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Geometric quantization and gravitational collapse

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The Kostant-Souriau method of geometric quantization is applied to homogeneous and isotropic cosmological models with positive intrinsic curvature and a massless Klein-Gordon scalar field. These models are studied because classically they collapse to a singularity. It is rigorously shown that the quantized models collapse as well (so that there is no "quantum bounce"). This work demonstrates the practical usefulness of geometric quantization for the study of physical systems.

I. INTRODUCTION

Ever since the first attempts were made to formulate spacetime physics in a manner compatible with the quantum principle, interest has focused on two possible effects: (1) that strong gravitational fields could create particles, and (2) that gravitationally induced spacetime collapse could be prevented. In recent years, there has been considerable progress in the study of the first of these two effects.[1,2] Most of this headway has been achieved via covariant quantization techniques which use the Feynman path integral to determine the transition probability from one state to another. It is not known how to calculate the necessary path integrals exactly, but apparently particle production can be effectively explored via semiclassical approximations and loop-by-loop Feynman diagram calculations.

Interest in the second possible effect has been heightened by the work of Parker and Fulling,[3,2] who show—at least in the semiclassical approximation—that quantum effects do indeed ward off collapse. However, this issue has yet to be adequately resolved within the full quantum theory. Here, the path-integral approach is not very effective, and so a different method of analysis is needed. The most widely used scheme has been that of "freezing out" all but a finite number of degrees of freedom (so that the spacetimes are homogeneous), canonically quantizing the resulting system, and then studying its quantum dynamics.[4,5] This type of analysis has met with only limited success and has given inconclusive answers to the question of whether or not gravitational collapse can be prevented.

The use of homogeneous cosmologies is mathematically attractive in studies of the quantization of gravity, since their phase spaces are finite-dimensional. By considering them instead of more general spacetime models, one can temporarily set aside the problems inherent to systems with an infinite number of degrees of freedom, and concentrate instead on problems associated with the choice of gauge (i.e., time), the constraints, and the nonlinearities of Einstein's theory. These problems are severe within the traditional canonical quantization program: (a) If we make no choice of gauge, we have a classically vanishing Hamiltonian. It can only generate quantum evolution via some Klein-Gordon perspective, with the attendant problems of finding positive-frequency wave functions and defining an inner product for the quantum Hilbert space. (b) If we do choose a gauge, then the Hamiltonian tends to be time-dependent, noncommuting for different times, and square root in form. The Schrödinger equation is then nearly impossible to solve. (c) Different choices of gauge give us various (usually inequivalent) quantizations. (d) Factor-ordering ambiguities also lead to different quantum systems. (e) There are serious questions regarding the physical interpretation of the resulting quantum dynamics.

Problems (c) and (e) above are in some sense physically fundamental and so are effectively independent of the particular type of analysis employed. The other difficulties (a), (b), and (d) are to a certain extent artifacts of the canonical quantization method, being derived from a subtle shortcoming of this procedure—its essentially local character. The rules of canonical quantization[6] have been designed for a physical system which is described classically by a Euclidean phase space. For many systems this limitation is unimportant; but a general system (such as one of the homogeneous spacetime models) need not have such simple geometrical and topological properties. So one should not expect canonical quantization to make sense in all generality, and one should not be surprised to encounter difficulties in applying canonical quantization to a nontrivial physical system.

The geometric Kostant-Souriau procedure[7,8] for obtaining a quantum description of a given physical system is essentially nothing more than a mathematically rigorous global generalization of the canonical quantization prescription. Indeed, "geometric quantization" strongly reflects the global structure of the classical phase space, which no longer need be Euclidean. Does the Kostant-Souriau procedure therefore overcome the other difficulties encountered by canonical quantization in analyzing homogeneous spacetimes? To a certain extent, yes. For example, there are no factor-ordering ambiguities in geometric quantization: To each quantizable classical observable there corresponds a unique well-defined quantum operator (cf. Sec. II). Furthermore, the Kostant-Souriau technique explicitly constructs a genuine Hilbert space in all cases, thus ensuring a proper probabilistic interpretation of the quantum state vectors. Previous studies with the homogeneous spacetime models[9] indicate that at least in the simpler cases, problem (b) is avoided as well—geometric quantization has no trouble handling the time-dependent, square-root Hamiltonians typical of these cosmologies. Moreover, by exploiting the full power of the Kostant-Souriau theory, it is easily possible to avoid the difficulties due to the noncommutativity of the Hamiltonian at different times.

While geometric quantization avoids many of the above-mentioned problems, it is no panacea. In particular, for certain physical systems it produces more than one quantum description, and there are systems for which the Kostant-Souriau calculations can be formidable. This first problem is expected: It is clear[10] that there cannot exist an "ideal" quantization procedure which produces a unique consistent quantum theory for every physical system given only its classical description. So the Kostant-Souriau procedure shares this "defect" with every method of quantization. A virtue of geometric quantization in this regard is that its multiple quantizations are to a large extent well-parametrized cohomologically. As for the second problem, only experience will tell whether or not geometric quantization is a practical procedure for a large number of systems.

In this paper, we discuss one class of systems for which (as we shall see) geometric quantization is certainly most useful—homogeneous and isotropic spacetimes containing a Klein-Gordon scalar field as well as a gravitational field satisfying the Einstein-Klein-Gordon field equations. Such spacetimes are interesting for a number of reasons: (1) They are compatible with the current observations and assumptions of cosmology; (2) they are well understood classically[11]; (3) some of the classical models collapse; (4) they are the simplest spacetimes which possess nontrivial dynamics (unlike the homogeneous and isotropic cosmologies which contain a fluid rather than a scalar field); and (5) certain of these systems have been canonically quantized.[11]

We have discussed elsewhere[9] the application of the Kostant-Souriau procedure to some of these "RWϕ" (Robertson-Walker with a scalar field ϕ) spacetime models. There, we focused on the ability of geometric quantization to handle systems which could not be treated by canonical techniques. But the models examined in Ref. 9 are inappropriate for studying quantum effects on collapse, since they do not collapse even classically. In this paper, we quantize the RWϕ models which do exhibit classical collapse, and we show (see Sec. VI) that the quantized versions of these models collapse as well.

Of course, our study does not settle the question of quantum collapse: We consider only an extremely limited class of spacetimes. We do not know whether or not the ansatz of freezing degrees of freedom, prior to quantization, is physically misleading.[12] And there remain a few problems regarding the physical interpretation of the quantum dynamics of these cosmologies (cf. Sec. VI). However, the Kostant-Souriau analysis does support the contention that at least in some (highly symmetric) spacetime models, quantum effects do *not* prevent collapse.

II. THE GEOMETRIC QUANTIZATION PROCEDURE

The two basic components of a quantum description of a physical system are (1) a Hilbert space $\mathcal{H}$ of quantum states and (2) a representation $\mathfrak{Q}$ of physical observables as self-adjoint operators on $\mathcal{H}$. *Geometric quantization*[7,8] is a procedure for constructing both $\mathcal{H}$ and $\mathfrak{Q}$ directly in terms of the underlying symplectic geometry of the classical system. Note that geometric quantization does not change the way in which quantum dynamics is analyzed (via the Schrödinger equation) or the way in which measurements are theoretically made in quantum mechanics.

In applying geometric quantization to a given physical system, one must build three structures: (1) a prequantization line bundle, (2) a polarization, and (3) a metalinear frame bundle. Roughly, the line bundle gives one a preliminary Hilbert space and a corresponding representation of the classical observables, the polarization serves to define a complete set of commuting observables, and the metalinear frame bundle provides one with a measure with which to define the quantum Hilbert-space inner product. We will describe each of these three structures in some detail (including criteria for existence and uniqueness), and then show how the geometric quantization procedure utilizes them to obtain the quantum Hilbert space $\mathcal{H}$ and the observable representation $\mathfrak{Q}$. First, however, since it is the basis of geometric quantization, let us recall the classical description of physical systems in the language of symplectic geometry.

A *symplectic manifold* (M, ω) is a $2n$-dimensional manifold M together with a distinguished closed nondegenerate two-form ω. The manifold M represents the phase space of a physical system (for which a global configuration space may or may not exist), while ω generalizes the Poisson brackets. The classical observables are realized as the set $C^\infty(M)$ of smooth real-valued functions on M. For any $F \in C^\infty(M)$, the corresponding *canonical* (or *Hamiltonian) vector field* ξ_F is defined by the equation

$$i(\xi_F)\omega = -dF\,, \tag{2.1}$$

where i denotes the left interior product. Such a vector field generates a symplectic automorphism (i.e., a canonical transformation) of (M, ω). For $F = H$, the physical Hamiltonian, ξ_H generates the classical evolution on M.

The first of the geometric quantization structures, the *prequantization line bundle*, consists of a complex line bundle $\pi\colon\ L \to M$ over phase space with a connection ∇ and a compatible Hermitian inner product (,) such that

$$\text{curvature } \nabla = -h^{-1}\omega \tag{2.2}$$

(h = Planck's constant). For a given symplectic manifold (M, ω), a prequantization line bundle $(L, \nabla, (\ ,\ ))$ exists iff the cohomology class $[h^{-1}\omega]$ of $h^{-1}\omega$ is *integral*, i.e., iff $[h^{-1}\omega]$ lies in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. If nonempty, the set of all prequantizations of (M, ω) is parametrized by the direct sum of a certain quotient group of the group $H^1(M, S^1)$ with the group of unitary characters of $\pi_1(M)$. Note that in the simple case in which M is simply connected and ω is exact (i.e., $[h^{-1}\omega] = 0$), the prequantization line bundle is unique and trivial.

Since the covariant derivative ∇ plays an important role in the quantum representation of the classical observables, it is useful to give a local expression for it. So let $U \subseteq M$ be open and contractible, and let λ_U be a local trivializing section of $\pi^{-1}(U)$ with the normalization

$$(\lambda_U, \lambda_U) = 1\,. \tag{2.3}$$

Then, if λ is any smooth section of L, there exists a smooth (complex-valued) function f_U on U such that $\lambda | U = f_U \lambda_U$. On U, it is always possible to choose a connection one-form γ_U such that

$$\omega | U = d\gamma_U \tag{2.4}$$

and

$$\nabla\lambda_U = -i\hbar^{-1}\gamma_U \otimes \lambda_U\,. \tag{2.5}$$

It follows that for any vector field X on M

$$(\nabla_X \lambda) | U = (X(f_U) - i\hbar^{-1}[i(X)\gamma_U] f_U)\lambda_U\,. \tag{2.6}$$

The prequantization line bundle provides us with a preliminary Hilbert-space representation of the classical observables. The prequantization Hilbert space is the completion of the space $\Gamma_0(L)$ of smooth compactly supported sections of L with respect to the inner product

$$\lambda, \chi \to \int_M (\lambda, \chi)\omega^n\,,$$

where ω^n is the canonical volume on M. Then if $F \in C^\infty(M)$ is a classical observable, its representation as an operator $\mathcal{P}F$ on $\Gamma_0(L)$ is defined by

$$\mathcal{P}F[\lambda] \equiv (-i\hbar\nabla_{\xi_F} + F)\lambda\,. \tag{2.7}$$

If the canonical vector field ξ_F of F is complete, then $\mathcal{P}F$ is essentially self-adjoint on $\Gamma_0(L)$.

Unfortunately (and this is why the prequantization Hilbert space and its associated representation of observables cannot be used as the complete quantization of a physical system), the prequantization Hilbert space is "too large."[13] To see this specifically, let M be the cotangent bundle of the configuration space C. Then ω is exact, L is the trivial bundle $T^*C \times \mathbb{C}$, and the prequantization Hilbert space may be identified with $L^2(T^*C, \omega^n)$. But quantum mechanics tells us that we should take "$L^2(C)$" to be the quantum Hilbert space, not $L^2(T^*C, \omega^n)$; otherwise we will violate the uncertainty principle.[14] The process of "polarizing" the classical phase space—the second step in the geometric quantization procedure—accomplishes the desired reduction of the prequantization representation.

A (real) *polarization* of a symplectic manifold (M, ω) is an involutive n-dimensional distribution $\mathfrak{F}$ such that ω restricted to directions in $\mathfrak{F}$ vanishes: $\omega(X, Y) = 0$ for all $X, Y \in \mathfrak{F}$. The polarization $\mathfrak{F}$ induces a foliation of M; we assume that the leaf space $M/\mathfrak{F}$ has a manifold structure such that the canonical projection $\mathfrak{k}: M \to M/\mathfrak{F}$ is a submersion. Fixing a polarization corresponds to choosing a quantum representation (position, momentum, or some mixture), while the leaf space $M/\mathfrak{F}$ is interpreted as a generalized configuration space.

The cotangent bundle T^*C of any manifold C has a distinguished polarization $\mathfrak{F}$, viz., the one spanned by vectors tangent to the fibers of the projection $T^*C \to C$. In this case, the polarization generalizes the position or Schrödinger representation and the leaf space $T^*C/\mathfrak{F}$ clearly can be identified with the configuration space C. Roughly, the quantum Hilbert space should be thought of as consisting of those sections λ of L which are covariantly constant along the leaves of the polarization, with the inner product given by integrating (λ, χ) over C.

In reducing the prequantization representation via polarization, we lose the measure ω^n and its induced inner product. Since the leaf space $M/\mathfrak{F}$ does not in general carry a canonically defined measure, we need yet an additional structure in order to define the quantum Hilbert space—the metalinear frame bundle and its concomitant bundle of half-forms.

Let $\mathcal{B}\mathfrak{F}$ denote the linear frame bundle of $\mathfrak{F}$, i.e., the collection of all ordered bases of $\mathfrak{F}$. It is a $\mathrm{GL}(n, \mathrm{R})$-principal bundle over M. Let $\mathrm{ML}(n, \mathrm{R})$ be the real $(n \times n)$-metalinear group and let ρ: $\mathrm{ML}(n, \mathrm{R}) \to \mathrm{GL}(n, \mathrm{R})$ be the 2 : 1 "covering" homomorphism. We denote by ζ the unique square root of the map $\det \circ \rho$ of $\mathrm{ML}(n, \mathrm{R})$ such that $\zeta(\tilde{1}) = 1$, where $\tilde{1}$ is the identity in $\mathrm{ML}(n, \mathrm{R})$.

A *metalinear frame bundle* of $\mathfrak{F}$ is an $\mathrm{ML}(n, \mathrm{R})$-principal bundle $\tilde{\mathcal{B}}\mathfrak{F}$ over M such that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{B}}\mathfrak{F} \times \mathrm{ML}(n, \mathrm{R}) & \longrightarrow & \tilde{\mathcal{B}}\mathfrak{F} \\ \Big\downarrow \tau \times \rho & & \Big\downarrow \tau \\ \mathcal{B}\mathfrak{F} \times \mathrm{GL}(n, \mathrm{R}) & \longrightarrow & \mathcal{B}\mathfrak{F} \end{array}$$

commutes, where the horizontal arrows denote the group actions, and τ is the 2 : 1 projection $\tilde{\mathcal{B}}\mathfrak{F} \to \mathcal{B}\mathfrak{F}$. A metalinear frame bundle of $\mathfrak{F}$ exists iff a certain class in $H^2(M, \mathrm{Z}_2)$ characteristic of $\mathcal{B}\mathfrak{F}$ vanishes; if nonempty, the set of all inequivalent metalinear frame bundles of a fixed polarization $\mathfrak{F}$ is parametrized by the cohomology group $H^1(M, \mathrm{Z}_2)$.

The *bundle* $\sqrt{\wedge^n \mathfrak{F}}$ *of half-forms* of $\mathfrak{F}$ is the fiber bundle over M associated to $\tilde{\mathcal{B}}\mathfrak{F}$ with typical fiber C on which $\mathrm{ML}(n, \mathrm{R})$ acts by multiplication by $\zeta(\tilde{G}), \tilde{G} \in \mathrm{ML}(n, \mathrm{R})$. A *half-form* ν, that is, a section of $\sqrt{\wedge^n \mathfrak{F}}$, can be uniquely identified with a complex-valued function ν^* on $\tilde{\mathcal{B}}\mathfrak{F}$ satisfying the condition

$$\nu^*(\underline{\tilde{b}} \cdot \tilde{G}) = \zeta(\tilde{G}^{-1})\nu^*(\underline{\tilde{b}}) ,$$

where $\underline{\tilde{b}} \in \tilde{\mathcal{B}}\mathfrak{F}$ and $\tilde{G} \in \mathrm{ML}(n, \mathrm{R})$.

We now give a convenient local expression for sections of $\sqrt{\wedge^n \mathfrak{F}}$. Let $U \subseteq M$ be a contractible open set, and choose a trivialization $\underline{\xi} : U \to \mathcal{B}\mathfrak{F}$ such that each of the vector fields ξ^j comprising $\underline{\xi}(m) = [m; \xi^1(m), \ldots, \xi^n(m)]$ is canonical. Since U is contractible, $\underline{\xi}$ may be lifted to a metalinear frame field $\underline{\tilde{\xi}} : U \to \tilde{\mathcal{B}}\mathfrak{F}$. We define $\nu_{\tilde{\xi}}$ to be the section of $\sqrt{\wedge^n \mathfrak{F}} \,|\, U$ such that the associated map $\nu_{\tilde{\xi}}^* : \tilde{\mathcal{B}}\mathfrak{F} \,|\, U \to \mathrm{C}$ satisfies

$$\nu_{\tilde{\xi}}^* \circ \underline{\tilde{\xi}} = 1 . \tag{2.8}$$

Every section ν of $\sqrt{\wedge^n \mathfrak{F}}$ may therefore be represented on U in the form

$$\nu \,|\, U = (\nu^* \circ \underline{\tilde{\xi}})\nu_{\tilde{\xi}} . \tag{2.9}$$

This expression enables us to define the "Lie derivative" of a half-form along canonical vector fields X which "preserve the polarization" in the sense that $[X, \mathfrak{F}] \subseteq \mathfrak{F}$. Associated with each such vector field X on M is a vector field v_X on $\tilde{\mathcal{B}}\mathfrak{F}$ obtained by lifting from $\mathcal{B}\mathfrak{F}$ the vertical vector field which generates the flow

$$\{\xi^1, \ldots, \xi^n\} \to \{\xi^1 + t[\xi^1, X], \ldots, \xi^n + t[\xi^n, X]\}$$

on $\mathcal{B}\mathfrak{F}$. The Lie derivative of a half-form ν is then defined by the formula

$$(\mathfrak{L}_X \nu)|U \equiv [X(\nu^* \circ \underline{\tilde{\xi}}) + v_X(\nu^*) \circ \underline{\tilde{\xi}}]\nu_{\tilde{\xi}} .$$

This expression may be simplified if we introduce the matrix $A(X)$ with components a^i_j given by

$$[X, \xi^i] = \sum_{j=1}^{n} a^i_j \xi^j ; \tag{2.10}$$

then we find

$$(\mathfrak{L}_X \nu) \,|\, U = [(X + \tfrac{1}{2}\mathrm{tr}A(X))(\nu^* \circ \underline{\tilde{\xi}})]\nu_{\tilde{\xi}} . \tag{2.11}$$

A half-form ν is *covariantly constant along* $\mathfrak{F}$ if $\mathfrak{L}_X \nu = 0$ for all canonical $X \in \mathfrak{F}$.

We now construct the quantum Hilbert space. Let $\bar{\Gamma}_0(L \otimes \sqrt{\wedge^n \mathfrak{F}})$ be the space of smooth compactly supported (modulo $\mathfrak{F}$) sections of $L \otimes \sqrt{\wedge^n \mathfrak{F}}$ which are covariantly constant along $\mathfrak{F}$ (via ∇ *and* $\mathfrak{L}$). The quantum Hilbert space $\mathcal{H}_{\mathfrak{F}}$ is the completion of $\bar{\Gamma}_0(L \otimes \sqrt{\wedge^n \mathfrak{F}})$ with respect to the inner product $\langle \,|\, \rangle$ which we define as follows: Let $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ be a basis of $T_m M$ such that

$$\underline{b} \equiv \{u_1, \ldots, u_n\} \in \mathcal{B}_m \mathfrak{F} ,$$

and

$$\omega(u_i, v_j) = \delta_{ij}, \quad \omega(v_i, v_j) = 0 .$$

It follows that $\underline{d} \equiv \{T\mathfrak{f}(v_1), \ldots, T\mathfrak{f}(v_n)\}$ is a basis for $T_{\mathfrak{f}(m)}(M/\mathfrak{F})$. If $\psi = \lambda \otimes \nu$ and $\sigma = \chi \otimes \mu$ are any two elements of $\bar{\Gamma}_0(L \otimes \sqrt{\wedge^n \mathfrak{F}})$, we pair ψ and σ to obtain a density $\langle \psi, \sigma \rangle$ on the leaf space $M/\mathfrak{F}$ by defining

$$\langle \psi, \sigma \rangle(\underline{d}) \equiv (\lambda(m), \chi(m))\nu^*(\underline{\tilde{b}})\overline{\mu^*(\underline{\tilde{b}})} . \tag{2.12}$$

Here $\tilde{\underline{b}}$ is a lift of $\underline{b}$ to $\tilde{\mathfrak{B}}_m\mathfrak{F}$ and $(,)$ is the Hermitian inner product on $\underline{L}$. Then we set

$$\langle\psi\,|\,\sigma\rangle \equiv \int_{M/\mathfrak{F}} \langle\psi,\sigma\rangle\,. \tag{2.13}$$

With this definition of the quantum Hilbert space $\mathcal{H}_{\mathfrak{F}}$, we complete the geometric quantization program by specifying the quantum representation of the classical observables. For $F\in C^\infty(M)$ such that the canonical vector field ξ_F is complete and preserves $\mathfrak{F}$, the quantum operator $\mathfrak{Q}F$ is defined by[15]

$$\mathfrak{Q}F[\lambda\otimes\nu] \equiv \mathcal{P}F[\lambda]\otimes\nu - i\hbar\lambda\otimes\mathfrak{L}_{\xi_F}\nu \tag{2.14}$$

for $\lambda\otimes\nu\in\tilde{\Gamma}_0(L\otimes\sqrt{\wedge^n\mathfrak{F}})$. The operator $\mathfrak{Q}F$ is essentially self-adjoint on $\tilde{\Gamma}_0(L\otimes\sqrt{\wedge^n\mathfrak{F}})$. If ξ_F is not complete, then we still define $\mathfrak{Q}F$ to be the quantum operator (2.14) corresponding to F, but the essential self-adjointness of $\mathfrak{Q}F$ need not follow, and must be checked on a case-by-case basis.

Formula (2.14) may be considerably simplified by using the local expressions derived earlier. Specifically, let $U\subseteq M$ be open and contractible and let $\underline{\xi}$ be a local metalinear frame field on U projecting onto a linear frame field ξ consisting of canonical vector fields spanning $\mathfrak{F}\,|\,U$. Without loss of generality, we may presume that the general wave function may be written in the form $\lambda\otimes\nu$, where $\nu\,|\,U=\nu_{\underline{\xi}}$ satisfies condition (2.8). Equations (2.7), (2.11), and (2.14) then yield

$$\mathfrak{Q}F[\lambda\otimes\nu]\,|\,U=\{[-i\hbar\nabla_{\xi_F}+F-\tfrac{1}{2}i\hbar\,\mathrm{tr}A(\xi_F)]\lambda\}\otimes\nu\,|\,U\,. \tag{2.15}$$

Geometric quantization has several surprising features, especially in comparison with the canonical theory. First, it is clear that the Kostant-Souriau method always produces a "genuine" Hilbert space as opposed to the indefinite inner product spaces which occasionally result from canonical quantization.[5,6] Second, Eq. (2.14) serves to define the quantum operator $\mathfrak{Q}F$ unambiguously; there can be no factor-ordering problems in geometric quantization. This is reflected by the fact that, in geometric quantization, it is generally not true that

$$\mathfrak{Q}[\eta(F,G,\ldots)]=\eta(\mathfrak{Q}F,\mathfrak{Q}G,\ldots)\,,$$

where η is any functional of the classical observables $F,G,\ldots$ (the "bracket goes to commutator" rule is of course still valid). Third, from (2.15) one sees that the quantum operators corresponding to observables whose canonical vector fields preserve the polarization are (at most) first-order differential operators.[16] Finally, it should be noted that the freedom in the choice of polarization can be used to great advantage in simplifying calculations, cf. Ref. 9. This concludes our brief (and unfortunately rather brutal) summary of those aspects of the Kostant-Souriau theory which will be needed in our discussions below.

III. THE CLASSICAL RWϕ MODELS

Before we can quantize the RWϕ spacetime models using the method of Kostant and Souriau, we must obtain a classical Hamiltonian formulation of them in the language of symplectic geometry. We do that here. Our first description is in unreduced form (i.e., the time is arbitrary and constraints are present). Then, since we have chosen in this paper to quantize only the fully reduced system, we briefly outline how one makes a choice of time and solves the constraints, and apply this reduction procedure to the RWϕ spacetimes.

An RWϕ model is a spatially homogeneous and isotropic spacetime, described by a metric g, which contains a scalar field ϕ such that g and ϕ jointly satisfy the Einstein-Klein-Gordon equations

$$G_{\mu\nu}=\tfrac{1}{2}\,\Box_\mu\phi\Box_\nu\phi-\tfrac{1}{4}g_{\mu\nu}\Box_\alpha\phi\Box^\alpha\phi+\tfrac{1}{4}m^2g_{\mu\nu}\phi^2 \tag{3.1}$$

and

$$\Box^2\phi-m^2\phi=0\,. \tag{3.2}$$

Here we choose units such that $16\pi G/c^4=1$, and denote by m the mass of the scalar field.

The symmetry conditions (homogeneity and isotropy) permit one to write the metric in the general form

$$g=-N^2(t)dt\otimes dt + \frac{R^2(t)}{[1+\frac{1}{4}k(x^2+y^2+z^2)]}\,(dx\otimes dx+dy\otimes dy+dz\otimes dz)\,. \tag{3.3}$$

This expression contains two time-dependent field variables $N(t)$ (the "lapse") and $R(t)$ (the "radius"), as well as the constant $k=+1$, 0, or -1. The choice of k determines the sign of the (constant) intrinsic curvature of the $t=$constant spatial surfaces. In addition, one finds (after integrating the classical evolution equations) that k determines the future of the spacetime: A $k=+1$ model collapses into a singularity while the $k=0$ and $k=-1$ versions expand forever.

The RWϕ models satisfy the MacCallum criteria[17] which permit one to substitute the symmetries into the spacetime action without affecting the dynamics. So we may analyze the RWϕ models using the simplifed action

$$S(g,\phi)=S(N,R,\phi)$$
$$=\int\left\{\int\frac{NR^3}{[1+\frac{1}{4}k(x^2+y^2+z^2)]^{3/2}}\times(-\tfrac{1}{24}\dot{R}^2+6kR+\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}m^2\phi^2)d^3x\right\}dt\,,$$

which is obtained by noting that $\phi(x,t)=\phi(t)$ and employing expression (3.3) for $g(x,t)$. Apparently N, R, and ϕ are the configuration-space parameters. Varying them, and carrying through the Dirac-Bergmann analysis,[18] we find that $N(t)$ is cyclic and remains undetermined, while $R(t)$ and $\phi(t)$ are dynamical variables with corresponding momenta $\pi_R(t)$ and $\pi_\phi(t)$. The effective phase space $\tilde{M}$ for the RWϕ models (regardless of the choice of the parameters k and m) is therefore $T^*\mathbb{R}^2_+$ with coordinates $\{R,\phi,\pi_R,\pi_\phi;R>0\}$; $N(t)$ is an auxiliary arbitrary function. The associated symplectic form is

$$\tilde{\omega}=d\pi_R\wedge dR+d\pi_\phi\wedge d\phi$$

everywhere on $\tilde{M}$, and the Hamiltonian is

$$\tilde{H}=-NK$$
$$=-N\left(\frac{1}{24R}\,\pi_R{}^2-\frac{1}{2R^3}\,\pi_\phi{}^2+6kR-\tfrac{1}{2}m^2\phi^2R^3\right). \tag{3.4}$$

K is a super-Hamiltonian and, as such, it is constrained to vanish. The system contains no other constraints.

This is the complete Hamiltonian formulation of the unreduced system. One could attempt to quantize the RWϕ models in this form, but we will work instead in the reduced formalism (primarily because the calculations are considerably simpler).

A proper reduction[18,19] consists of a choice of time followed by an elimination of the constraint $K=0$ [from (3.4)] in a manner compatible with the time choice. "Choosing the time" means specifying a relation between the arbitrary variable t and one of the dynamical fields (R, π_R, ϕ, π_ϕ, or some functional thereof). This may be done either directly (e.g., $t=R$) or by first specifying $N(t)$ explicitly and then solving one of the evolution equations for the dynamical variable of interest (call it "q_t"). Regardless of how one obtains q_t, the outcome is that q_t is eliminated from the ranks of the dynamical variables, and $N(t)$ is no longer arbitrary.

To solve the constraint "in a manner compatible with the choice of time" means to solve $K=0$ for the variable p_t canonically conjugate to q_t. One does this in order to obtain a reduced formalism which is in symplectic form. The reduced phase space M is the level surface $K^{-1}\{0\}$ in $\tilde{M}$ with q_t quotiented out; it is a two-dimensional manifold which in general differs from $\mathbb{R}^2$. The symplectic form $\tilde{\omega}$ on $\tilde{M}$ induces a closed two-form ω on M which, by construction, is nondegenerate.

The Hamiltonian for the reduced system is p_t, regarded as a function on M. That is, if one solves $K=0$ for $p_t=H$, then H is the effective Hamiltonian. It is a genuine Hamiltonian (as opposed to a super-Hamiltonian) and it generates the classical evolution of the RWϕ universe (as a path in M parametrized by t).

As an example of the reduction procedure, we consider the time choice $t=\phi$ which will actually be used in the subsequent quantization analysis. This choice, together with the evolution equation

$$d\phi/dt=-N\pi_\phi/R^3\,,$$

fixes

$$N=-R^3/\pi_\phi\,.$$

Since π_ϕ is conjugate to ϕ, we solve $K=0$ for π_ϕ obtaining

$$\pi_\phi=R(\tfrac{1}{12}\pi_R{}^2+12kR^2-m^2\phi^2R^4)^{1/2}\,. \tag{3.5}$$

The reduced phase space is now $M=\mathbb{R}^2_+$, with (global) coordinates R and π_R. The symplectic form is

$$\omega=d\pi_R\wedge dR \tag{3.6}$$

and the effective Hamiltonian is $H=\pi_\phi$ from (3.5). This completes the reduction.

At this point, one can proceed to find the classical evolution of the RWϕ models. Leaving out the details,[11] we find (as noted above) that the $k=+1$ models collapse while the $k=0,-1$ models do not. Therefore, we concentrate in this paper on the quantization of the $k=+1$ models. We use the "matter-time" $t=\phi$ in these models for two reasons: (1) unlike, e.g., the "intrinsic" time $t=R$, ϕ-time covers the entire classical evolution of the $k=+1$ RWϕ models; and (2) with ϕ-time, we may quantize the observable R^2 and monitor the asymptotic temporal behavior of its expectation value as a test for collapse. Classically, one finds that the radius R may be expressed parametrically as a function of ϕ as follows:

$$R(\phi)=\frac{R_{\max}}{\left[\cosh\left(\frac{\phi-\phi_0}{\sqrt{3}}\right)\right]^{1/2}}\,. \tag{3.7}$$

IV. QUANTIZATION OF THE MASSLESS MATTER-TIME MODEL

We now apply the geometric quantization procedure outlined in Sec. II to the $k=+1$ RWϕ model cosmologies with massless Klein-Gordon scalar field. As seen in Sec. III, if we choose the "matter-time" $t=\phi$ and reduce, we obtain for our phase space the open half-plane R^2_+ corresponding to the classically allowed values $(0,\infty)$ and $(-\infty,\infty)$ of the canonically conjugate variables R and π_R. The symplectic form on this space is $\omega = d\pi_R \wedge dR$ [Eq. (3.6)], and the reduced Hamiltonian [Eq. (3.5)] may be written in the form

$$H=\beta R({\pi_R}^2+\alpha^2R^2)^{1/2}, \tag{4.1}$$

where $\beta\equiv(2\sqrt{3})^{-1}$ and $\alpha\equiv 12$. Note that H is both positive-definite and time-independent.

It is convenient to introduce on R^2_+ the polar coordinates

$$r=({\pi_R}^2+\alpha^2R^2)^{1/2}$$

and

$$\theta=\frac{1}{2i}\ln\left(\frac{\pi_R+i\alpha R}{\pi_R-i\alpha R}\right).$$

The ranges of r and θ are $(0,\infty)$ and $(0,\pi)$, respectively; the values $r=0$ and $\theta=0,\pi$ represent classically singular states. The symplectic form is now

$$\omega=\frac{r}{\alpha}dr\wedge d\theta, \tag{4.2}$$

while the Hamiltonian (4.1) becomes

$$H=\frac{\beta}{\alpha}r^2\sin\theta. \tag{4.3}$$

The canonical vector field of H is

$$\xi_H=2\beta\sin\theta\frac{\partial}{\partial\theta}-\beta R\cos\theta\frac{\partial}{\partial r}. \tag{4.4}$$

Since R^2_+ is contractible both $H^2(R^2_+,R)$ and $H^1(R^2_+,S^1)$ vanish, so that the prequantization line bundle L is unique and trivial. Let λ be the trivializing section of $L\approx R_+\times C$ which is normalized to unity:

$$\langle\lambda,\lambda\rangle=1. \tag{4.5}$$

The symplectic form is exact, i.e., $\omega=d\gamma$ with

$$\gamma=\frac{r^2}{2\alpha}d\theta,$$

and consequently the covariant derivative [cf. Eq. (2.5)] is given by

$$\nabla\lambda=\frac{r^2}{2i\hbar\alpha}d\theta\otimes\lambda. \tag{4.6}$$

The choice of polarization $\mathfrak{F}$ of R^2_+ is dictated by the desire that the Hamiltonian H be directly quantizable[20] in the representation $\mathfrak{K}_{\mathfrak{F}}$; thus $\mathfrak{F}$ must be such that $[\xi_H,\mathfrak{F}]\subseteq\mathfrak{F}$. A suitable choice is the radial polarization $\mathfrak{F}$ spanned by the canonical vector field $(1/r)\partial/\partial r$, since

$$\left[\xi_H,\frac{1}{r}\frac{\partial}{\partial r}\right]=\frac{2\beta}{r}\cos\theta\frac{\partial}{\partial r}. \tag{4.7}$$

Turning now to the construction of the half-forms, we find that $H^2(R^2_+,Z_2)$ and $H^1(R^2_+,Z_2)$ both vanish so that the metalinear frame bundle $\tilde{\mathfrak{B}}\mathfrak{F}$ of $\mathfrak{F}$ is unique and trivial. The section $\underline{\xi}$ of $\mathfrak{B}\mathfrak{F}$ defined by

$$\underline{\xi}(m)=\left(m,\frac{1}{r}\frac{\partial}{\partial r}\right) \tag{4.8}$$

trivializes $\mathfrak{B}\mathfrak{F}$ and induces the trivializing section

$$\underline{\tilde{\xi}}(m)=(m,\underline{1}) \tag{4.9}$$

of $\tilde{\mathfrak{B}}\mathfrak{F}\approx R^2_+\times ML(1,R)$; clearly, $\tau\circ\underline{\tilde{\xi}}=\underline{\xi}$. The section $\underline{\tilde{\xi}}$ of $\tilde{\mathfrak{B}}\mathfrak{F}$ in turn induces via (2.8) a preferred trivializing section $\nu_{\tilde{\xi}}$ of the bundle of half-forms $\sqrt{\wedge^1\mathfrak{F}}\approx R^2_+\times C$.

To form the quantum representation space $\mathfrak{K}_{\mathfrak{F}}$, we must identify those sections ψ of $L\otimes\sqrt{\wedge^1\mathfrak{F}}$ which are covariantly constant along $\mathfrak{F}$. From (4.6) and (4.8) we have $\nabla_{\underline{\xi}}\lambda=0$ so that λ is covariantly constant. The matrix $A(\underline{\xi})$ is trivially zero and so using (2.8) and (2.11), we find $\mathcal{L}_{\xi}\nu_{\tilde{\xi}}=0$. Consequently, the most general section ψ of $L\otimes\sqrt{\wedge^1\mathfrak{F}}$ which is covariantly constant along $\mathfrak{F}$ has the form $\psi=f\lambda\otimes\nu_{\tilde{\xi}}$, where the complex-valued function f depends only upon θ.

The leaf space $R^2_+/\mathfrak{F}$ is the interval $(0,\pi)$ parametrized by θ, and the projection $\mathfrak{f}$: $R^2_+\to(0,\pi)$ is given by $\mathfrak{f}(r,\theta)=\theta$. Consider the basis $\{(1/r)\partial/\partial r,\ \alpha\partial/\partial\theta\}$ of TR^2_+, and let $\psi=f\lambda\otimes\nu_{\tilde{\xi}}$ and $\sigma=g\lambda\otimes\nu_{\tilde{\xi}}$ be two elements of $\bar{\Gamma}_0(L\otimes\sqrt{\wedge^1\mathfrak{F}})$. Then, according to (2.12), (4.5), and (2.8),

$$\langle\psi,\sigma\rangle\left[T\mathfrak{f}\left(\alpha\frac{\partial}{\partial\theta}\right)\right]=f\bar{g}.$$

The inner product (2.13) is thus

$$\langle\psi|\sigma\rangle=\frac{1}{\alpha}\int_0^\pi f\bar{g}\,|d\theta| \tag{4.10}$$

and the association $f\lambda\otimes\nu_{\tilde{\xi}}\to f$ defines an isomorphism of $\mathfrak{K}_{\mathfrak{F}}$ with $L^2(0,\pi)$.

V. QUANTUM DYNAMICS

Regardless of the quantization procedure employed, the quantum dynamics of a physical system is determined by Schrödinger's equation. Thus, it is necessary to quantize the Hamiltonian (4.3). Since ξ_H preserves $\mathfrak{F}$, formula (2.15) is applicable. If $f\lambda\otimes\nu_{\tilde{\xi}}\in\bar{\Gamma}_0(L\otimes\sqrt{\wedge^1\mathfrak{F}})$, then (2.6), (4.6), and (4.4) give

$$\nabla_{\xi_H}(f\lambda)=[\xi_H(f)-(\beta/\alpha)r^2\sin\theta f]\lambda. \tag{5.1}$$

From (2.10) and (4.7) we calculate $A(\xi_H)=2\beta\cos\theta$, so that by (4.3), (4.4), (5.1), and (2.15),

$$\mathfrak{Q}H[f\lambda\otimes\nu_{\underline{\mathfrak{F}}}]=-i\hbar\left[2\beta\sin\theta\frac{\partial f}{\partial\theta}+\beta\cos\theta f\right]\lambda\otimes\nu_{\underline{\mathfrak{F}}}.$$

Identifying $\mathcal{H}_{\mathfrak{F}}$ and $L^2(0,\pi)$, we may thus view the quantum Hamiltonian as the first-order differential operator

$$\mathfrak{Q}H=-i\hbar\beta\left(2\sin\theta\frac{d}{d\theta}+\cos\theta\right)\qquad(5.2)$$

on $L^2(0,\pi)$.

Since the vector field ξ_H is not complete on $\mathbb{R}^2_+$, we cannot *a priori* conclude that $\mathfrak{Q}H$ is essentially self-adjoint on $C_0^\infty(0,\pi)$ (cf. Sec. II). Nonetheless, this turns out to be the case, and in fact $\mathfrak{Q}H$ is a self-adjoint differential operator on the Sobolev space

$$\mathcal{H}^1_{\mathfrak{Q}H}(0,\pi)=\{f\in L^2(0,\pi)\,|\,\mathfrak{Q}H(f)\in L^2(0,\pi)\}\subset L^2(0,\pi).$$

The time-independent Schrödinger equation is

$$\mathfrak{Q}H[\psi]=E\psi.$$

For $\psi=f\lambda\otimes\nu_{\underline{\mathfrak{F}}}$, this eigenvalue equation becomes

$$2\sin\theta\frac{df}{d\theta}+\cos\theta f=\frac{iE}{\hbar\beta}f,$$

which has the distributional solutions

$$f_E(\theta)=C_E\sin^{-1/2}\theta(\tan\tfrac{1}{2}\theta)^{iE/2\hbar\beta},\qquad(5.3)$$

where the C_E are constants. The wave functions (5.3) are energy normalized:

$$\begin{aligned}\langle f_{E'}\lambda\otimes\nu_{\underline{\mathfrak{F}}}|f_E\lambda\otimes\nu_{\underline{\mathfrak{F}}}\rangle&=\frac{1}{\alpha}C_{E'}\overline{C}_E\int_0^\pi\sin^{-1}\theta\left(\tan\frac{\theta}{2}\right)^{(i/2\hbar\beta)(E'-E)}d\theta\\&=\frac{C_{E'}\overline{C}_E}{\alpha}4\pi\hbar\beta\,\delta(E'-E)\\&=\delta(E'-E)\end{aligned}$$

provided we set

$$C_E=\frac{1}{2}\left(\frac{\alpha}{\pi\hbar\beta}\right)^{1/2}.$$

The spectrum of $\mathfrak{Q}H$ is $(-\infty,\infty)$, and the eigendistributions $\{f_E|E\in\mathbb{R}\}$ form a complete set. Thus, the general solution of the time-dependent Schrödinger equation is

$$\psi(t)=\left[\int_{-\infty}^{+\infty}g(E)f_E(\theta)e^{(iE/\hbar)(t-t_0)}dE\right]\lambda\otimes\nu_{\underline{\mathfrak{F}}},\qquad(5.4)$$

where $g(E)$ is a Fourier amplitude.

We point out that there can be no ambiguity in the interpretation of the Hamiltonian operator (5.2) or the resulting Schrödinger equation. This is, we feel, a practical advantage of the Kostant-Souriau theory *vis-à-vis* the canonical framework, in which the quantum Hamiltonian is realized as the square root of a second-order differential operator. In the latter formalism, it is not clear how the quantum-mechanical system is to evolve[5,11]: via the Schrödinger equation taken "as is" (with the proper functional analytic definition of the square root), according to a Klein-Gordon-type equation, or perhaps even a version of the Dirac equation.

VI. GRAVITATIONAL COLLAPSE

We now show that the quantized RWϕ ($k=+1$, massless) spacetime models all collapse to a singularity (as they do classically). We prove this by calculating the time evolution of the expectation value of the radius (squared) of the model cosmologies, and verifying that for large times, this quantity necessarily decays to zero. That is, for any quantum state $\psi(t)$ we will demonstrate that

$$\lim_{t\to\infty}\langle\psi(t)|\mathfrak{Q}R^2|\psi(t)\rangle=0,\qquad(6.1)$$

where $\psi(t)$ takes the general form (5.4).

Actually, it is more convenient to work in the Heisenberg picture rather than the Schrödinger picture. Thus, in lieu of calculating (6.1), we compute the equivalent quantity

$$\langle\psi(t_0)|\mathfrak{Q}R^2(t)|\psi(t_0)\rangle,\qquad(6.2)$$

where $\psi(t_0)$ represents the state of the universe at the "initial time" t_0. The time-evolving Heisenberg operator $\mathfrak{Q}R^2(t)$ is defined via

$$\mathfrak{Q}R^2(t)=U^{-1}(t-t_0)\mathfrak{Q}R^2U(t-t_0),\qquad(6.3)$$

where $U(t-t_0)$ is the time-development operator generated by $\mathfrak{Q}H$.

We must first calculate $\mathfrak{Q}R^2$ explicitly. In polar coordinates

$$R=\frac{1}{\alpha}r\sin\theta,$$

so that

$$\xi_{R^2}=\frac{2}{\alpha}\sin\theta\left(\sin\theta\frac{\partial}{\partial\theta}-r\cos\theta\frac{\partial}{\partial r}\right).$$

Since

$$\left[\xi_{R^2},\frac{1}{r}\frac{\partial}{\partial r}\right]=\frac{4}{\alpha}\sin\theta\cos\theta\frac{1}{r}\frac{\partial}{\partial r},\qquad(6.4)$$

ξ_{R^2} preserves the radial polarization and hence R^2 may be directly quantized[21] via (2.15). Let $f\lambda\otimes\nu_{\underline{\mathfrak{F}}}\in\tilde{\Gamma}_0(L\otimes\sqrt{\Lambda^1\mathfrak{F}})$ be a quantum wave function. From (2.6) and (4.6) we get

$$\nabla_{\xi_{R^2}}(f\lambda)=\frac{\sin^2\theta}{\alpha}\left(2\frac{\partial f}{\partial\theta}+\frac{r^2}{i\hbar\alpha}f\right)\lambda\,. \tag{6.5}$$

It follows from (6.4) and (2.10) that $A(\xi_{R^2})=(4/\alpha)\sin\theta\cos\theta$. Combining this with (2.15) and (6.5), we obtain the desired result

$$\mathfrak{Q}R^2[f\lambda\otimes\nu_{\bar\xi}]=-\frac{2i\hbar}{\alpha}\sin\theta\left(\sin\theta\frac{\partial f}{\partial\theta}+\cos\theta f\right)\lambda\otimes\nu_{\bar\xi}\,.$$

Consequently, $\mathfrak{Q}R^2$ may be viewed as the ordinary differential operator

$$\mathfrak{Q}R^2=-\frac{2i\hbar}{\alpha}\sin\theta\left(\sin\theta\frac{d}{d\theta}+\cos\theta\right) \tag{6.6}$$

on $L^2(0,\pi)$.

The Heisenberg equation of motion for the operator $\mathfrak{Q}R^2(t)$ is

$$i\hbar\frac{d}{dt}[\mathfrak{Q}R^2(t)]=[\mathfrak{Q}R^2(t),\mathfrak{Q}H]\,, \tag{6.7}$$

subject to the initial condition $\mathfrak{Q}R^2(t_0)=\mathfrak{Q}R^2$. To calculate the right-hand side of (6.7), we use

$$[\mathfrak{Q}R^2(t),\mathfrak{Q}H]=U^{-1}(t-t_0)[\mathfrak{Q}R^2,\mathfrak{Q}H]U(t-t_0)\,. \tag{6.8}$$

From (5.2) and (6.6),

$$[\mathfrak{Q}R^2,\mathfrak{Q}H]=-\frac{2\hbar^2\beta}{\alpha}\left(\sin^3\theta+\frac{\alpha}{i\hbar}\cos\theta\,\mathfrak{Q}R^2\right)\,, \tag{6.9}$$

so that by (6.3) and (6.8) it remains to calculate the time-development of the operators $\sin^3\theta$ and $\cos\theta$. Let us use the convenient notation

$$A(t)\equiv U^{-1}(t-t_0)\sin^3\theta\, U(t-t_0)$$

and

$$B(t)\equiv U^{-1}(t-t_0)\cos\theta\, U(t-t_0)\,.$$

Then, solving the Heisenberg operator equations for $A(t)$ and $B(t)$ we obtain

$$A(t)=\left[1-\left(\frac{1-\tan^2(\frac{1}{2}\theta)e^{4\beta(t-t_0)}}{1+\tan^2(\frac{1}{2}\theta)e^{4\beta(t-t_0)}}\right)^2\right]^{3/2} \tag{6.10}$$

and

$$B(t)=\frac{\cot^2(\frac{1}{2}\theta)e^{-4\beta(t-t_0)}-1}{\cot^2(\frac{1}{2}\theta)e^{-4\beta(t-t_0)}+1} \tag{6.11}$$

Consequently, (6.9)–(6.11) give for (6.8)

$$\frac{d}{dt}[\mathfrak{Q}R^2(t)]-2\beta B(t)\mathfrak{Q}R^2(t)=\frac{2i\hbar\beta}{\alpha}A(t)\,.$$

This linear inhomogeneous equation has the factor-ordered solution

$$\mathfrak{Q}R^2(t)=\frac{e^{-2\beta(t-t_0)}}{\sin^2(\frac{1}{2}\theta)[1+\cot^2(\frac{1}{2}\theta)e^{-4\beta(t-t_0)}]}\times\left(\mathfrak{Q}R^2-\frac{4i\hbar}{\alpha}\tan\tfrac{1}{2}\theta\cos^2(\tfrac{1}{2}\theta)\times\{[1+\tan^2(\tfrac{1}{2}\theta e^{4\beta(t-t_0)}]^{-1}-[1+\tan^2(\tfrac{1}{2}\theta)]^{-1}\}\right). \tag{6.12}$$

We now show that

$$\lim_{t\to\infty}\langle\psi(t_0)|\mathfrak{Q}R^2(t)|\psi(t_0)\rangle=0 \tag{6.13}$$

for any initial state $\psi(t_0)\in L^2(0,\pi)$, thus proving that *all physically well-defined states of the* ($k=1$, $m=0$) RWϕ *universe eventually collapse*. Let $\{|n\rangle, n\in\mathbf{Z}\}$ denote the complete orthornormal basis

$$|n\rangle=\left(\frac{\alpha}{\pi}\right)^{1/2}e^{-2in\theta} \tag{6.14}$$

of $L^2(0,\pi)$ with respect to the inner product (4.10). A Fourier analysis of the initial state

$$\psi(t_0)=\sum_{n=-\infty}^{+\infty}C_n|n\rangle$$

permits us to rewrite the expectation value (6.2) as

$$\langle\psi(t_0)|\mathfrak{Q}R^2(t)|\psi(t_0)\rangle=\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}C_k\bar{C}_n\langle k|\mathfrak{Q}R^2(t)|n\rangle\,.$$

To prove (6.13), it therefore suffices to show that

$$\lim_{t\to\infty}\langle k|\mathfrak{Q}R^2(t)|n\rangle=0$$

for all n and k.

We first consider the diagonal matrix elements. From (4.10) and (6.14), we find

$$\langle n|\mathfrak{Q}R^2(t)|n\rangle=\frac{1}{\pi}\int_0^\pi e^{-2in\theta}\mathfrak{Q}R^2(t)e^{2in\theta}d\theta\,. \tag{6.15}$$

Since $\mathfrak{Q}R^2(t)$ is a symmetric operator the matrix element (6.15) must be real. Consequently, the quantity with coefficient $4i\hbar/\alpha$ in the expression (6.12) for $\mathfrak{Q}R^2(t)$ can be ignored since it contributes only to the imaginary part of (6.15), which must vanish overall. Thus, according to (6.12) and (6.6), the integral (6.15) becomes

$$-\frac{2i\hbar}{\alpha\pi}e^{-2\beta(t-t_0)}\int_0^\pi e^{-2in\theta}\left(\frac{\sin^{-2}(\frac{1}{2}\theta)\sin\theta}{1+\cot^2(\frac{1}{2}\theta)e^{-4\beta(t-t_0)}}\right)\times\left(\sin\frac{d}{d\theta}+\cos\theta\right)e^{2in\theta}d\theta\,.$$

From this expression, we see that the $\cos\theta$ term in $\mathfrak{Q}R^2$ contributes only to the imaginary part of the matrix element and hence may be excluded; we need consider only the $\sin\theta\, d/d\theta$ term. Performing the indicated differentiation and simplifying, we obtain finally

$$\langle n|\mathfrak{Q}R^2(t)|n\rangle=\frac{16\hbar n}{\alpha\pi}e^{-2\beta(t-t_0)}\int_0^\pi\frac{\sin^2(\frac{1}{2}\theta)}{\tan^2(\frac{1}{2}\theta)+e^{-4\beta(t-t_0)}}d\theta\,. \tag{6.16}$$

The integral in (6.16) may be put into the standard form (A1) discussed in the Appendix by making the substitution $y=-2\ln(\tan\frac{1}{2}\theta)$, whence we obtain

$$\int_{-\infty}^{+\infty} \frac{e^{-(3/2)y}}{(e^{-4\beta(t-t_0)}+e^{-y})(1+e^{-y})^2}dy .$$

Conditions (A2) and (A3) on the parameters

$$\mu=\tfrac{3}{2}, \quad \delta=-4\beta(t-t_0), \quad \nu=1, \quad \gamma=0, \quad \rho=2$$

are satisfied, so this integral converges. By (A4), it equals

$$e^{4\beta(t-t_0)}B(\tfrac{3}{2},\tfrac{3}{2})_2F_1(1,\tfrac{3}{2},3;1-e^{4\beta(t-t_0)}) ,$$

which we can transform into

$$B(\tfrac{3}{2},\tfrac{3}{2})_2F_1(1,\tfrac{3}{2},3;1-e^{-4\beta(t-t_0)})$$

by using (A7). Thus we obtain

$$\langle n|\mathcal{Q}R^2(t)|n\rangle=\frac{2\hbar n}{\alpha}e^{-2\beta(t-t_0)}\,{}_2F_1(1,\tfrac{3}{2},3;1-e^{-4\beta(t-t_0)}) , \tag{6.17}$$

where $B(\frac{3}{2},\frac{3}{2})=\pi/8$.

Since $|1-e^{-4\beta(t-t_0)}|<1$ for $t>t_0$, we may series expand the hypergeometric function in (6.17) via (A5), obtaining

$$\langle n|\mathcal{Q}R^2(t)|n\rangle=\frac{2\hbar n}{\alpha}e^{-2\beta(t-t_0)}\sum_{k=0}^{\infty}P_k(1-e^{-4\beta(t-t_0)})^k , \tag{6.18}$$

where

$$P_k=(\tfrac{3}{2})_k/k(3)_k ,$$

cf. the Appendix. As $t\to\infty$, the summation on the right-hand side of (6.18) tends to

$$\sum_{k=0}^{\infty}P_k ,$$

which is just the series expansion of ${}_2F_1(1,\frac{3}{2},3;1)$. Condition (A6) is satisfied so that this sum converges. Therefore,

$$\lim_{t\to\infty}\langle n|\mathcal{Q}R^2(t)|n\rangle=\frac{2\hbar n}{\alpha}\left[\lim_{t\to\infty}e^{-2\beta(t-t_0)}\right]\sum_{k=0}^{\infty}P_k=0 .$$

We now compute the off-diagonal matrix elements. Since $\langle k|\mathcal{Q}R^2(t)|n\rangle=\overline{\langle n|\mathcal{Q}R^2(t)|k\rangle}$, it suffices to consider only the case $n>k$. Equations (4.10), (6.14), (6.12), and (6.6) give

$$\langle k|\mathcal{Q}R^2(t)|n\rangle=-\frac{2i\hbar}{\alpha\pi}e^{-2\beta(t-t_0)}\int_0^{\pi}e^{-2ik\theta}\frac{\sin^{-2}(\frac{1}{2}\theta)}{[1+\cot^2(\frac{1}{2}\theta)e^{-4\beta(t-t_0)}]}\left\{\sin^2\theta\frac{d}{d\theta}+\sin\theta\cos\theta + 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}\left[\left(1+\tan^2\frac{\theta}{2}e^{4\beta(t-t_0)}\right)^{-1} -\left(1+\tan^2\frac{\theta}{2}\right)^{-1}\right]\right\}e^{2in\theta}d\theta$$

$$=-\frac{2i\hbar}{\alpha\pi}e^{-2\beta(t-t_0)}\int_0^{\pi}\frac{\sin\theta\, e^{2i(n-k)\theta}}{\sin^2(\frac{1}{2}\theta)[1+\cot^2(\frac{1}{2}\theta)e^{-4\beta(t-t_0)}]} \times\{2in\sin\theta+\cos\theta-\cos^2(\tfrac{1}{2}\theta)+[1+\tan^2(\tfrac{1}{2}\theta)e^{+4\beta(t-t_0)}]^{-1}\}d\theta .$$

Making the substitution $y=-2\ln(\tan\frac{1}{2}\theta)$, defining $p\equiv n-k$, and using the identity

$$e^{2ip\theta}\equiv[(1+2ie^{-y/2}-e^{-y})/(1+e^{-y})]^{2p} ,$$

we can rewrite this as

$$\langle k|\mathcal{Q}R^2(t)|n\rangle=-\frac{4i\hbar}{\alpha\pi}e^{-2\beta(t-t_0)}\int_{-\infty}^{+\infty}\frac{e^{-y}}{(e^{-4\beta(t-t_0)}+e^{-y})}(1+e^{-y})^{-(2p+1)}(1+2ie^{-y/2}-e^{-y})^{2p} \times\left[2in\frac{e^{-y/2}}{1+e^{-y}}-\frac{e^{-y}}{1+e^{-y}}+(1+e^{-y}e^{4\beta(t-t_0)})^{-1}\right]dy . \tag{6.19}$$

Since p is a positive integer, we may binomially expand

$$(1+2ie^{-y/2}-e^{-y})^{2p}$$

twice to obtain

$$\sum_{j=0}^{2p}\sum_{l=0}^{2p-j}\binom{2p}{j}\binom{2p-j}{l}(2i)^j(-1)^l e^{-(l+j/2)y} .$$

Substituting this into (6.19) and simplifying, we get

$$\langle k|\mathfrak{Q}R^2(t)|n\rangle = -\frac{4i\hbar}{\alpha\pi}e^{-2\beta(t-t_0)}\sum_{j=0}^{2p}\sum_{l=0}^{2p-j}\binom{2p}{j}\binom{2p-j}{l}(2i)^j(-1)^l$$
$$\times\left[2in\int_{-\infty}^{+\infty}\frac{e^{-(l+j/2+3/2)y}}{(e^{-4\beta(t-t_0)}+e^{-y})(1-e^{-y})^{2p+2}}dy - \int_{-\infty}^{+\infty}\frac{e^{-(l+j/2+2)y}}{(e^{-4\beta(t-t_0)}+e^{-y})(1-e^{-y})^{2p+2}}dy + e^{-4\beta(t-t_0)}\int_{-\infty}^{+\infty}\frac{e^{-(l+j/2+1)y}}{(e^{-4\beta(t-t_0)}+e^{-y})^2(1-e^{-y})^{2p+1}}dy\right].$$

The three integrals in the above expression may be calculated using the Appendix; it is straightfoward to check that the convergence conditions (A2) and (A3) are satisfied in each case. Therefore the matrix element becomes

$$\langle k|\mathfrak{Q}R^2(t)|n\rangle = -\frac{4i\hbar}{\alpha\pi}e^{-2\beta(t-t_0)}\sum_{j=0}^{2p}\sum_{l=0}^{2p-j}\binom{2p}{j}\binom{2p-j}{l}(2i)^j(-1)^l$$
$$\times[2in\,e^{4\beta(t-t_0)}B(l+\tfrac{1}{2}j+\tfrac{3}{2},2p-l-\tfrac{1}{2}j+\tfrac{3}{2})\,{}_2F_1(1,l+\tfrac{1}{2}j+\tfrac{3}{2},2p+3;1-e^{4\beta(t-t_0)})$$
$$-e^{4\beta(t-t_0)}B(l+\tfrac{1}{2}j+2,2p-l-\tfrac{1}{2}j+1)\,{}_2F_1(1,l+\tfrac{1}{2}j+2,2p+3;1-e^{4\beta(t-t_0)})$$
$$+e^{4\beta(t-t_0)}B(l+\tfrac{1}{2}j+1,2p-l-\tfrac{1}{2}j+2)\,{}_2F_1(2,l+\tfrac{1}{2}j+1,2p+3;1-e^{4\beta(t-t_0)})].$$

An application of (A7) to the three hypergeometric functions yields finally

$$\langle k|\mathfrak{Q}R^2(t)|n\rangle = -\frac{4i\hbar}{\alpha\pi}\sum_{j=0}^{2p}\sum_{l=0}^{2p-j}\binom{2p}{j}\binom{2p-j}{l}(2i)^j(-1)^l$$
$$\times e^{-2\beta(t-t_0)}[2inB(l+\tfrac{1}{2}j+\tfrac{3}{2},2p-l-\tfrac{1}{2}j+\tfrac{3}{2})\,{}_2F_1(1,2p-l-\tfrac{1}{2}j+\tfrac{3}{2},2p+3;1-e^{-4\beta(t-t_0)})$$
$$-B(l+\tfrac{1}{2}j+2,2p-l-\tfrac{1}{2}j+1)\,{}_2F_1(1,2p-l-\tfrac{1}{2}j+1,2p+3;1-e^{-4\beta(t-t_0)})$$
$$+e^{-4\beta(t-t_0)}B(l+\tfrac{1}{2}j+1,2p-l-\tfrac{1}{2}j+2)\,{}_2F_1(2,2p-l-\tfrac{1}{2}j+2,2p+3;1-e^{-4\beta(t-t_0)})].$$

We now investigate the asymptotic temporal behavior of this matrix element. Typically, the time dependence of the various terms in $\langle k|\mathfrak{Q}R^2(t)|n\rangle$ is of the form

$$e^{-s\beta(t-t_0)}{}_2F_1(a,b,c;1-e^{-4\beta(t-t_0)})$$

for some positive integer s. Since $|1-e^{-4\beta(t-t_0)}|<1$ for $t>t_0$, we may expand the hypergeometric function as in (A5), thereby obtaining

$${}_2F_1(a,b,c;1-e^{-4\beta(t-t_0)}) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r r!}(1-e^{-4\beta(t-t_0)})^r.$$

In all cases $c>a+b$ so that

$$\lim_{t\to\infty}\left(\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r r!}(1-e^{-4\beta(t-t_0)})^r\right)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r r!}$$

is finite. Consequently, as $t\to\infty$, the terms in $\langle k|\mathfrak{Q}R^2(t)|n\rangle$ approach

$$e^{-s\beta(t-t_0)}\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r r!},$$

so finally

$$\lim_{t\to\infty}\langle k|\mathfrak{Q}R^2(t)|n\rangle = 0.$$

The asymptotic decay in time of the off-diagonal matrix elements, combined with that of the diagonal ones, suffices to prove that (6.13) is satisfied for any square-integrable initial state $\psi(t_0)$. This is our previously mentioned result: The quantum models collapse. It is interesting to compare the explicit rate of this collapse with that of the classical $k=+1$ models in ϕ-time. As $t\to\infty$, the dominant terms in $\langle\psi(t_0)|\mathfrak{Q}R^2(t)|\psi(t_0)\rangle$ decay at the rate of $e^{-2\beta(t-t_0)}$, or $e^{-(t-t_0)/\sqrt{3}}$ if we replace β by its numerical value. But from (3.7), the classical radius $R^2(\phi)$ evolves as

$$\frac{R^2_{max}}{\cosh[(\phi - \phi_0)/\sqrt{3}]}$$

so that for large ϕ, $\cosh[(\phi - \phi_0)/\sqrt{3}] \to e^{-(\phi-\phi_0)/\sqrt{3}}$. Thus the quantum collapse rate matches that of the classical models exactly.

One might question whether or not the result (6.13) really implies that the quantized ($k = +1$, $m = 0$) models collapse. Since we have made the choice of time $t = \phi$, this result only says that as ϕ *classically* grows very large, $\langle \mathcal{Q}R^2(\phi)\rangle$ goes to zero. Why—quantum mechanically—should ϕ grow? More precisely, why should the quantum state $\psi(\phi)$ evolve to "$\phi = \infty$"? Is this just an artifact of our classical treatment of ϕ? Might not quantum effects, e.g., a "bounce," prevent this?

To resolve this issue, we recall that the Hamiltonian operator (5.2) is self-adjoint. It follows that probability is conserved, i.e., the norm of any state $\psi(\phi)$ is independent of ϕ. Therefore, there is no "leakage" of the states even as $\phi \to \infty$, and so any given quantum RWϕ model necessarily evolves to the $\phi \to \infty$ limit. The point of our analysis is to show that as this limit is approached, the expectation value of the radius of the given model decays to zero.

One might still object to the inequivalent treatment of ϕ and R in this quantum model of the RWϕ system. However, ϕ and R cannot both be treated as dynamical variables in any reduced Hamiltonian formulation. Therefore they cannot both be quantized, unless one retains the unreduced classical formalism. We shall address the problem of directly quantizing the unreduced system in a future work.

Others have looked at the RWϕ and similar cosmologies and have attempted to delineate alternative criteria for determining whether or not collapse occurs in the quantum system. For example, one may try to thwart collapse by imposing the boundary condition that the quantum wave function vanish at the classical singularity.[22] In the matter-time RWϕ models, however, it is clear from the form (5.3) of the energy eigenfunctions that this boundary condition is inappropriate. Regardless, it is not entirely clear that such a boundary condition—assuming that it can be imposed—actually signifies the absence of collapse. Presumably, as Blyth and Isham[11] point out, one should also consider the behavior of the quantity P_ϵ—the probability that if a measurement of R is made then it will lie in the interval $[0, \epsilon]$—as $\epsilon \to 0$. Unfortunately, it is not clear which values of $dP_\epsilon/d\epsilon$ represent collapse and which do not. The approach to the problem employed in this paper—calculating the asymptotic temporal behavior of the expectation value of R^2—seems to be at least as convincing as any other. Furthermore, as far as we know, ours is the first rigorous fully quantum-mechanical determination (using any criteria) of whether or not collapse occurs.

VII. CONCLUSION

The two main goals of the work described in this paper were (1) to find out whether or not a fully quantized simple model cosmology (specifically, the $k = 1$, $m = 0$ RWϕ model) would be prevented from collapsing into a singularity by quantum effects, and (2) to determine how useful the Kostant-Souriau geometric quantization procedure is in studying homogeneous cosmologies. Our results conclusively answer both questions: The quantized models *do* collapse, and geometric quantization *is* a very effective tool for studying it.

Starting from each of these two conclusions, there are a number of directions in which further research would be useful. Regarding the problem of gravitational collapse, we list the following.

(1) RWϕ models in unreduced form. To avoid any spurious effects due to the choice of time, one should quantize the unreduced system described in the first part of Sec. III.

(2) Massive RWϕ models. Parker and Fulling,[3] using perturbation techniques, claim that this model (RWϕ with $k = +1$ and $m \neq 0$) exhibits quantum bounce. This result should be checked using the (nonperturbative) method of geometric quantization.

(3) Other homogeneous cosmologies. The RWϕ models are among the simplest of the homogeneous cosmologies. One can permit anisotropy and add other fields (e.g., electromagnetism) without losing the simplicity of a finite number of degrees of freedom.[23] Those more complicated homogeneous cosmologies which exhibit collapse classically (e.g., those with S^3 spatial topology) should be studied in quantum form.

(4) Inhomogeneous cosmologies. The only way to decide conclusively whether or not the freezing of degrees of freedom is physically realistic is to quantize models with an infinite number of degrees of freedom.

The first three of these projects are not particularly difficult conceptually and are currently being carried out by the authors. The last requires a considerable amount of new insight.

Let us turn now to the research which is less concerned with the phenomenon of collapse and focuses rather on the usefulness of the geometric quantization procedure.

(1) More complicated models (with or without

collapse). There have been very few applications of geometric quantization to physical systems with more than one or two degrees of freedom. The homogeneous cosmologies are a fertile source of such systems. Many of these cosmologies do not collapse classically, but should nonetheless be interesting as test systems for the Kostant-Souriau procedure.

(2) Choice of time. When are the quantizations of a given cosmology with two different choices of time unitarily equivalent? Study via geometric quantization might help us to answer this question.

(3) Quantizations using different bundles. When the classical symplectic geometry of a given physical system is nontrivial, there may be more than one possible choice of prequantization line bundle and half-form bundle. Are the quantizations based on these choices generally inequivalent? Since many homogeneous cosmologies have such nontrivial classical descriptions, this question may be studied using them.[24]

(4) Choice of polarization. In all physical systems, different choices of polarization are available. Again, the equivalence of the resulting quantizations should be studied.

(5) Systems with singularities in the classical phase space. Many homogeneous cosmologies exhibit linear instabilities and therefore have singular phase spaces.[25] The geometric quantization of such systems should be attempted and compared with perturbation calculations.[26]

Previous work by the authors[9] pertains to some of these issues. In particular, noncollapsing ($k=0$ or -1) RWϕ models with and without mass have been studied with various choices of time and polarization. In the massless case, the classical system is "simple" and so the appropriate bundles are unique, but there are (at least) two polarizations of interest. The resulting quantizations are equivalent, and are also consistent with a canonical quantization of the system (thereby demonstrating that as a geometrical generalization of canonical quantization, the Kostant-Souriau method gives identical results for simple systems). The massive case is more complicated and has not (to our knowledge) been successfully quantized by canonical techniques. The Kostant-Souriau treatment is successful, however. Indeed, it produces two quantizations (choice of two metalinear frame bundles), which one shows are inequivalent (different spectra).

An interesting possibility suggested by this earlier work is the possibility that a different choice of time may be compensated by a different choice of polarization. This phenomenon is currently under study.

Much work obviously remains to be done. Geometric quantization is a new and apparently very useful procedure which should be exploited much more than it has been so far in physics.

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APPENDIX

The integral

$$\int_{-\infty}^{+\infty} \frac{e^{-\mu y}}{(e^{\delta}+e^{-y})^{\nu}(e^{\gamma}+e^{-y})^{\rho}}\, dy \qquad \text{(A1)}$$

converges provided the parameters μ, δ, ν, γ, and ρ satisfy the conditions

$$|\operatorname{Im}\delta| < \pi, \quad |\operatorname{Im}\gamma| < \pi \qquad \text{(A2)}$$

and

$$0 < \operatorname{Re}\mu < \operatorname{Re}(\nu+\rho)\,. \qquad \text{(A3)}$$

It has the following representation[27]:

$$e^{[\gamma(\mu-\rho)-\delta\gamma]}B(\mu, \nu+\rho-\mu)\,{}_2F_1(\nu, \mu, \nu+\rho; 1-e^{\gamma-\delta})\,, \qquad \text{(A4)}$$

where B is the beta function and ${}_2F_1$ is Gauss's hypergeometric function.

Several properties of ${}_2F_1(a,b,c;z)$ are worth noting. It may be expanded in series as follows:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\,, \qquad \text{(A5)}$$

where $(a)_n = a(a+1)\times\cdots\times(a+n-1)$. This series converges if $|z|<1$ and also if $z=1$ provided[28]

$$c > a+b\,. \qquad \text{(A6)}$$

Furthermore,[29]

$${}_2F_1\left(a,b,c,1-\frac{1}{x}\right) = x^a\,{}_2F_1(a,c-b,c;1-x)\,. \qquad \text{(A7)}$$

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