczyjew, and Lichnerowicz.[11]

Let N be a submanifold of the presymplectic manifold (M,ω) with inclusion j. The manifold N is called a *constraint* submanifold, and the triple (M,ω,N) is called a *canonical system*. We define the *symplectic complement* $TN^\perp$ of $\underline{TN}$ in TM to be

$$TN^\perp=\{Z\in TM|N \text{ such that } \omega|N(X,Z)=0 \text{ for all } X\in\underline{TN}\}.$$

For our purposes this is not the most convenient characterization of $TN^\perp$. We prefer that given by the following.

Proposition 1: $TN^\perp=\{Z\in TM|N \text{ such that } j^*[\iota(Z)\omega]=0\}$.

Proof: Let $Z\in TN^\perp$. Then for any $W\in TN$,

$$0=\omega|N(j_*W,Z)=j^*\langle j_*W|\iota(Z)\omega\rangle=\langle W|j^*[\iota(Z)\omega]\rangle.$$

As this is true for all $W\in TN$, it follows that $j^*[\iota(Z)\omega]=0$. Conversely, if $j^*[\iota(Z)\omega]=0$, then the equality is established by reversing the above calculation. Q.E.D.

If S is a subspace of a Banach space E, we define $S^{\vdash}\subset E^*$, the *annihilator* of S, to be the set of all $\beta\in E^*$ such that $\langle v|\beta\rangle=0$ for all $v\in S$. Similarly, if Λ is a subspace of E^*, we define $\Lambda^{\vdash}\subset E$ to be the collection of all $v\in E$ such that $\langle v|\lambda\rangle=0$ for all $\lambda\in\Lambda$. If E is reflexive, then it is possible to show[30] that $(\Lambda^{\vdash})^{\vdash}=\bar{\Lambda}$. We shall say that ω is *topologically closed* provided the map $\flat$ is a closed map, i.e., $\flat$ maps closed sets into closed sets. We note that if ω is strongly nondegenerate, then it is necessarily topologically closed. We can now prove the following important fact[31]:

Proposition 2: If M is reflexive and ω is topologically closed, then $(TN^\perp)^{\vdash}=\underline{TN}^\flat$.

Proof: With obvious shorthand notation,

$$W\in(\underline{TN}^\flat)^{\vdash}\Leftrightarrow\omega/N(W|\underline{TN})=0\Leftrightarrow W\in TN^\perp$$

thereby proving that $(\underline{TN}^\flat)^{\vdash}=TN^\perp$. As ω is topologically closed, $\underline{TN}^\flat$ is closed in T^*M. The desired result follows from the above by taking $\Lambda=\underline{TN}^\flat$. Q.E.D.

The constraint submanifold N is said to be

(i) *isotropic* if $\underline{TN}\subset TN^\perp$,

(ii) *coisotropic* or *first-class* if $TN^\perp\subset\underline{TN}$,

(iii) *weakly symplectic* or *second class* if $\underline{TN}\cap TN^\perp=\{0\}$, and

(iv) *Lagrangian* if $\underline{TN}=TN^\perp$.

Clearly, $\underline{TN}\cap TN^\perp=\underline{\ker\omega_N}$, where $\ker\omega_N$ is the set of all $W\in TN$ such that $\iota(W)\omega_N=0$. If N does not happen to fall into any of these categories, then N is said to be a *mixed* constraint manifold.

Locally, a first-class constraint submanifold can be described by the vanishing of a collection of functions A such that for all $f\in A$, $W[f]|N=0$ for all $W\in TN^\perp$. If (M,ω) happens to be strongly nondegenerate, this is easily seen to be equivalent to Dirac's requirement that A be in involution, i.e., $\{f,g\}|N=0$ for all $f,g\in A$.

The functions $f\in A$ of the preceding paragraph are called *first-class constraint functions*. More generally, any function f (resp. 1-form γ) on M such that $f|N=0$ (resp. $j^*\gamma=0$) is called a *constraint function* (resp. *constraint form*), and any function g (resp. 1-form σ) on M such that $W[g]|N=0$ (resp. $\langle W|\sigma\rangle|N=0$) for all $W\in TN^\perp$ is said to be *first class*. Functions (resp. forms) which are not first-class are called *second class*. A second-class constraint submanifold, then, can be locally described by second-class constraint functions. In general, a mixed or isotropic constraint submanifold will require both first- as well as second-class constraint functions for its local description.

As an example of a second-class constraint submanifold, let $C\subset Q$, where Q is configuration space. Then T^*C is a weakly symplectic submanifold of (T^*Q,Ω), hence it is second class. Furthermore, the constraint submanifold $\pi^{-1}(C)\subset T^*Q$ is first class. The former is an example of a *holonomic* constraint.

We have discussed some simple properties of submanifolds of presymplectic manifolds in the above, but we have not yet indicated their origin. It is to this task that we now turn our attention.

IV. THE CONSTRAINT ALGORITHM

We begin by taking a presymplectic manifold (M_1,ω_1) to be the phase space of some physical system. Let H_1 be the Hamiltonian of the system. We inquire as to under what conditions and by what methods we can solve the canonical equations of motion $\iota(X)\omega_1=dH_1$. Actually, we can be somewhat more general[32] and write the Hamilton equations as

$$\iota(X)\omega_1=\alpha_1, \tag{4.1}$$

where α_1 is a closed 1-form, the *Hamiltonian form*. Locally, as α_1 is closed, we can always find a Hamiltonian function corresponding to α_1. As was mentioned in the Introduction, if α_1 is in the range of $\flat:TM_1\to T^*M_1$, then Eq. (4.1) is consistent as it stands and can be solved directly for X.[33]

In the generic case, however, this will not be so. But there may exist points of M_1 (such points being assumed to form a submanifold M_2 of M_1),[34] for which $\alpha_1|M_2$ is in the range of $\flat|M_2$. We are thus led to try and solve Eq. (4.1) restricted[35] to M_2, i.e.,

$$(\iota(X)\omega_1-\alpha_1)\circ j_2=0, \tag{4.2}$$

where j_2: $M_2\to M_1$ is the inclusion.

Equation (4.2) evidently possesses solutions, but this is not the whole story. Physically, we must demand that the motion of the system be constrained to lie in M_2, if this concept is to have any meaning. Thus, the locally Hamiltonian vector field X appearing in (4.2) must be tangent to M_2, that is, X must be of the form $X=j_{2*}\hat{X}$ with $\hat{X}\in TM_2$, or else the equations of motion will try to evolve the system off M_2.

This requirement will not necessarily be satisfied, forcing us to further restrict (4.1) to the submanifold M_3 of M_2 defined by

$$M_3:=\{m\in M_2 \text{ such that } \alpha_1(m)\in \underline{TM_2}^{\flat}\}.$$

We must now ensure that the solution to (4.1) restricted to M_3 is in fact tangent to M_3; this will in general necessitate yet further restrictions.

It is now clear how the algorithm must proceed. We generate a string of submanifolds

$$\cdots\to M_3 \xrightarrow{j_3} M_2 \xrightarrow{j_2} M_1$$

defined as follows

$$M_{l+1}:=\{m\in M_l \text{ such that } \alpha_1(m)\in \underline{TM}_l^{\flat}\}.$$

Once the constraint algorithm so defined is set into motion, only one of three distinct possibilities may occur.[36] They are:

Case 1: There exists a K such that $M_K=\phi$,

Case 2: Eventually, the algorithm produces a submanifold $M_K\neq\phi$ such that $\dim M_K=0$, and

Case 3: There exists a K such that $M_K=M_{K+1}$ with $\dim M_K\neq 0$.

In Case 1, $M_K=\phi$ means that the Hamilton equations (4.1) have no solutions at all in any sense. In principle, this means that (M_1,ω_1,α_1) does not accurately describe the dynamics of any system.

The second possibility results in a constraint submanifold which consists of isolated points. The equations (4.1) are consistent, but the only possible solution is $X=0$ and there is no dynamics.

For Case 3, we have a constraint submanifold M_K and completely consistent equations at motion on M_K of the form

$$(\iota(X)\omega_1-\alpha_1)|M_K=0. \tag{4.3}$$

It is this submanifold M_K (the *final* constraint submanifold) which corresponds to the submanifold N discussed in Sec. III.

If the algorithm terminates, then *by construction* we are assured that at least one solution X to the canonical equations exists and furthermore that this solution is tangent to M_K. We note that X need not be unique, for we can add to it any element of $\ker\omega_1\cap\underline{TM}_K$. In addition, it is obvious, again by construction, that the final constraint submanifold is *unique* in the following sense: if N is any other submanifold along which the equations (4.1) are satisfied, then $N\subset M_K$.

The algorithm we have proposed provides a geometrically intuitive and conceptually simple method for defining and solving consistent equations of motion on a presymplectic manifold. The algorithm is of very general applicability, requiring only that the phase spaces involved be Banach manifolds.

For many purposes, the algorithm as presented above is too "abstract." More precisely, it is somewhat difficult to use in practice, the calculation of the constraint submanifolds occasionally being a rather formidable task. In addition, the present form of the algorithm is too awkward to be useful for comparison with the Dirac–Bergmann theory. Consequently, we will now recast the algorithm into a form which is more tractable in these regards.

We begin by recharacterizing the constraint submanifold M_2. We can typify the inconsistency of Eq. (4.1) as follows: Consider the set $TM_1^{\perp}$ of vector fields characterized as in Proposition 1. If Eq. (4.1) is to be solvable, then $W\in TM_1^{\perp}$ implies that the left-hand side of (4.1) vanishes and consequently it follows that $\langle W|\alpha_1\rangle$ vanishes. On the other hand, if $W\in TM_1^{\perp}$ implies that $\langle W|\alpha_1\rangle=0$, then $\alpha_1\in(TM_1^{\perp})^{\perp}$. If ω_1 is topologically closed and if M_1 is reflexive, then by Proposition 2 we have that $\alpha_1\in(TM_1^{\flat})$. Thus, the points of M_1 where (4.1) is inconsistent are exactly those points for which $\langle W|\alpha_1\rangle$ is nonzero. Subject to the above asumptions, then, M_2 can alternatively characterized as follows

$$M_2:=\{m\in M_1 \text{ such that } \langle TM_1^{\perp}|\alpha_1\rangle(m)=0\}$$

with obvious shorthand notation. The consistency conditions $\langle TM_1^{\perp}|\alpha_1\rangle=0$ are called, after Dirac and Bergmann, *secondary constraints*.

Returning to the problem of solving (4.2), the demand that the solution X be tangent to M_2 leads to further consistency conditions (*tertiary* constraints) as follows: If there exists an X tangent to M_2 such that (4.2) holds, then for $W \in TM_1$,

$$0 = [\iota(W)\iota(X)\omega_1 - \iota(W)\alpha_1]\circ j_2$$
$$= -j_2^*\langle j_{2*}\hat{X} \,|\, \iota(W)\omega_1\rangle - \langle W \,|\, \alpha_1\rangle\circ j_2$$
$$= -\langle X \,|\, j_2^*[\iota(W)\omega_1]\rangle - \langle W \,|\, \alpha_1\rangle\circ j_2$$

where $\hat{X} \in TM_2$ with $X = j_{2*}\hat{X}$. Consequently, consistency of (4.2) demands that if W is such that $j_2^*[\iota(W)\omega_1] = 0$ (i.e., $W \in TM_2^{\perp}$), then $\langle W|\alpha_1\rangle\circ j_2 = 0$. This, again, may not always be the case and we must correspondingly restrict the equation (4.2) to those points of M_2 where $\langle TM_2^{\perp}|\alpha_1\rangle = 0$.

The algorithm then proceeds as before, generating a sequence of submanifolds

$$\cdots \rightarrow M_3 \xrightarrow{j_3} M_2 \xrightarrow{j_2} M_1$$

defined as follows

$$M_{l+1} := \{m \in M_l \text{ such that } \langle TM_l^{\perp}|\alpha_1\rangle(m) = 0\},$$

where

$$TM_l^{\perp} = \{W \in TM_1 \text{ such that } k_l^*[\iota(W)\omega_1] = 0\}$$

for $l \geqslant 1$ with $k_l := j_2\circ j_3\circ\cdots\circ j_l$ and $k_1 := \mathrm{id}|M_1$. The constraint functions on M_{l-1} which define M_l are called *l-ary constraints* and are always of the form $\langle TM_{l-1}{}^{\perp}|\alpha_1\rangle = 0$. Sometimes, for convenience, all l-ary constraints are (for $l \geqslant 2$) simply called *secondary*.

If the algorithm terminates, we are faced with the same three possibilities as before. In the second or third case, we now explicitly show that (4.3) possesses solutions. We note that, as the algorithm terminates with M_K, $\langle TM_K{}^{\perp}|\alpha_1\rangle = 0$.

Theorem: The canonical equations

$$(\iota(X)\omega_1 = \alpha_1)|M_K$$

possess solutions tangent to M_K iff

$$\langle TM_K^{\perp}|\alpha_1\rangle = 0.$$

Proof: $\Rightarrow$ Let $X \in TM_K$ be a solution, and suppose that $W \in TM_K^{\perp}$. Then

$$\langle W \,|\, \alpha_1\rangle = \langle W, k_K{}^*X|\omega_1\rangle \circ k_K$$

$$= -k_K{}^*\langle k_{K*}X|\iota(W)\omega_1\rangle$$

$$= -\langle X|k_K{}^*[\iota(W)\omega_1]\rangle \quad (\text{as } X \in TM_K)$$

$$= 0$$

by Proposition 1.

$\Leftarrow$ Suppose $W \in TM_K^{\perp}$. Then $\langle W|\alpha_1\rangle = 0$, so that $\alpha_1|M_K \in (TM_K^{\perp})^{\vdash}$. But by Proposition 2, $(TM_K^{\perp})^{\vdash} = \underline{TM}_K^{\flat}$. Thus, $\alpha_1|M_K \in \underline{TM}_K^{\flat}$, that is, there exists an $X \in \underline{TM}_K$ such that $[\iota(X)\omega_1 = \alpha_1]|M_K$. Q.E.D.

It is of interest to note that the above theorem is actually independent of the constraint algorithm. In fact, if N is *any* submanifold of a presymplectic manifold (M,ω), then the equations $(\iota(X)\omega - \alpha)|N = 0$ possess solutions tangent to N iff $\langle TN^{\perp}|\alpha\rangle = 0$.

We now turn to the uniqueness of the final constraint manifold M_K. For suppose there exists some other submanifold N along which the equations (4.1) are satisfied, that is, let $X = j_*\hat{X}$, $\hat{X} \in TN$ be such that

$$[\iota(X)\omega_1 - \alpha_1]|N = 0,$$

where $j: N \rightarrow M$ is the inclusion. Then if $W \in TM_1^{\perp}$, we have from the above that $\langle W|\alpha_1\rangle\circ j = 0$, so that $N \subset M_2$. Let $\tilde{j}_2: N \rightarrow M_2$ be the inclusion; then $j = j_2\circ\tilde{j}_2$. For $Y \in TM_2^{\perp}$,

$$0 = [\iota(Y)\iota(X)\omega_1 - \iota(Y)\alpha_1]\circ j$$
$$= -\langle \hat{X} \,|\, j^*[\iota(Y)\omega_1]\rangle - \langle Y|\alpha_1\rangle\circ j$$

Now $j^*[\iota(Y)\omega_1] = \tilde{j}_2{}^*\circ j_2{}^*[\iota(Y)\omega_1] = 0$ as $Y \in TM_2^{\perp}$, so $\langle Y|\alpha_1\rangle\circ j = 0$, and thus $N \subset M_3$. Continuing in this fashion, we see that $N \subset M_K$.

This version of the algorithm, while perhaps not quite as intuitive as the earlier construction, is still geometrically natural and much better suited to calculation. However, it is important to bear in mind that this version can be used only when the model space for M_1 is reflexive and ω_1 is topologically closed; otherwise one might obtain spurious results.

The canonical system (M_1,ω_1,M_K) and the equations of motion (4.3) are the end results of the constraint algorithm. The further development of the theory (Dirac brackets, the reduced phase space, quantization) follows from the geometric constraint formalism of Śniatycki, Tulczyjew and Lichnerowicz. But now we must turn to a thorough investigation of our geometric algorithm vis-a-vis the Dirac–Bergmann theory.

V. RELATION TO THE DIRAC–BERGMANN THEORY OF CONSTRAINTS[5–7,14]

We now compare the constraint algorithm presented in the last section with the Dirac–Bergmann theory, and show that ours does in fact generalize the latter. We also contrast our our method with similar algorithms presented by Shanmugadhasan, Kundt, and Hinds and point out that these algorithms disagree with ours and consequently with the Dirac–Bergmann theory as well.

We first briefly sketch the Dirac–Bergmann algorithm, displaying the correspondence between their techniques and our more geometric ones.

We start with a Lagrangian L and a reflexive configuration space Q. Changing to canonical form via the fiber derivative FL, we find that the motion of the system is constrained to the submanifold $M_1 := FL\,(TQ)$ of the strongly symplectic manifold T^*Q. Locally, on some neighborhood U, we can describe $U_1 := M_1 \cap U$ by a set of primary constraints $\{\phi^A\}$. Using these, Dirac argues that the Hamiltonian on U should be of the form

$$h = \bar{H}_1 + u_A\phi^A, \tag{5.1}$$

where $\bar{H}_1$ is any extension to U of the Hamiltonian H_1 induced on M_1 by FL and the u_A are yet to be determined Lagrange multipliers.[37]

Translating into symplectic terms, Dirac then searches for solutions to

$$(\iota(X)\Omega - dh)|U_1 = 0, \tag{5.2}$$

where Ω is the canonical symplectic form on T^*Q. As Ω is nondegenerate, solutions X certainly exist, but Dirac notes that the constraints ϕ^A must be preserved, that is,

$X[\phi^A]|U_1 = 0$. Geometrically, this means that X must be tangent to U_1. In terms of the Poisson bracket associated with Ω via (2.6), this requirement translates into a set of conditions

$$\dot{\phi}^A|U_1 = 0, \tag{5.3}$$

where

$$\dot{\phi}^A = \{\phi^A, \overline{H}_1\} + u_B\{\phi^A, \phi^B\}. \tag{5.4}$$

The vanishing of the expressions (5.4) by virtue of (5.3) will, in general, give some information about the u_A and will also give a number of additional constraints. To see this, consider all possible linear combinations of (5.3). Some of these linear combinations will be satisfied trivially, others will fix some of the Lagrange multipliers u_B, and the remaining ones will be independent of the u_B.

These latter conditions take the form $f_A^\alpha \dot{\phi}^A$ where

$$f_A^\alpha\{\phi^A, \phi^B\}|U_1 = 0$$

by (5.4), thus yielding

$$f_A^\alpha\{\phi^A, \overline{H}_1\}|U_1 = 0 .$$

In general, of course, these last equations will not be satisfied except on a local submanifold U_2 of U_1. These conditions are therefore secondary constraints.

Denoting the quantities $f_A^\alpha\{\phi^A, \overline{H}_1\}$ by ζ^α, we see that the preservation of these secondary constraints requires that

$$\dot{\zeta}^\alpha|U_2 = 0,$$

where

$$\dot{\zeta}^\alpha = \{\zeta^\alpha, \overline{H}_1\} + u_B\{\zeta^\alpha, \phi^B\}.$$

As before, the linear combinations of the above conditions which are independent of the u_B, i.e., those linear combinations $g_\alpha^a \dot{\zeta}^\alpha$ such that

$$g_\alpha^a\{\zeta^\alpha, \phi^B\}|U_2 = 0, \tag{5.5}$$

will yield tertiary constraints

$$g_\alpha^a\{\zeta^\alpha, \overline{H}_1\}|U_2 = 0. \tag{5.6}$$

One then iterates this procedure, arriving at some final local constraint submanifold U_K (if the problem is solvable) and a solution X to the equations of motion of the form

$$\iota(X)\Omega = d\overline{H}_1 + u_\mu d\chi^\mu + u_i d\xi^i \tag{5.7}$$

restricted to U_K, where the χ^μ are first-class primary constraints (the Lagrange multipliers u_μ being arbitrary) and the ξ^i are second-class primary constraints (the u_i being fixed).[38]

Furthermore, it was shown that the first-class *primary* constraints are generating functions of motions (i.e., gauge transformations) which leave the physical state invariant (this is, of course, related to the fact that the u_μ are arbitrary). This led Dirac to conjecture that the first-class *secondary* constraints may also generate physically irrelevant motions and hence they should also (for the sake of completeness) be included in the Hamiltonian.[39] Dirac therefore proposed adjoining the first-class secondary constraints ψ^a with arbitrary multipliers λ_a to h thereby obtaining the "extended" Hamiltonian

$$h_E = \overline{H}_1 + u_\mu\chi^\mu + u_i\xi^i + \lambda_a\psi^a. \tag{5.8}$$

Thus, Dirac reasoned that the solutions of

$$(\iota(X)\Omega - dh_E)|U_K = 0 \tag{5.9}$$

would give the most general evolution of the system.

This, then, is the essence of the Dirac–Bergmann theory. With regard to our construction, the first important fact is that each Dirac–Bergmann local constraint submanifold U_l is an open submanifold of the M_l produced by our algorithm. To see this, consider the lth step of the Dirac–Bergmann algorithm, and let ζ^α be (at most) l-ary constraints. Define, as Ω is strongly nondegenerate, the vector field Y^a on U_l by

$$\iota(Y^a)\Omega = g_\alpha^a d\zeta^\alpha. \tag{5.10}$$

Using (2.6), Eqs. (5.5) become

$$0 = g_\alpha^a\{\zeta^\alpha, \phi^B\}|U_l$$

$$= -\iota(Y^a)\, d\phi^B|U_l$$

and consequently $Y^a \in \underline{TU_1}$, as the ϕ^B are primary constraints. Thus, if $j_1: M_1 \rightarrow T^*Q$ is the inclusion,

$$(j_1 \circ k_l)^*[\iota(Y^a)\Omega] = 0$$

by (5.10), so that $Y^a \in TU_l^{\perp\perp}$ by Proposition 1. Consequently,[38]

$$Y^a \in TU_l^{\perp\perp} \cap \underline{TU_1} = \underline{TU_l^\perp}.$$

Similarly, one can show that every vector field $Y \in \underline{TM_l^\perp}$ induces a condition of the form (5.6). Consequently, the same equations which define the local submanifold U_l also locally generate the constraint submanifold M_l.

Therefore, it is clear that the Dirac–Bergmann algorithm is just a local version of our algorithm. Even so, the algorithm we have presented has one significant advantage over the Dirac–Bergmann method in that it is of considerably more general applicability. It is apparent how crucially the Dirac–Bergmann algorithm depends upon the existence of the primary constraints. Our geometric algorithm, by way of contrast, requires only M_1 and its presymplectic structure for its utilization. The manifold M_1 *never* need be a primary constraint submanifold of some other strongly nondegenerate manifold.

But one important difference yet remains. Dirac solved the equations of motion on T^*Q along M_K, whereas we have done so on M_1 along M_K. We now show that we can lift our equations of motion (4.3) to T^*Q obtaining the equations (5.7) and thereby *proving* the formal equivalence of the two algorithms, and thus substantiating the Dirac–Bergmann procedure.

To find the analog of (4.3) on T^*Q, we write

$$\iota(X)\Omega-\alpha_0=\beta_0 \tag{5.11}$$

along M_K, where X is some solution of (4.3) and α_0 is any 1-form on T^*Q such that $\alpha_1=j_1^*\alpha_0$. As X solves (4.3), pulling (5.11) back to M_1 gives $j_1^*\beta_0=0$ so that β_0 is a primary constraint form. Locally, β_0 can be decomposed (nonuniquely) in the form

$$\beta_0=l_\mu d\chi^\mu+g_i d\xi^i.$$

Thus, (5.11) becomes locally

$$\iota(X)\Omega-\alpha_0=l_\mu d\chi^\mu+g_i d\xi^i. \tag{5.12}$$

Now, (4.3) only determines X up to vector fields in $TM_K\cap\underline{\ker\omega_1}=\underline{\ker\omega_K}\cap\underline{\ker\omega_1}$. Letting $Y\in\underline{\ker\omega_K}\cap\underline{\ker\omega_1}$, we see that $X-Y$ must satisfy (5.11) as well, and since $\iota(Y)\Omega$ is a first-class primary constraint form, it can be locally expressed as $f_\mu d\chi^\mu$. Substituting into (5.12), we can write along M_K

$$\iota(X)\Omega=\alpha_0+(l_\mu+f_\mu)d\chi^\mu+g_i d\xi^i. \tag{5.13}$$

From this we see that the second-class piece $g_i d\xi^i$ of (5.12) is insensitive to the choice of X. Moreover, the first-class part $l_\mu d\chi^\mu$ is uniquely determined only for fixed X. Consequently, as X is not unique, the functions l_μ are arbitrary; on the other hand, the g_i are independent of the choice of X and hence are completely determined. Thus, we have reproduced Dirac's result (5.7).

It remains to discuss the "extended" equations of motion (5.9). We notice that nowhere in (5.13) do secondary constraints appear, nor is there any *a priori* reason why they should, at least from the geometric arguments presented above.

The ultimate resolution of this problem depends upon whether or not the first-class secondary constraints generate gauge transformations.[40] This, in turn, depends crucially upon one's definition of "physical state" and "gauge transformation." In other words, how "gauge" the first-class secondary constraints are depends upon the *physical interpretation* of the algorithm and consequently is not strictly amenable to proof.[41]

For example, in the "orthodox" interpretation of the algorithm,[14] all the first-class secondary constraints ψ^a are *assumed* to be gauge. In this case, one could append these constraints to the Hamiltonian as in (5.8) without changing the physical content of the theory; however, in practice one may not always want to do this. The reason is that one may have fixed a gauge (either inadvertently or by design) in the Hamiltonian; some of the ψ^a will then generate physically irrelevant motions that will not respect the gauge condition. If one wishes to retain this choice of gauge in the description of the system, then one cannot attach these constraints to the Hamiltonian. On the other hand, there may be certain other ψ^a which will generate gauge transformations which do not break the gauge; these can be included without reservation in the Hamiltonian—in fact, they are "already there" in some sense (for an example, see Sec. VI). Thus, from the standpoint of the usual interpretation of the algorithm, one in general does not need, or perhaps want, to append the first-class secondary constraints to the Hamiltonian: Some of the ψ^a will break the gauge choice, and those that do not are already present in the Hamiltonian.

There do exist other "unorthodox" interpretations of the algorithm in which certain of the first-class secondary constraints are *not* gauge. Consequently, these constraints certainly cannot be included in the Hamiltonian. The remaining ψ^a which do generate physically irrelevant motions may or may not be attached to the Hamiltonian as discussed above.

For a more detailed presentation of these points and examples thereof, consult Ref. 14.

In 1965, Hinds[21] presented an algorithm which, like ours, was stated in geometric language. Rather than consider this algorithm in detail, we merely point out the major differences between Hinds' approach and ours. Basically, the crux of the matter is that, at the lth step of the algorithm, Hinds attempts to solve the equation ($\omega_l:=k_l^*\omega_1$ etc.)

$$\iota(X)\omega_l=\alpha_l \tag{5.14}$$

in contrast to our equation

$$[\iota(X)\omega_1]\circ k_l=\alpha_1\circ k_l. \tag{5.15}$$

The conditions for the existence of solutions to an equation of the type (5.14) are less restrictive than those required for Eq. (5.15). To see this, note that the sets of vector fields which generate Hinds algorithm are $\ker\omega_l$, whereas ours are $TM_l^\perp$, and $\underline{\ker\omega}_l\subset TM_l^\perp$. The upshot of this is that after the $l=2$ step, Hinds' algorithm and ours diverge: The constraint submanifolds M_l for $l\geqslant 2$ are no longer the same in both algorithms. If one attempts to reproduce the Dirac–Bergmann results using Hinds's scheme, one obtains

$$h_E=\overline{H}_1+u_\mu\chi^\mu+u_i\xi^i+\lambda_a\psi^a+\lambda_\Delta\theta^\Delta,$$

where the coefficients λ_Δ of the second-class secondary constraints θ^a do not necessarily vanish.

A simple example which illustrates the above is the following: Take $TQ=TR^4$ with coordinates $\{q^i,v^i\}$ with Lagrangian

$$L(q,v)=\tfrac{1}{2}m(v^1)^2-\tfrac{1}{2}k(q^1)^2 -b(q^3q^4)+\tfrac{1}{2}c(v^4-aq^2)^2.$$

A somewhat different scheme was proposed by Shanmugadhasan[19] to rectify an alleged oversight in the Dirac–Bergmann theory. Shanmugadhasan for the most part works on velocity phase space and deals directly with the Lagrange equations. He claims that the Dirac–Bergmann theory overlooks certain subsidiary conditions arising from the degeneracy of the Hessian matrix ($\partial^2L/\partial v^i\partial v^j$); this of course is not the case as these subsidiary conditions are none other than primary constraints (Sec. II). Furthermore, Shanmugadhasan completely ignores the possibility that secondary constraints might occur in the theory, and of course it is these which really form the core of the problem. In fact, Shanmugadhasan's method cannot cope with the perfectly consistent (if somewhat strange) Lagrangian given above.

Kundt[20] also quarrels with the Dirac–Bergmann algorithm and has offered his own interpretation of their theory[17] which, curiously enough, requires all the primary constraints to be first class. Kundt's theory fails for the Proca field.

VI. AN EXAMPLE: ELECTROMAGNETISM[7,16]

The Maxwell theory provides a nice illustration of both the geometric calculations involved in the algorithm and the application of modern infinite-dimensional techniques to symplectic geometry. Throughout this section, we shall closely follow the notation of Chernoff and Marsden.[16] We shall also sacrifice mathematical rigor (i.e., we shall ignore certain infinite-dimensional technicalities) in favor of a clearer exposition.

The 3+1 decomposed Maxwell Lagrangian can be written as

$$L(A,\dot A)=\tfrac{1}{2}\textstyle\int_{R^3}[(\vec\nabla A_\perp)^2 + 2(\vec\nabla A_\perp)\cdot\dot{\vec A}+\dot{\vec A}^2-(\vec\nabla\times\vec A)^2]d\mu, \quad (6.1)$$

where the vector potential is decomposed $A=(A_\perp,\vec A)$, R^3 denotes a constant-time Cauchy surface in Minkowski spacetime, and μ is some measure on R^3.

We must first decide on a choice for velocity phase space TQ. The configuration space should be some Hilbert space of all 4-vectors $(A_\perp,\vec A)$. As L contains at most first spatial derivatives of A, an appropriate choice for Q is

$$Q=H^1_\perp\oplus\vec H^1$$

with the obvious notational shorthand, where H^1 is the first Sobolev space on R^3. Velocity phase space, that is, the manifold of all $(A,\dot A)$ is then

$$TQ=Q\oplus(L^2_\perp\oplus\vec L^2) \quad (6.2)$$

as no spatial derivatives of $\dot A$ appear in L. The measure μ can then be taken to be the ordinary L^2 measure on R^3. We note that Q is reflexive, so that the symplectic form Ω on T^*Q is strongly nondegenerate and hence topologically closed.

To translate into the Hamiltonian language, we must calculate the fiber derivative FL. By definition, $FL|Q=\mathrm{id}|Q$ so

$$FL(A,\dot A)\cdot(A,\dot B)=(A,\dot DL(A,\dot A)\cdot\dot B), \quad (6.3)$$

where $\dot D$ denotes the Frechét derivative along the fiber. An easy calculation shows that

$$\dot DL(A,\dot A)\cdot\dot B=\textstyle\int_{R^3}[\dot{\vec A}\cdot\dot{\vec B}+(\vec\nabla A_\perp)\cdot\dot{\vec B}]d\mu. \quad (6.4)$$

If we define the natural pairing $\langle\,|\,\rangle: TQ\times T^*Q\to R$ by

$$\langle(A,\dot A)\,|\,(A,\pi)\rangle=\textstyle\int_{R^3}[\dot{\vec A}\cdot\vec\pi+\dot A_\perp\pi_\perp]d\mu, \quad (6.5)$$

where $(A,\pi)\in T^*Q$, then (6.3) becomes, using (6.4)

$$FL(A,\dot A)=(A,\dot{\vec A}+\vec\nabla A_\perp). \quad (6.6)$$

Defining the "canonical field momentum" $\vec\pi$ by

$$\vec\pi:=\dot{\vec A}+\vec\nabla A_\perp, \quad (6.7)$$

it is suggestive that $\pi_\perp$ does not appear in (6.6). In fact, if one defines the projection $\mathrm{pr}^2_\perp$ on the second factor by

$$\mathrm{pr}^2_\perp(A,\pi)=\pi_\perp,$$

then it follows that

$$\mathrm{pr}^2_\perp\circ FL(A,\dot A)=0. \quad (6.8)$$

Thus, $\pi_\perp=0$ is a primary constraint. The primary constraint submanifold M_1 of T^*Q is then

$$M_1=Q\oplus\vec L^{2*}. \quad (6.9)$$

We now apply the algorithm. The strong symplectic form Ω on T^*Q is given by (2.1a),

$$\Omega(a\oplus\pi,b\oplus\tau)=\langle a|\tau\rangle-\langle b|\pi\rangle \quad (6.10)$$

with $a\oplus\pi$, $b\oplus\tau\in T(T^*Q)$.[25] If j_1 is the inclusion of M_1 into T^*Q, we have $\omega_1=j_1{}^*\Omega$. Consequently, as Ω is topologically closed, ω_1 is also. This, combined with the fact that M_1 is reflexive, allows us to use the second version of the algorithm presented in Sec. IV.

The first thing we must do is calculate $TM_1^\perp=\ker\omega_1$. That is, we search for vectors $b\oplus\tau\in TM_1$ which annihilate all other vectors $a\oplus\pi$ in TM_1. Using (6.5) and (6.10), we find that $b\oplus\tau\in TM_1^\perp$ iff

$$b\oplus\tau=(b_\perp,\vec 0)\oplus 0. \quad (6.11a)$$

In other words,

$$TM_1^\perp=H^1_\perp\vec 0. \quad (6.11b)$$

The Hamiltonian H_1 induced on M_1 by FL is, according to (2.5), (6.1), and (6.6)

$$H_1(A,\pi)=\textstyle\int_{R^3}[\tfrac{1}{2}\vec\pi^2-(\vec\nabla A_\perp)\cdot\vec\pi+\tfrac{1}{2}(\vec\nabla\times\vec A)^2]d\mu. \quad (6.12)$$

Consequently, if $b\oplus\tau\in TM_1$,

$$dH_1(A,\pi)\cdot(b\oplus\tau)=\textstyle\int_{R^3}[\vec\pi\cdot\vec\tau+b_\perp(\vec\nabla\cdot\vec\pi)+A_\perp(\vec\nabla\cdot\vec\tau) +(\vec\nabla\times\vec A)\cdot(\vec\nabla\times\vec b)]d\mu. \quad (6.13)$$

To continue with the algorithm, it is necessary to make sure that the primary constraint (6.8) is preserved. Thus, we demand that $\langle TM_1^\perp|dH_1\rangle=0$. Letting $b\oplus\tau\in TM_1^\perp$, we have from (6.11a) upon substitution into (6.13)

$$dH_1(A,\pi)\cdot(b\oplus\tau)=\textstyle\int_{R^3}b_\perp(\vec\nabla\cdot\vec\pi)d\mu.$$

This expression will be zero provided

$$\vec\nabla\cdot\vec\pi=0, \quad (6.14)$$

as $b_\perp$ is arbitrary. We thus pick up a secondary constraint, M_2 being the submanifold of M_1 along which (6.14) is satisfied.

Pursuing the algorithm, we must now find $\underline{TM}_2^\perp$. For $a\oplus\pi$ in $\underline{TM}_2$ and $b\oplus\tau\in TM_1$,

$$\omega_1(a\oplus\pi,b\oplus\tau)=\textstyle\int_{R^3}[\vec\tau\cdot\vec a-\vec\pi\vec b]d\mu \quad (6.15)$$

by (6.9). In general, the right-hand side of (6.15) will vanish iff $\vec\tau=\vec 0$ and $\vec b=\vec\nabla g$ for some function g, making use of (6.14) and an integration by parts. Consequently,

$$TM_2^\perp=\{b\oplus 0\in TM_1 \text{ such that } \vec b=\vec\nabla g,\ g\in H^1\}. \quad (6.16)$$

At this point, the algorithm terminates. To see this, let $b\oplus 0$ be as in (6.16). Substitution into (6.13) gives

$$dH_1(A,\pi)\cdot(b\oplus 0)=\textstyle\int_{R^3}[b_\perp(\vec\nabla\cdot\vec\pi) +(\vec\nabla\times\vec A)\cdot(\vec\nabla\times\vec\nabla g)]d\mu.$$

The first term vanishes by (6.14) and the second also as curl(grad)$=0$. Consequently, $\langle TM_2^{\perp} | dH_1\rangle=0$ and M_2 is the final constraint submanifold.

Thus the Maxwell canonical system is (M_1,ω_1,M_2). We now investigate the nature of this final constraint submanifold M_2. First of all, we claim that $TM_2^{\perp}\subset \underline{TM_2}$. Indeed, if $b\oplus\tau\in TM_2^{\perp}$, then $\tau_\perp=0$ so that $b\oplus\tau\in \overline{TM_1}$ and moreover, $\vec{\tau}=\vec{0}$ so that (6.14) is satisfied. Furthermore, $TM_2^{\perp}\neq \underline{TM_2}$. This is easily understood, as $b\oplus(0,\vec{\tau})$ with $\vec{\nabla}\cdot\vec{\tau}=0$ is a member of $\underline{TM_2}$ but not of $TM_2^{\perp}$ unless $\vec{\tau}=0$. Hence, M_2 is strictly coisotropic, and the canonical system (M_1,ω_1,M_2) is first-class. In particular, the constraint $\vec{\nabla}\cdot\vec{\pi}=0$ is first-class.

The basic theorem of Sec. IV assures us that solutions to Hamilton's equations

$$(\iota(X)\omega_1-dH_1)\circ j_2=0 \tag{6.17}$$

exist. To find these solutions, write $X=a\oplus\sigma$, and let $b\oplus\tau\in TM_1$ be arbitrary, $(A,\pi)\in M_2$. The equations of motion can then be written

$$\omega_1(a\oplus\sigma,b\oplus\tau)|(A,\pi)=dH_1(A,\pi)\cdot(b\oplus\tau). \tag{6.18}$$

Using (6.13) and (6.15), the above becomes

$$\int_{R^3}[\vec{\tau}\cdot\vec{a}-\vec{\sigma}\cdot\vec{b}]d\mu=\int_{R^3}[\vec{\tau}\cdot\vec{\pi}+(b_\perp)(\vec{\nabla}\cdot\vec{\pi})+A_\perp(\vec{\nabla}\cdot\vec{\tau}) +(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times b)]d\mu. \tag{6.19}$$

As $(A,\pi)\in M_2$, the second term on the right-hand side of (6.19) drops out. After a rearrangement of the last term and an integration by parts, the right-hand side becomes

$$\int_{R^3}[\vec{\tau}\cdot(\vec{\pi}-\vec{\nabla}A_\perp)+(\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\Delta\vec{A})\cdot\vec{b}]d\mu.$$

Comparing the left-hand side of (6.19) with this, we obtain

$$\frac{d\vec{A}}{dt}:=\vec{a}=\vec{\pi}-\vec{\nabla}A_\perp,$$

$$\frac{d\vec{\pi}}{dt}:=\vec{\sigma}=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\Delta\vec{A}, \tag{6.20}$$

$$\frac{dA_\perp}{dt}:=a_\perp=\text{undetermined}.$$

These are, of course, just Maxwell's equations. Performing a transverse-longitudinal decomposition of $\vec{A},\vec{\pi}$ we obtain

$$\begin{aligned}&\frac{dA_\perp}{dt}=\text{undetermined}\\ &\frac{d\vec{A}_L}{dt}=-\nabla A_\perp,\\ &\frac{d\vec{A}_T}{dt}=\vec{\pi}_T,\\ &\vec{\pi}_L=0,\\ &\frac{d\vec{\pi}_T}{dt}=-\Delta\vec{A}_T.\end{aligned} \tag{6.21}$$

Consequently, these equations determine $\vec{A}_T,\vec{\pi}_T$ uniquely from given initial data, but the evolution of $A_\perp$ and $\vec{A}_L$ is arbitrary.

Let us compare the equations of motion (6.21) and the known gauge freedom of the electromagnetic field with the predictions of the algorithm. In particular, (6.17) shows that the Hamiltonian vector field X is unique only up to elements of $\ker\omega_1\cap\underline{TM_2}=\ker\omega_1$. Consequently, vector fields in $\ker\omega_1$ necessarily generate gauge transformations; if $V\in\ker\omega_1$, then V is of the form $(V_\perp,\vec{0})\oplus 0$ and its effect is to generate arbitrary changes in the evolution of $A_\perp$. This is clearly consistent with the field equations. Turning now to the first-class secondary constraint (6.14), we wonder if it is the generator of physically irrelevant motions. From the geometric point of view, we are really asking whether or not the vector fields in $\underline{\ker\omega_2}=TM_2^{\perp}$ are gauge vector fields. If $W=(0,-\vec{\nabla}g)\oplus 0$, then

$$\iota(W)\omega_1\cdot(b\oplus\tau)=-\int_{R^3}\vec{\tau}\cdot(\vec{\nabla}g)d\mu.$$

Demanding that $X-W$ satisfy (6.17) as well as X has the effect of replacing the second of equations (6.21) by

$$\frac{d\vec{A}_L}{dt}=-\vec{\nabla}A_\perp-\vec{\nabla}g$$

and leaving the others invariant. As $A_\perp$ is arbitrary to begin with, it is evident that this equation is completely equivalent to (6.21). The addition of $-\vec{\nabla}g$ to the right-hand side of this equation has no physical effect whatsoever. Thus, $\underline{\ker\omega_2}$ consists of gauge vector fields.[42]

From another standpoint, rather than writing

$$\iota(X-W)\omega_1=dH_1$$

along M_2, we can put

$$\iota(X)\omega_1=dH_1+\iota(W)\omega_1.$$

Effectively, we are adding a term $g(\vec{\nabla}\cdot\vec{\tau})$ to the right-hand side of (6.13). In terms of the Hamiltonian itself, we are replacing $-(\vec{\nabla}A_\perp)\cdot\vec{\pi}$ by $-[\vec{\nabla}(A_\perp+g)]\cdot\vec{\pi}$. An integration by parts finally gives

$$dH_1+\iota(W)\omega_1=d[H_1+\int_{R^3}g(\vec{\nabla}\cdot\vec{\pi})d\mu].$$

The function whose differential appears on the right-hand side of this equation is none other than the pullback to M_1 to Dirac's extended Hamiltonian (5.8). With respect to the discussion in the last section, the above arguments show that for ordinary electromagnetism, one *can* add the first-class secondary constraints to the Hamiltonian since (i) these constraints are gauge, and (ii) no choice of gauge has been fixed in the Lagrangian (6.1). Notice also that we know (i) to be true regardless of the physical interpretation of the algorithm; in fact, we have not really physically interpreted the algorithm at all. As may be expected, this is due to the fact that the Maxwell theory is so "simple."

In the generic case, result (i) above will *not* be independent of the physical interpretation of the algorithm. Neither will (ii) be the case in general. One need not look far or long for a Lagrangian which has both of these problems, for consider

$$L(A,\dot{A})=\int_{R^3}\left[\tfrac{1}{2}(\partial_\mu A^\nu)(\partial^\mu A_\nu) -A_\mu\partial^\mu\phi-\frac{\lambda}{2}\phi^2\right]d\mu.$$

Is this Lagrangian to be regarded as electromagnetism in the Lorentz gauge, or is it an entirely different (massless, divergence-free, spin 1) field? This particular Lagrangian is discussed further in Ref. 14.

One should compare the above calculation with that given by Dirac.[7] Although this is not really a "working physicist" type calculation, these rigorous infinite-dimensional techniques are capable of rapidly producing results—in fact, they are indispensible when one discusses purely presymplectic systems. In finite dimensions, this geometric formalism is every bit as convenient to use as are the standard techniques.

VII. CONCLUSION

The algorithm we have presented completely solves, from a mathematical point of view, the problem of constrained symplectic systems (in both the finite- and infinite-dimensional cases). Even more significantly, it allows us to solve the Hamilton equations in the hitherto untreated presymplectic case. Combined with the geometric constraint theory of Śniatycki, Tulczyjew, and Lichnerowicz, it furnishes a powerful physical tool.

In addition to generalizing the Dirac–Bergmann theory of constraints, the algorithm has the advantage of being a global, manifestly coordinate-free theory. The algorithm is presented in a mathematically rigorous fashion which we feel is geometrically natural, intuitive, and useful from a practical (calculational) standpoint.

The algorithm provides insight into the old "controversy" of whether or not first-class secondary constraints really generate gauge transformations. It can be shown[14] that the algorithm cannot actually *prove* that all such constraints will beget physically irrelevant motions; nonetheless, equipped with a suitable physical interpretation, this algorithm furnishes a superior framework for discussing such questions. Consequently, these techniques may be of great value for the consideration of theories whose gauge properties at this time are poorly understood.

Our algorithm can also be adapted[43,44] to the Lagrangian case. Here, the Dirac–Bergmann formalism cannot be applied at all, and other proposed schemes have met with only limited success.[18,19] From the standpoint of this paper, the Lagrangian case can be regarded as a specific example of a presymplectic manifold $(TQ, FL^*\Omega)$, where Ω is the canonical symplectic structure on T^*Q and hence can be dealt with by the algorithm presented here. In this way the formal equivalence of the Hamiltonian and Lagrangian formalisms can be established even in the degenerate case.[43,45]

Since this algorithm enables us to treat *a priori* presymplectic systems as well as ordinary constrained symplectic systems, this work may engender motivation for inquiring as to how to quantize such presymplectic systems,[17] perhaps from the viewpoint of the geometric Kostant–Souriau quantization program.[46]

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[21] G. Hinds, "Foliations and the Dirac Theory of Constraints," Ph.D. thesis, Univ. of Maryland, 1965.
[22] Throughout this paper we assume that all physical systems under consideration are time-independent.
[23] A. Weinstein, *Lectures on Symplectic Manifolds*, AMS Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics #29 (1977).
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[25] Herein we establish our notation and terminology. All manifolds appearing in this paper are assumed to be C^∞-Banach manifolds. In particular, if N is a submanifold of a Banach manifold M, then N is assumed to be a Banach manifold in its own right, and hence must be *closed* as a subset of M. If $N \subset M$, then $\overline{N}$ denotes the topological closure of N in M. A manifold is said to be reflexive if its model space is reflexive. Sign and normalization conventions for the tensor algebras (differential forms and vector fields) are those of Ref. 24. We denote by the same symbol TM both the tangent bundle of M and the space of all vector fields tangent to M, etc. We designate the natural pairing $TM \times T^*M \rightarrow R$ by $\langle \mid \rangle$. The symbol d denotes the exterior derivative, L_X the Lie derivative along a vector field X, ι the interior product, and D the Frechét derivative. If γ is a p-form, and $X_1,...,X_p$ are vector fields, then we have the following equivalence class of notations

$\iota(X_1)\cdots\iota(X_p)\gamma \equiv \langle X_1,\ldots,X_p|\gamma\rangle \equiv \gamma(X_p,\ldots,X_1)$.

We define $\ker\gamma = \{Y\in TM$ such that $\iota(Y)\gamma=0\}$. The symbol "$|N$" denotes "restriction to the submanifold N." If $N\subset M$ with inclusion j and γ is a form on M, we denote by either $\gamma|N$ or $\gamma\circ j$ the restriction of γ to points of N, and by γ_N the pullback $j^*\gamma$ of γ to N. If $S\subset TN$, we put $\underline{S} := j_*S\subset TM$. We now briefly explain how one calculates locally, following Refs. 1 and 16. If $U\subset E$ is a chart on a manifold Q, then $T^*U=U\times E^*$ is a chart on T^*Q, and a point $m\in T^*Q$ has the local representation $m=(x,\sigma)$ where $x\in U$, $\sigma\in E^*$. A chart on $T(T^*Q)$ is $T(T^*U)=(U\times E^*)\oplus(E\times E^*)$. Thus a tangent vector X to T^*Q has the local representation $X(m)=(x,\sigma)\oplus(a,\pi)$ where $a\in E$ and $\pi\in E^*$. We will often suppress the base point (x,σ) and simply write this as $X=:a\oplus\pi$. Thus, for example, if α is a 1-form on T^*Q, the interior product $\iota(X)\alpha(m)$ is written locally $\alpha(x,\sigma)\cdot(a\oplus\pi)$.

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[30]This can be seen as follows: Let S be a subspace of E; we claim that $(S^\perp)^\perp=\overline{S}$. For if $\beta\in S^\perp$, then $\beta\in\overline{S}^\perp$ by continuity, and consequently $\overline{S}\subset(S^\perp)^\perp$. Conversely, suppose that $\bar{v}\in(S^\perp)^\perp-\overline{S}$. As $\bar{v}\notin\overline{S}$, by the Hahn-Banach theorem there exists a $\beta\in E^*$ such that $\langle v|\beta\rangle=0$ for all $v\in S$ with $\langle\bar{v}|\beta\rangle\neq0$. But this contradicts the fact that $\bar{v}\in(S^\perp)^\perp$. Thus $(S^\perp)^\perp\subset\overline{S}$ and the claim is proven. However, we see that the corresponding result $(\Lambda^\perp)^\perp=\overline{\Lambda}$ for $\Lambda\subset E^*$ will not be true in general as $\Lambda^\perp$ is defined as a subspace of E, not E^{**}, and consequently we cannot apply the Hahn-Banach theorem as before. On the other hand, if E is reflexive so that $E=E^{**}$, then the above arguments apply and in this case, $(\Lambda^\perp)^\perp=\overline{\Lambda}$.

[31]An interesting corollary to this result is the following: Let M be reflexive, and suppose that ω is topologically closed. Then weak nondegeneracy is equivalent to strong nondegeneracy.

[32]Insofar as the algorithm *itself* is concerned, the equations (4.1) can be completely general, i.e., ω_1 can be *any* 2-tensor and α_1 *any* 1-form. It is not known to the authors whether or not there is any physical significance in replacing the exact 1-form dH_1 by α_1, i.e., are there any physical systems which require for their description a Hamiltonian 1-form rather than a Hamiltonian function?

[33]As was mentioned in the Introduction, we must then integrate X. In finite dimensions, the equations defining the integral curves of X are ordinary differential equations and there is no problem in obtaining unique (local) solutions. In the infinite-dimensional case, these will be partial differential equations, and the situation is correspondingly much more complicated.

[34]We assume that all of the M_l appearing in the algorithm are in fact Banach submanifolds. In practice, of course, this assumption will not generally be valid. In such case, one must resort to standard tricks, e.g., cut the manifold into pieces where everything is nice and work locally. For a nice discussion of how to proceed see Hinds (Ref. 21) and Śniatycki (Ref. 10).

[35]In this context, we must emphasize that we are restricting (4.1) to M_2. Equation (4.2) is *not* the pullback of (4.1).

[36]A fourth possibility arises in the infinite-dimensional case where the algorithm may not terminate at all. In this situation the final constraint submanifold can be taken to be the intersection of all the submanifolds M_l.

[37]This is contrary to Kundt (Ref. 20) who views the u_A appearing in (5.1) as determined from the outset.

[38]Here, "first-" and "second-class" are defined as in Sec. III but with respect to the Ω-symplectic complement of TM_l in $T(T^*Q)$, which we denote by $TM_l^{\perp\perp}$. It is trivial to establish that $\underline{TM}_l^\perp=TM_l^{\perp\perp}\cap\underline{TM}_l$.

[39]Dirac, *op. cit.* Ref. 6, pp. 25–6.

[40]In the following discussion a number of results will be quoted without proof or extensive explanation. For further details, consult Ref. 14.

[41]This is in contradistinction to Dirac's intimation that a proof of the gaugeness of the first-class secondary constraints should exist (Ref. 12). In fact, Dirac's claim that one can generate first-class secondary constraints by taking Poisson brackets of first-class primary constraints is incorrect, as the Poisson bracket of two first-class primary constraints is necessarily another primary constraint.

[42]There is a much easier way to tell whether or not a given vectorfield is gauge than the type of argument presented here. This technique, which we call the "gauge vector field algorithm" is capable of generating all the gauge vector fields in a particular theory (or at least all those vector fields which preserve any choices of gauge made in the Hamiltonian). For details, see Ref. 14.

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