NONUNIMODULARITY AND THE QUANTIZATION OF THE PSEUDO-RIGID BODY

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ABSTRACT. We discuss the pseudo-rigid body as an example of a classical system with nonunimodular gauge symmetries. We show that the standard Dirac quantization prescription must be modified for such systems, and illustrate how this can be accomplished by means of "unimodularization."

1. Introduction

When quantizing a constrained classical system with nonunimodular gauge symmetries, the usual Dirac prescription for selecting the gauge invariant states of the system must be modified [2, 11]. Otherwise, one may lose the correlation between the classical and quantum gauge invariant states, and the quantization of the original system may lead to incorrect results.

To be more precise, let (M,ω) be a symplectic manifold, H a Lie group acting symplectically on M on the left, and J an Ad^*-equivariant momentum mapping for this action Φ. The invariance of the system with respect to the gauge group H demands that J vanish [5]. On the quantum level, these constraints are traditionally implemented by means of the *Dirac conditions*

$$QJ_\xi\,[\Psi] = 0 \tag{1}$$

for all $\xi \in \mathfrak{h}$, the Lie algebra of H, where QJ_ξ is the quantum operator corresponding to the function $J_\xi = \langle J, \xi \rangle$. The space $\mathcal{H}_0$ of all such Ψ is supposed to comprise the physically admissible quantum states of the system.

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1 Recall that a group H is unimodular if it carries a bi-invariant volume. Infinitesimally, this is equivalent to $tr(ad_\xi) = 0$ for all $\xi \in \mathfrak{h}$.

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In general, one expects the Hilbert space $\overline{\mathcal{H}}$ obtained by quantizing the classical reduced phase space $\overline{M} = H\backslash J^{-1}(0)$ to coincide with $\mathcal{H}_0$. Put another way, the processes of reduction and quantization should commute. Provided the reducing group H is unimodular[1] this is indeed the case, at least under mild restrictions which are satisfied by most systems of physical interest. (See [2, 4] and references cited therein.)

When H is not unimodular this equivalence is no longer guaranteed, and in fact can be shown to fail [2, 3, 11]. In practical terms, however, this has not been a matter of concern, as nonunimodular symmetries are rare. Nonetheless, there are bona fide mechanical systems which have such symmetries, and for which the Dirac prescription (1) leads to demonstrably erroneous conclusions. In this paper we will study one such system: the *pseudo-rigid body*.

To regain the correspondence between the original and reduced phase space quantizations, a theory has been developed in [2] which for our purposes can be regarded as a means of "unimodularizing" a classical system with nonunimodular symmetries. One thereby obtains a new unimodular system which is "classically equivalent" to the original system and to which the results cited above apply. Thus reduction and quantization will commute when one considers this new system. This theory is in essence a "bosonic" counterpart to the classical BRST formalism, but is phrased purely in terms of the underlying symplectic geometry of the system.

A comprehensive discussion of these matters is given in [2]. It is enough here to sketch the results. One replaces the original Hamiltonian system (M,ω,H,Φ,J) by an "extended" Hamiltonian system $\left(\tilde{M},\tilde{\omega},\tilde{H},\tilde{\Phi},\tilde{J}\right)$, where:

$$\begin{aligned} \tilde{M} &= M \times (\mathfrak{h}^* \oplus \mathfrak{h}) \\ \tilde{\omega} &= \omega + \omega_{\mathfrak{h}^*} \\ \tilde{H} &= T^*H \approx H \ltimes \mathfrak{h}^* \\ \tilde{\Phi}_{(h,\nu)}(m,\eta,\rho) &= \left(\Phi_h(m), Ad^*_{h^{-1}}(\eta) + \nu, Ad_h\rho\right) \\ \tilde{J}_{(\xi,\nu)}(m,\eta,\rho) &= J_\xi(m) - \langle ad^*_\xi(\eta), \rho\rangle + \langle \nu, \rho\rangle . \end{aligned} \tag{2}$$

In these expressions $\omega_{\mathfrak{h}^*}$ is the canonical symplectic structure on $\mathfrak{h}^* \oplus \mathfrak{h} \approx T^*\mathfrak{h}^*$ and we have used the canonical left trivialization to endow $T^*H \approx H \times \mathfrak{h}^*$ with a semi-direct product structure. (H acts on $\mathfrak{h}^*$ by the coadjoint representation.)

This construction has three salient features: (*i*) The reduced spaces

$$\tilde{H}\backslash\tilde{J}^{-1}(0,0) \approx H\backslash J^{-1}(0),$$

so that this new extended system is classically equivalent to the original one. (*ii*) The extended reducing group $\tilde{H} = T^*H$ is *always unimodular*, so that the quantization of $\tilde{M}$ will be compatible with that of the reduced space $\overline{M}$. (*iii*) It is canonical and symplectically natural. That is, the definition of $\left(\tilde{M}, \tilde{\omega}, \tilde{H}, \tilde{\Phi}, \tilde{J}\right)$ in (2) is not arbitrary, but rather is the result of applying symplectic induction to H viewed as a subgroup of $T^*H \approx H \ltimes \mathfrak{h}^*$.

Since the extended system is unimodular, we may apply the Dirac prescription (1) to it, obtaining

$$\tilde{Q}\tilde{J}_{(\xi,\nu)}[\tilde{\Psi}] = 0 \tag{3}$$

for all $(\xi, \nu) \in \tilde{\mathfrak{h}} \approx \mathfrak{h} \ltimes \mathfrak{h}^*$. Thus the physically admissible quantum states of the system are those which are $\tilde{H}$-invariant. In terms of the original system, (3) effectively reduces to the *modified Dirac prescription*

$$QJ_\xi\,[\Psi] = -\frac{i}{2}\,tr\left(ad_\xi\right)\Psi. \tag{4}$$

It follows that from the standpoint of the nonuniumodular group H, the physically admissible quantum states are actually *quasi*-invariant.

We use the pseudo-rigid body to illustrate these results as well as the formalism developed in [2]. We show that our unimodularization technique (or, equivalently, the modified Dirac prescription (4)) yields the expected — and correct — results, whereas the original Dirac prescription (1) goes awry.

Our conclusions here may have interesting implications regarding the quantization of field theories, as one would not expect their (infinite-dimensional) gauge groups to be unimodular in any reasonable sense. This issue has only begun to be explored [7].

2. The Classical Pseudo-Rigid Body

Consider a pseudo-rigid body, that is, an elastic body moving in $\mathbb{R}^3$ which is allowed to undergo linear deformations. The configuration space for such a body is the group $G = GL(3,\mathbb{R})_+ \ltimes \mathbb{R}^3$, where the identity component $GL(3,\mathbb{R})_+$ of $GL(3,\mathbb{R})$ consists of shears, dilations and rigid rotations, and the $\mathbb{R}^3$ factor comprises rigid translations. The Lie algebra $\mathfrak{g}$ of G consists of pairs (a,b) with $a \in \mathfrak{gl}(3,\mathbb{R})$ and $b \in \mathbb{R}^3$; similarly, we may represent its dual $\mathfrak{g}^*$ by pairs (α,β) with $\alpha \in \mathfrak{gl}(3,\mathbb{R})$ and $\beta \in \mathbb{R}^3$. The pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is defined by

$$\langle(\alpha,\beta),(a,b)\rangle = tr(\alpha a) + \beta b, \tag{5}$$

where we regard β and b as row and column vectors, respectively.

The "true" configuration space (or the *space of shapes*) of the body is clearly the homogeneous space $S = H\backslash G$, where $H = \left(\mathbf{R}_+^* \times SO(3)\right) \ltimes \mathbf{R}^3$. Geometrically, this means that we identify configurations that differ only by rigid rotations, homogeneous dilations and spatial translations. The crucial point for the present discussion is that *the reducing group H is nonunimodular:* for $(c,d) \in \mathfrak{h}$, we have $tr\left(ad_{(c,d)}\right) = tr\ c$. Nonunimodularity actually comes from the dilations via the semi-direct product structure on H.

Let us now describe the details of the classical reduction. We use the canonical left trivialization to identify the phase space T^*G with $M = G \times \mathfrak{g}^*$. (That is, we work in "body coordinates" or the "material representation.") The canonical 1–form on M is $\theta(g,\mu) = \langle \mu, g^{-1}\mathrm{d}g\rangle$, where $g = (A,B)$, $g^{-1}\mathrm{d}g$ denotes the Maurer-Cartan form of G and $\mu = (\alpha,\beta) \in \mathfrak{g}^*$. We thus have

$$\theta = tr\left(\alpha A^{-1}\mathrm{d}A\right) + \beta A^{-1}\mathrm{d}B. \tag{6}$$

The Hamiltonian action of H on M is $(h,(g,\mu)) \rightsquigarrow (hg,\mu)$ with momentum mapping $J : M \to \mathfrak{h}^*$ given by

$$\begin{aligned} J(A,B,\alpha,\beta) &= i^*\left(Ad^*_{(A,B)^{-1}}(\alpha,\beta)\right) \\ &= i^*\left(Ad^*_{A^{-1}}\left(\alpha + A^{-1}B\beta\right), \beta A^{-1}\right), \end{aligned} \tag{7}$$

where $i : \mathfrak{h} \to \mathfrak{g}$ is the inclusion (*cf.* [1], §4.4). From (7) and (5) we see that the vanishing of $\langle J(A,B,\alpha,\beta),(c,d)\rangle$ for all $(c,d) \in \mathfrak{h}$ implies $\beta = 0$ and

$$tr\left(Ad^*_{A^{-1}}(\alpha)\,c\right) = 0. \tag{8}$$

Since $c \in Lie\left(\mathbf{R}_+^* \times SO(3)\right)$, c must be of the form $rI + K$, where $r \in \mathbf{R}$, I is the 3×3 identity and K is skew-symmetric. Taking $K = 0$, (8) yields $tr\ \alpha = 0$. Similarly, taking $r = 0$ we find that $Ad^*_{A^{-1}}(\alpha) = A\alpha A^{-1}$ must be symmetric, so that

$$\alpha^t = \left(A^tA\right)\alpha(A^tA)^{-1}. \tag{9}$$

The constraint submanifold $J^{-1}(0)$ therefore consists of triples (A,B,α), where α is traceless and satisfies (9).

Now every matrix $A \in GL(3,\mathrm{R})_+$ admits a unique polar decomposition $A = RQ$, where $Q = \sqrt{A^tA}$ is positive-definite symmetric and $R \in SO(3)$. Set $\bar{p} = Q\alpha Q^{-1}$ and $\bar{q} = Q/(det\ Q)^{\frac{1}{3}}$. From this and (9) we deduce that the reduced manifold $\overline{M} = H\backslash J^{-1}(0)$ consists of pairs $(\bar{q}, \bar{p})$ of symmetric 3×3 matrices with $\bar{q} > 0$ & $det\,\bar{q} = 1$ and $tr\,\bar{p} = 0$. The symplectic form on $\overline{M}$ is $\bar{\omega} = -\mathrm{d}\bar{\theta}$, where from (6)

$$\bar{\theta} = tr\left(\bar{q}^{-1}\bar{p}\mathrm{d}\bar{q}\right). \tag{10}$$

Evidently

$$\overline{M} \approx T^*S \approx \mathrm{R}^5 \times \mathrm{R}^5.$$

Remarks: (*i*) These results could also have been obtained by means of the cotangent bundle reduction theorem [8]. In this regard we point out that the symplectic structure on $\overline{M} \approx T^*S$ *is* canonical, although the coordinates $(\bar{q}, \bar{p})$ are not (*cf.* (10)). This is because we are working in the material representation of T^*S.

(*ii*) The reduced symplectic manifold T^*S also arises in the liquid drop model of the nucleus [6, 9, 10]. In this context, the cotangent bundle of $S = SO(3)\backslash SL(3,\mathrm{R})$ appears as a coadjoint orbit of the 15–dimensional group $CM(3)$ of (lower) block triangular matrices in $Sp(3,\mathrm{R})$. The fact that $\overline{M}$ can be viewed as a reduction of $T^*SL(3,\mathrm{R})$ has proved quite useful in deriving the general form of "collective" Hamiltonians. Here, however, the reducing group $SO(3)$ is unimodular.

3. Unimodularization and Quantization

We now unimodularize the pseudo-rigid body and quantize it. For the necessary background regarding quantization theory in this context, see [2]. Since we will be working exclusively in the cotangent category, it suffices to consider only the half-density quantization.

Let $\tilde{G} = G \times \mathfrak{h}^*$ be the "extended" configuration space. Then the smooth quantum state space associated to the extended phase space $\tilde{M} \approx T^*\tilde{G}$ is $\Delta^{\frac{1}{2}}(\tilde{G})$, the linear space of 1/2–densities on $\tilde{G}$. We may explicitly describe these half-densities as follows.

Fix an everywhere nonvanishing *right* G–invariant half-density δ_G on G and let $\varepsilon_{\mathfrak{h}^*}$ denote the Lebesgue volume on $\mathfrak{h}^*$. Then

$$\delta_{\tilde{G}} = \delta_G \otimes |\varepsilon_{\mathfrak{h}^*}|^{\frac{1}{2}} \tag{11}$$

is a half-density on the extended configuration space which is invariant under the *left* action of $\tilde{H}$ on $\tilde{G}$ given by[2]

$$\big((h,\nu),(g,\eta)\big) \rightsquigarrow \big(hg, Ad^*_{h^{-1}}(\eta)+\nu\big).$$

Indeed, invariance follows from the observations that (*i*) for $h \in H$, $det(Ad_h)$ is the same regardless of whether one computes the adjoint action with respect to G or its subgroup H,[3] and (*ii*) $\left[det(Ad_h)\right]\left[det(Ad^*_{h^{-1}})\right] = 1$. Thus wave functions $\tilde{\Psi} \in \Delta^{\frac{1}{2}}(\tilde{G})$ take the form $\tilde{\Psi} = \tilde{\psi}(A,B,\eta)\,\delta_{\tilde{G}}$.

Since $\delta_{\tilde{G}}$ is $\tilde{H}$-invariant it follows that $\mathfrak{L}_{(\xi,\nu)_{\tilde{G}}}\delta_{\tilde{G}} = 0$ for all $(\xi,\nu) \in \tilde{\mathfrak{h}}$. (Here $(\xi,\nu)_{\tilde{G}}$ denotes the fundamental vector field on $\tilde{G}$ corresponding to $(\xi,\nu) \in \tilde{\mathfrak{h}}$.) Thus $\tilde{J}_{(\xi,\nu)}$ quantizes as

$$\tilde{Q}\tilde{J}_{(\xi,\nu)}[\tilde{\Psi}] = -i\left[(\xi,\nu)_{\tilde{G}}\,\tilde{\psi}\right]\delta_{\tilde{G}}. \tag{12}$$

Setting $\xi = 0$, the Dirac prescription (3) yields $\tilde{\psi} = \tilde{\psi}(A,B)$ only. Similarly, setting $\nu = 0$ we obtain $\tilde{\psi} = \tilde{\psi}\left(Q/(det\ Q)^{\frac{1}{3}}\right)$. Thus the space $\tilde{\mathcal{H}}_0$ of $\tilde{H}$–invariant quantum states of the pseudo-rigid body consists of wave functions of the form

$$\tilde{\Psi} = \tilde{\psi}\left(Q/(det\ Q)^{\frac{1}{3}}\right)\delta_{\tilde{G}}. \tag{13}$$

We remark that elements of $\tilde{\mathcal{H}}_0$ are not square integrable over $\tilde{G}$.

It is instructive to compare (12) and (13) with what one would obtain by directly quantizing the original system. Every element $\Psi \in \Delta^{\frac{1}{2}}(G)$ can be written $\Psi = \psi(A,B)\,\delta_G$. Note, however, that δ_G is *not* invariant under the action of H on G; in fact,

$$\mathfrak{L}_{\xi_G}\delta_G = \frac{1}{2}\,tr\big(ad_\xi\big)\,\delta_G \tag{14}$$

for all $\xi \in \mathfrak{h}$. Thus we have

$$QJ_\xi[\Psi] = -i\left[\xi_G\psi + \frac{1}{2}\,tr\big(ad_\xi\big)\,\psi\right]\delta_G. \tag{15}$$

[2] Note that the action $\tilde{\Phi}$ in (2) is just the cotangent lift of this action to $\tilde{M} \approx T^*\tilde{G}$.

[3] The same can then be said for $tr(ad_\xi)$ for all $\xi \in \mathfrak{h}$. These coincidences result from the particular forms of G and H.

Consequently the modified quantization condition (4) reduces to $\xi_G \psi = 0$, since the right hand side of (4) exactly cancels the half-density contribution in (15). (This is, of course, why we chose δ_G as we did. Here we are again making crucial use of the coincidence that $tr\left(ad_\xi\right)$ for $\xi \in \mathfrak{h}$ is the same regardless of whether ad_ξ is computed relative to H or G.) The space $\mathcal{H}_*$ of quasi H–invariant states therefore consists of wave functions of the form $\Psi = \psi\left(Q/\left(det\ Q\right)^{\frac{1}{3}}\right)\delta_G$, consistent with (13) and (11).[4]

Next we turn to the quantization of the reduced phase space. The corresponding wave functions $\bar{\Psi} \in \Delta^{\frac{1}{2}}(S)$ may be expressed as $\bar{\Psi} = \bar{\psi}\left(\bar{q}\right)\delta_S$, where δ_S is the natural half-density on S induced from $\delta_{\tilde{G}}$ on $\tilde{G}$ as follows. Since $\tilde{H}$ is unimodular, $\tilde{\mathfrak{h}}$ carries an Ad–invariant volume.[5] Fix a frame $\tilde{\vartheta}$ for $\tilde{\mathfrak{h}}$ of unit volume. Now $\tilde{H}$ acts freely on $\tilde{G}$, so that $\tilde{G} \to S$ is a principal $\tilde{H}$–bundle. Thus if $\bar{\zeta}$ is a frame for TS, we may define

$$\delta_S\left(\bar{\zeta}\right) = \delta_{\tilde{G}}(\zeta, \tilde{\vartheta}_{\tilde{G}}), \tag{16}$$

where ζ is any lift of $\bar{\zeta}$ to TG. Since $\delta_{\tilde{G}}$ is left $\tilde{H}$–invariant and $\tilde{H}$ is unimodular, we see that this definition makes sense. Moreover, δ_S so defined is independent of the choice of frame $\tilde{\vartheta}$ for $\tilde{\mathfrak{h}}$.

The association $\widetilde{\mathcal{H}}_0 \approx \overline{\mathcal{H}}$ given by

$$\tilde{\psi}\left(Q/\left(det\ Q\right)^{\frac{1}{3}}\right)\delta_{\tilde{G}} \leftrightarrow \tilde{\psi}\left(\bar{q}\right)\delta_S \tag{17}$$

therefore provides a *canonical* identification of the $\tilde{H}$–invariant quantum states of the extended system with the quantum states of the reduced system, as guaranteed by the Smooth Equivalence Theorem of [2, 4].

Using this result we can endow the space $\widetilde{\mathcal{H}}_0$ with an inner product. (Recall that elements of $\widetilde{\mathcal{H}}_0$ are not square integrable over $\tilde{G}$.) We simply use the inner product on $\overline{\mathcal{H}} \approx L^2(S)$ along with (17) to define

$$\langle\tilde{\Psi}, \tilde{\Phi}\rangle = \int_S \tilde{\psi}\left(\bar{q}\right)^* \tilde{\phi}\left(\bar{q}\right)\delta_S{}^2.$$

Thus our unimodularization technique enables us to make the space of $\tilde{H}$–invariant states into a Hilbert space.

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4 Although the coefficient function ψ is H–invariant, the total wave function $\Psi = \psi\delta_G$ is *not*, since δ_G is not.

5 In fact, since $\tilde{H}$ is a cotangent bundle there is a *canonical* choice for this volume. Simply demand that it give one when evaluated on frames of the form (ϑ, ϑ^*) for $\tilde{\mathfrak{h}} \approx \mathfrak{h} \ltimes \mathfrak{h}^*$, where ϑ is any frame for $\mathfrak{h}$ and ϑ^* is the dual frame for $\mathfrak{h}^*$.

4. On the Dirac Quantization Conditions

The computations above illustrate how one unimodularizes (and then quantizes) a system with nonunimodular gauge symmetries. We wish to emphasize here the fact that one *must* unimodularize the original system if there is to be an intrinsically defined correspondence between the gauge invariant and reduced quantum states of the system. Why this is so is explained in detail in [11]; see also [2]. But one can gain an appreciation for what goes wrong in the presence of nonunimodular gauge symmetries in general by again examining the specific case of the pseudo-rigid body. One basic problem is that there is no *intrinsic* way to associate H–invariant (or even quasi H–invariant) half-densities on G with half-densities on S; indeed, there is no strict analogue of (16) in either of these contexts. And it is this relation which is the key ingredient in the canonical equivalence (17) of the extended and reduced phase space quantizations.

In the absence of unimodularity — as in the quantization of the original phase space — one can try to mimic the approach of the last section. For instance, fix a frame ϑ for $\mathfrak{h}$; then one may verify from (16) and (11) (scaling ϑ as appropriate) that

$$\delta_S\left(\bar{\zeta}\right) = \delta_G\left(\zeta, \vartheta_G\right). \tag{18}$$

This result gives rise to an isomorphism $\mathcal{H}_* \approx \overline{\mathcal{H}}$, but it is not intrinsically defined as — in contrast to (16) — it depends on the choice of frame ϑ for $\mathfrak{h}$. Even worse, one must resort to an ad hoc construction in order to identify the space $\mathcal{H}_0$ of H–invariant states with $\overline{\mathcal{H}}$. (An example of such an ad hoc isomorphism is provided by (24) below.)

Having extolled the virtues of quantizing the extended unimodular system, let us now study the quantization of the original system. Our analysis thus far does not distinguish in any significant way between the original and modified Dirac quantization conditions; as we have indicated above, both $\mathcal{H}_*$ and $\mathcal{H}_0$ can be (noncanonically) identified with $\overline{\mathcal{H}}$. Thus it may seem that either quantization prescription will lead to results which are (again, noncanonically) equivalent to those obtained by quantizing the reduced space. In fact this is true if one uses the modified Dirac prescription (4) as was demonstrated in [11]. But this is certainly not the case if one uses instead the *uncorrected* Dirac prescription (1): we will show that the resulting quantization is actually *inconsistent* with the quantization of the reduced phase space. The simplest way to do this is via the quantization of observables on M.

If the quantizations of the original and reduced systems are to be equivalent, there must exist an isomorphism $\sigma : \mathcal{H}_0 \to \overline{\mathcal{H}}$ which intertwines the quantizations of (a certain class of) observables. That is, suppose that f is an observable on M which is H–invariant and so reduces to an observable $\bar{f}$ on $\overline{M}$; then we require

$$\sigma\left(\mathcal{Q}f\,[\Psi]\right) = \bar{\mathcal{Q}}\bar{f}\,[\sigma(\Psi)] \tag{19}$$

for all $\Psi \in \mathcal{H}_0$. The class of observables to which (19) applies must include those which are at most linear in the momenta. We now prove, in the case of the pseudo-rigid body, that there does not exist such an isomorphism σ.

For suppose there did. Set $\delta = \sigma^{-1}(\delta_S)$, where δ_S is given by (18). Since $\delta \in \mathcal{H}_0$ it is left H–invariant.

On the other hand, let us consider the action of G on itself by *right* translations. Denote by K the momentum map for the lifted action on M; observe that K is linear in the momenta. Since right translations commute with the left H–action, the G–action descends to S with corresponding momentum map $\bar{K}$ on $\overline{M}$. Using (18) and the fact that δ_G is right–invariant, it follows that δ_S is right-invariant. Thus $\bar{\mathcal{Q}}\bar{K}_\mu\,[\delta_S] = 0$ for all $\mu \in \mathfrak{g}^*$. Then (19) implies that $\mathcal{Q}K_\mu\,[\,\delta\,] = 0$ so that δ is right-invariant as well. Then δ must be a constant multiple of δ_G. But according to (14), δ_G is *not* left H–invariant unless $tr\left(ad_\xi\right)$ vanishes for all $\xi \in \mathfrak{h}$, which it does not. Thus we have a contradiction.

As a concrete illustration, consider the right action of $GL\,(3,\mathbf{R})$ on G given by $\left(W,(A,B)\right) \rightsquigarrow (AW, B)$. The lifted action on M is

$$\left(W,(A,B,\alpha,\beta)\right) \rightsquigarrow \left(AW, B, W^{-1}\alpha W, \beta W\right).$$

The corresponding momentum map $K : M \to \mathfrak{gl}\,(3,\mathbf{R})^*$ can be interpreted as the "material momentum." Analogous to (7) we obtain from (5) that

$$K_\Omega\,(A,B,\alpha,\beta) = tr(\alpha\Omega) \tag{20}$$

for $\Omega \in \mathfrak{gl}\,(3,\mathbf{R})$. Since K_Ω is linear in the momenta, its quantization is routine; using the fact that δ_G is right–invariant, we compute

$$\mathcal{Q}K_\Omega\,[\Psi] = -i\left[tr\left((A\Omega)^t\,\frac{\partial\psi}{\partial A}\right)\right]\delta_G. \tag{21}$$

Extend the $GL(3,\mathbb{R})$–action to $\tilde{G} = G \times \mathfrak{h}^*$ by letting it act trivially on the second factor. Then the momentum map $\tilde{K} : \tilde{M} \to \mathfrak{gl}(3,\mathbb{R})^*$ for the lifted action on $\tilde{M}$ is given by the same formula (20), and it quantizes according to essentially the same formula (21), *viz.*,

$$\tilde{Q}\tilde{K}_\Omega[\tilde{\Psi}] = -i\left[tr\left((A\Omega)^t\frac{\partial\tilde{\psi}}{\partial A}\right)\right]\delta_{\tilde{G}}. \tag{22}$$

On the other hand, as it is H–invariant, K_Ω descends to $\overline{M} \approx T^*S$ as $\bar{K}_\Omega(\bar{q},\bar{p}) = tr(\bar{q}^{-1}\bar{p}\bar{q}\Omega)$. We then find that

$$\bar{Q}\bar{K}_\Omega\,[\bar{\Psi}] = -i\left[tr\left(\kappa\frac{\partial\bar{\psi}}{\partial\bar{q}}\right)\right]\delta_S, \tag{23}$$

where the symmetric matrix κ is the unique solution of

$$\kappa\bar{q} + \bar{q}\kappa = \Omega^t\bar{q}^2 + \bar{q}^2\Omega^t - \frac{2}{3}(tr\,\Omega)\,\bar{q}^2.$$

It is then straightforward, albeit tedious, to directly verify that the isomorphism $\widetilde{\mathcal{H}}_0 \approx \overline{\mathcal{H}}$ unitarily intertwines the operators $\tilde{Q}\tilde{K}_\Omega$ and $\bar{Q}\bar{K}_\Omega$. (Alternatively, this follows from the Smooth Equivalence Theorem.) Similarly one shows using [11] that the isomorphism $\mathcal{H}_* \approx \overline{\mathcal{H}}$ induced by the association (18) intertwines the operators (21) and (23), and that the isomorphism $\widetilde{\mathcal{H}}_0 \approx \mathcal{H}_*$ induced by the association (11) intertwines the operators (21) and (22).

Now suppose that we use instead the uncorrected quantization condition (1). Furthermore assume that the isomorphism $\sigma : \mathcal{H}_0 \to \overline{\mathcal{H}}$ is chosen in such a way that

$$\begin{aligned}\delta(A,B) &= \left[det\left(Ad_{(A,B)}\right)\right]^{-\frac{1}{2}}\delta_G(A,B)\\ &= (det\,A)^{-\frac{1}{2}}\,\delta_G(A,B)\,.\end{aligned} \tag{24}$$

Note that δ is *the* left-invariant helf-density on G associated to the right-invariant half-density δ_G. Then from (21) we compute

$$QK_\Omega[\delta] = \frac{i}{2}(tr\,\Omega)\,\delta,$$

and this does not correspond under σ to $\bar{Q}\bar{K}_\Omega(\delta_S)$, which vanishes. Thus this particular isomorphism σ is not an equivalence of the original and reduced quantum systems. Of course, our result above shows that this is the case for *any* $\sigma : \mathcal{H}_0 \to \overline{\mathcal{H}}$.

We close by presenting an entirely different argument as to why the uncorrected Dirac prescription (1) is untenable. Suppose that we fix the center of mass of the body at the outset, so that the spatial translations in G and H are eliminated. Then H is unimodular! If we now quantize this system and its reduction, we obtain results which coincide exactly with those presented here using the modified quantization condition (4). As one does not expect that fixing the center of mass of the body will substantially alter its quantum behavior, one must again conclude that the "$-\frac{i}{2}\, tr\left(ad_\xi\right)$" correction is essential.

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