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MARK J. GOTAY & LEN BOS

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Abstract

We algebraically reduce the system consisting of a nonrelativistic particle moving in $\mathbb{R}^n$ with vanishing angular momentum $\mathcal{J}$. After analyzing the conical structure of the constraint set $\mathcal{J}^{-1}(0)$, we use algebraic geometric techniques to explicitly construct the reduced Poisson algebra of rotationally invariant observables. This procedure enables us to completely identify the effects of the singularity in $\mathcal{J}^{-1}(0)$ on the system. We then group-theoretically reduce the system and compare our results with those obtained algebraically.

0. Introduction

In celestial mechanics, rotational invariance allows one to eliminate four variables from Lagrange's equations. This procedure, Jacobi's celebrated "elimination of the node," has been generalized by Marsden and Weinstein [8] to the case when an arbitrary symmetry group acts on the phase space of a Hamiltonian system. The idea is as follows.

Consider a constraint of the form $\mathcal{J} = \text{constant}$, where $\mathcal{J}$ is a momentum mapping for the group action. Then one may reduce the number of degrees of freedom of the system by dividing out the symmetries of the constraint set. Subject to certain technical assumptions, Marsden and Weinstein showed that the resulting "reduced phase space" of invariant states is in fact a symplectic manifold.

However, in many interesting situations the Marsden-Weinstein reduction procedure is not applicable and one must use instead the algebraic reduction technique of Śniatycki and Weinstein [12]. This yields a "reduced Poisson algebra" of invariant observables which contains all essential components of the reduced canonical formalism.

Here, following Jacobi, we reduce the system consisting of a nonrelativistic particle moving in $\mathbb{R}^n$ with fixed angular momentum $\mathcal{J} = l$, where $\mathcal{J}: \mathbb{R}^{2n} \rightarrow \mathfrak{so}(n)^*$ is the momentum map for the cotangent action of $\mathrm{SO}(n)$ on the phase space $\mathbb{R}^{2n}$. When $l \neq 0$ the constraint set $\mathcal{J}^{-1}(l)$ is smooth and the reduced canonical formalism is given by the symplectic structure on the Marsden-Weinstein reduced phase space. This case, which is classical and well understood, is discussed in [1]. In this paper we consider the critical case of *zero* angular momentum. Then $\mathcal{J}$ is “singular” in the sense that $\mathcal{J}^{-1}(0)$ fails to be a manifold, and reduction must now proceed on the algebraic level.

Certain adapted coordinates were used in [4] to compute the reduced Poisson algebra when $n = 2$. For $n > 2$ the situation is somewhat more complicated and requires substantially different techniques (although the results are similar) and it is this case which is essentially our subject here.

We begin in §1 by analyzing the structure of the constraint set $\mathcal{J}^{-1}(0)$ and then carry out the algebraic reduction procedure of Śniatycki and Weinstein in §§2–4. Next in §5, we construct the orbit space $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ which surprisingly turns out to be a symplectic V -manifold as well as a singular symplectic manifold. Finally, we compare the group-theoretic and algebraic reductions of this system in §6. A number of tangential results and proofs are relegated to the Appendixes.

This paper is the third in a series (cf. [4] and [5]) devoted to studying the algebraic reduction procedure, both in itself and vis-à-vis its group-theoretic counterpart. This work indicates that these two reductions are often very closely related, so that one may ask under what conditions they will be equivalent for singular momentum mappings (in the nonsingular case equivalence is immediate [12]). It is likely that many of the techniques of this paper can be generalized to other group actions and should therefore prove useful in establishing such equivalence.

1. The structure of the constraint set

The phase space for the particle is $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ with coordinates (x, p) . The standard symplectic form on $\mathbb{R}^{2n}$ is

$$(1.1) \quad \omega = \sum_{k=1}^n dp_k \wedge dx_k$$

and the associated Poisson bracket on $C^\infty(\mathbb{R}^{2n})$ is

$$(1.2) \quad \{f, g\} = \sum_{k=1}^n [f, g]_{x_k, p_k},$$

where

$$[f, g]_{u,v} := (\partial f / \partial u)(\partial g / \partial v) - (\partial f / \partial v)(\partial g / \partial u).$$

The cotangent action of the rotation group on the phase space is $(A, (\mathbf{x}, \mathbf{p})) \rightarrow (A\mathbf{x}, A\mathbf{p})$. Upon identifying $\mathfrak{so}(n)^*$ with $\mathbb{R}^{n(n-1)/2}$, the Ad^* -equivariant angular momentum map for this action may be written

$$(1.3) \quad \mathcal{J}(\mathbf{x}, \mathbf{p}) = \mathbf{x} \wedge \mathbf{p}.$$

We first claim that $\mathcal{J}: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n(n-1)/2}$ has rank $n-1$ everywhere on $\mathcal{J}^{-1}(0)$ except at $\mathbf{0}$. As

$$D\mathcal{J}(\mathbf{x}, \mathbf{p}) \cdot (\mathbf{u}, \mathbf{v}) = \mathbf{u} \wedge \mathbf{p} + \mathbf{x} \wedge \mathbf{v}$$

we may partition the Jacobian of $\mathcal{J}$ into two $n(n-1)/2 \times n$ blocks $[A|B]$, where A is the matrix of the map $a(\mathbf{u}) = \mathbf{u} \wedge \mathbf{p}$ and B that of $b(\mathbf{v}) = \mathbf{x} \wedge \mathbf{v}$. Now suppose $(\mathbf{x}, \mathbf{p}) \in \mathcal{J}^{-1}(0)$ so that $\mathbf{x} = s\mathbf{q}$ and $\mathbf{p} = t\mathbf{q}$ for some unit vector $\mathbf{q}$ and scalars s, t not both zero. Then $A = tC$ and $B = -sC$ for some matrix C . Since clearly $\ker C = \langle \mathbf{q} \rangle$ has dimension one, it follows that $\text{rk}[A|B] = \text{rk}[tC|-sC] = n-1$. Thus $\mathcal{J}^{-1}(0) \subset \mathbb{R}^{2n}$ is an $(n+1)$ -dimensional manifold everywhere except possibly at the origin, where, according to general principles [2], we expect it to have a conical singularity.

Henceforth it is convenient to regard $\mathbb{R}^{2n} = \mathbb{C}^n$ with coordinate $\mathbf{z} = \mathbf{x} + i\mathbf{p}$. Then the $\text{SO}(n)$ -action on $\mathbb{C}^n$ is $(A, \mathbf{z}) \rightarrow A\mathbf{z}$ and $\mathcal{J}$ becomes

$$(1.4) \quad \mathcal{J}(\mathbf{z}) = (i/2)(\mathbf{z} \wedge \bar{\mathbf{z}}).$$

Theorem 1.1. $\mathcal{J}^{-1}(0)$ is a (complex) cone over $\mathbb{R}P^{n-1}$, i.e.,

$$\mathcal{J}^{-1}(0) = C_{\mathbb{C}}(\mathbb{R}P^{n-1}).$$

Proof. First note that $\mathcal{J}(\lambda \mathbf{z}) = |\lambda|^2 \mathcal{J}(\mathbf{z})$. Now consider the fibration:

$$(1.5) \quad \begin{array}{ccc} \mathbb{C} & \rightarrow & \mathbb{C}^n \\ & \searrow \pi_{\mathbb{C}} & \\ & & \mathbb{C}P^{n-1} \end{array}$$

The constraint $\mathbf{z} \wedge \bar{\mathbf{z}} = 0$, when written in terms of affine coordinates $v_i = z_i/z_n$ on $\mathbb{C}P^{n-1}$, becomes a set of reality conditions

$$v_i - \bar{v}_i = 0, \quad i = 1, \dots, n-1.$$

In other words,

$$\pi_{\mathbb{C}}(\mathcal{J}^{-1}(0) - \{\mathbf{0}\}) \subseteq \mathbb{R}P^{n-1} \xrightarrow{\iota} \mathbb{C}P^{n-1},$$

where ι is the natural inclusion. Furthermore, if $[\mathbf{y}] \in \mathbb{R}P^{n-1}$, then $\mathbf{z} = \mathbf{y} + \mathbf{0}i$ belongs to $\mathcal{J}^{-1}(0)$, so $\pi_{\mathbb{C}}$ actually maps $\mathcal{J}^{-1}(0) - \{\mathbf{0}\}$ onto $\mathbb{R}P^{n-1}$. q.e.d.

Before proceeding with our study of $\mathcal{J}^{-1}(0)$ we need some topological results. View S^1 , S^{n-1} , and S^{2n-1} as the unit spheres in $\mathbb{C}$, $\mathbb{R}^n$, and $\mathbb{C}^n$, respectively. Let $S^1 \times_{\mathbf{Z}_2} S^{n-1}$ be the double antipodal identification of $S^1 \times S^{n-1}$ given by

$$(1.6) \quad (\lambda, y) \sim (-\lambda, -y),$$

and define $\text{pr}_{\mathbb{C}}: S^1 \times_{\mathbf{Z}_2} S^{n-1} \rightarrow \mathbb{R}P^{n-1}$ by $\text{pr}_{\mathbb{C}}([\lambda, y]) = [\mathbf{y}]$, where $[\cdot]$ denotes $\sim$ equivalence classes. Then it is straightforward to check that the S^1 -action

$$(1.7) \quad (\eta, [\lambda, y]) \rightarrow [\eta\lambda, y]$$

on $S^1 \times_{\mathbf{Z}_2} S^{n-1}$ makes

$$(1.8) \quad \begin{array}{c} S^1 \rightarrow S^1 \times_{\mathbf{Z}_2} S^{n-1} \\ \downarrow \text{pr}_{\mathbb{C}} \\ \mathbb{R}P^{n-1} \end{array}$$

into a principal circle bundle.

Now consider the Hopf fibration

$$(1.9) \quad \begin{array}{c} S^1 \rightarrow S^{2n-1} \\ \downarrow \pi_{\mathbb{C}} \\ \mathbb{C}P^{n-1} \end{array}$$

associated to (1.5).

Theorem 1.2. *The fibration (1.8) is just the restriction of (1.9) to $\mathbb{R}P^{n-1}$ in $\mathbb{C}P^{n-1}$.*

Proof. Consider the injection $\tilde{S}: S^1 \times_{\mathbf{Z}_2} S^{n-1} \rightarrow S^{2n-1}$ given by

$$(1.10) \quad [\lambda, y] \rightarrow \lambda(y + \mathbf{0}i).$$

A short calculation shows that the diagram

$$\begin{array}{ccc} S^1 \times_{\mathbf{Z}_2} S^{n-1} & \xrightarrow{\tilde{S}} & S^{2n-1} \\ \downarrow \text{pr}_{\mathbb{C}} & & \downarrow \pi_{\mathbb{C}} \\ \mathbb{R}P^{n-1} & \xrightarrow{i} & \mathbb{C}P^{n-1} \end{array}$$

commutes and, moreover, that $\tilde{S}$ is equivariant with respect to the S^1 -actions on $S^1 \times_{\mathbf{Z}_2} S^{n-1}$ and S^{2n-1} . Thus $\tilde{S}$ is a principal bundle map and it follows from [13, §10] that (1.8) is equivalent to the pullback of (1.9) to $\mathbb{R}P^{n-1}$ via i . q.e.d.

From Theorems 1.1 and 1.2 we have that

$$\mathcal{J}^{-1}(0) \cap S^{2n-1} \approx S^1 \times_{\mathbf{Z}_2} S^{n-1},$$

i.e., spherical sections of $\mathcal{J}^{-1}(0)$ are twisted n -handles and are thus *connected*. Furthermore, since $\pi_1(S^1 \times_{\mathbf{Z}_2} S^{n-1}) \approx \mathbf{Z}$ and $\pi_1(\mathbb{R}P^1 \times \mathbb{R}P^{n-1}) \approx \mathbf{Z} \oplus \mathbf{Z}_2$ for $n > 2$, we see that

$$S^1 \times_{\mathbf{Z}_2} S^{n-1} \neq \mathbb{R}P^1 \times \mathbb{R}P^{n-1}.$$

The bundle (1.8) is therefore *nontrivial* if $n > 2$. The exception is when $n = 2$, in which case $S^1 \times_{\mathbf{Z}_2} S^1 \approx T^2$.

Remark. $\mathcal{J}^{-1}(0)$ may also be viewed as a *real cone* over $\mathbb{R}P^1 \times \mathbb{R}P^{n-1}$; this is intimately related to the algebraic geometry of Segre embeddings. As these results are not relevant to our later work, they are presented in the Geometric Appendix.

2. The representation theorem

We now begin the algebraic reduction procedure. Reverting for the time being to real coordinates, let $\mathfrak{S}$ be the ideal in $C^\infty(\mathbb{R}^{2n})$ generated by the $n(n-1)/2$ components $\mathcal{J}_{ij} = x_i p_j - x_j p_i$ of $\mathcal{J}$. The first step is to compute the quotient $C^\infty(\mathbb{R}^{2n})/\mathfrak{S}$ which, if $\mathcal{J}^{-1}(0)$ were a manifold, would simply be $C^\infty(\mathcal{J}^{-1}(0))$. Instead we have the next best thing:

Representation Theorem. $C^\infty(\mathbb{R}^{2n})/\mathfrak{S} \approx W^\infty(\mathcal{J}^{-1}(0))$, the smooth functions on $\mathcal{J}^{-1}(0)$ in the sense of Whitney.

Here, a function on $\mathcal{J}^{-1}(0)$ is “smooth in the sense of Whitney” if it is the restriction of a smooth function on $\mathbb{R}^{2n}$.

Proof. It suffices to show that $f \in \mathfrak{S}$ iff $f|_{\mathcal{J}^{-1}(0)} = 0$. The obverse is immediate. For the converse, suppose $f|_{\mathcal{J}^{-1}(0)} = 0$. We will prove that there exists an open set $\mathfrak{U}$ about each $m \in \mathbb{R}^{2n}$ such that $f|_{\mathfrak{U}} \in \mathfrak{S}|_{\mathfrak{U}}$. The theorem then follows by patching these local results together with a partition of unity.

There are three cases to consider:

- (i) $m \notin \mathcal{J}^{-1}(0)$,
- (ii) $m \in \mathcal{J}^{-1}(0)$, $m \neq \mathbf{0}$,
- (iii) $m = \mathbf{0}$.

Case (i). Choose $\mathfrak{U}$ such that $\mathfrak{U} \cap \mathcal{J}^{-1}(0) = \emptyset$. Let $\mathcal{J}^2$ denote $\sum \mathcal{J}_{ij}^2$. Since $\mathcal{J}^2 \neq 0$ on $\mathfrak{U}$, $g = (f|_{\mathfrak{U}})/\mathcal{J}^2$ is smooth on $\mathfrak{U}$. Then

$$f|_{\mathfrak{U}} = g\mathcal{J}^2 = \sum (g\mathcal{J}_{ij})\mathcal{J}_{ij} \in \mathfrak{S}|_{\mathfrak{U}}.$$

Case (ii). Choose $\mathfrak{U}$ such that $\mathbf{0} \notin \mathfrak{U}$. Since $\mathcal{J}$ has rank $n - 1$ at m , we may use $n - 1$ of the $\mathcal{J}_{ij}$, say $\mathcal{J}_1, \dots, \mathcal{J}_{n-1}$, as coordinates on $\mathfrak{U}$. By Hadamard's Lemma, $f|(\mathfrak{U} \cap \mathcal{J}^{-1}(0)) = 0$ implies that $f| \mathfrak{U} = \sum_{i=1}^{n-1} a_i \mathcal{J}_i$ for some smooth functions a_i .

Case (iii). By a theorem of Malgrange-Martinet [9, §3.5], we need only show that the Taylor series $T_0 f \in \mathfrak{S}_0$, where $\mathfrak{S}_0$ is the ideal generated by $\{T_0 \mathcal{J}_{ij} = \mathcal{J}_{ij}\}$ in the ring of formal power series at the origin. We first define some notation and state several algebraic facts.

Let

$$(2.1) \quad q_{1k} = x_k \quad \text{and} \quad q_{2k} = p_k$$

for $k = 1, \dots, n$, and set

$$q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ q_{21} & \cdots & q_{2n} \end{bmatrix} \in \mathbb{R}^{2 \times n}, \quad i = \begin{bmatrix} i_{11} & \cdots & i_{1n} \\ i_{21} & \cdots & i_{2n} \end{bmatrix} \in \mathbb{N}^{2 \times n}.$$

For any such i , define the row sum $\rho(i)$ of i to be

$$\rho(i) = \left[\sum_{k=1}^n i_{1k}, \sum_{k=1}^n i_{2k} \right]^t$$

and the column sum $\chi(i)$ of i to be

$$\chi(i) = [i_{11} + i_{21}, \dots, i_{1n} + i_{2n}].$$

Consider the polynomial ring $\mathbb{R}[q] = \mathbb{R}[q_{11}, \dots, q_{2n}]$. Any $P \in \mathbb{R}[q]$ may be written in the form

$$P(q) = \sum_i a_i q^i,$$

where each $a_i \in \mathbb{R}$ and $q^i = q_{11}^{i_{11}} \cdots q_{2n}^{i_{2n}}$.

If $\mathfrak{S}$ is the ideal in $\mathbb{R}[q]$ generated by the polynomials $\mathcal{J}_{ij}$, we have

$$V(\mathfrak{S}) = \{q : P(q) = 0 \quad \forall P \in \mathfrak{S}\} = \mathcal{J}^{-1}(0).$$

Now, $I(V(\mathfrak{S})) = \{P \in \mathbb{R}[q] : P(q) = 0 \quad \forall q \in V(\mathfrak{S})\}$. In the Algebraic Appendix we establish that

$$(2.2) \quad I(V(\mathfrak{S})) = \mathfrak{S}.$$

Moreover, we have the explicit representation

$$(2.3) \quad I(V(\mathfrak{S})) = \left\{ P = \sum_i a_i q^i : \sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i = 0 \quad \text{for all } \mathbf{r} \in \mathbb{R}^2, \mathbf{k} \in \mathbb{R}^n \right\}.$$

With these results we are ready to prove case (iii). Write

$$(2.4) \quad T_0 f = \sum_i a_i q^i = \sum_{l=0}^{\infty} S_0^l,$$

where

$$S_0^l = \sum_{r_1+r_2=l} \left(\sum_{\rho(i)=\mathbf{r}} a_i q^i \right)$$

is the homogeneous part of the l th Taylor polynomial of f at $\mathbf{0}$. We may further decompose this as

$$(2.5) \quad S_0^l = \sum_{\substack{r_1+r_2=l \\ k_1+\dots+k_n=l}} \left(\sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i q^i \right).$$

We claim that $S_0^l \in \mathfrak{S}$ for all l . Indeed, define $g: \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ by

$$(2.6) \quad g(u_1, \dots, u_{n+2}) = f(u_{n+1}\mathbf{u}, u_{n+2}\mathbf{u}),$$

where $\mathbf{u} = (u_1, \dots, u_n)$. Since $u_{n+1}\mathbf{u}$ and $u_{n+2}\mathbf{u}$ are linearly dependent, $u_{n+1}\mathbf{u} \wedge u_{n+2}\mathbf{u} = 0$, that is, $(u_{n+1}\mathbf{u}, u_{n+2}\mathbf{u}) \in \mathcal{J}^{-1}(0)$. But as $f|_{\mathcal{J}^{-1}(0)} = 0$, $g \equiv 0$ on $\mathbb{R}^{n+2}$, and so $T_0 g \equiv 0$. Now if $T_0 f$ is given by (2.4), then, according to (2.6), $T_0 g$ consists of sums of terms of the form

$$\left(\sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i \right) (u_{n+1}^{r_1} u_{n+2}^{r_2} u_1^{k_1} \dots u_n^{k_n})$$

each of which must vanish. Thus for any fixed $\mathbf{r}$ and $\mathbf{k}$ we must have

$$\sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i = 0.$$

By (2.3) and (2.5), then, $S_0^l \in I(V(\mathfrak{S}))$ for all l . From (2.2), $S_0^l \in \mathfrak{S}$ for all l .

In summary, we have shown that

$$T_0 f = \sum_{l=0}^{\infty} \sum_{1 \leq i < j \leq n} f_{ij}^l \mathcal{J}_{ij}$$

with $f_{ij}^l \mathcal{J}_{ij}$ homogeneous of degree l . Since each $\mathcal{J}_{ij}$ is homogeneous of degree 2, we may assume that each f_{ij}^l is homogeneous of degree $l - 2$. Thus

$$T_0 f = \sum_{1 \leq i < j \leq n} \left(\sum_{l=0}^{\infty} f_{ij}^l \right) \mathcal{J}_{ij},$$

where, for each (i, j) , $\sum_{l=0}^{\infty} f_{ij}^l$ is a formal power series. By the Borel extension lemma [6, p. 98] these series are Taylor series. Thus, finally, $T_0 f \in \mathfrak{S}_0$.

3. Resolution of the singularity

Returning to complex notation, we have established that $C^\infty(\mathbb{C}^n)/\mathfrak{S} \approx W^\infty(\mathcal{J}^{-1}(0))$. However, this representation of the quotient is not very convenient. Before proceeding with the reduction we obtain a more manageable representation by resolving the singularity in the constraint set.

A. The universal line bundle. View $\mathbb{C}P^{n-1}$ as the set of all complex lines l through the origin in $\mathbb{C}^n$. Let

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & \Gamma \\ & \downarrow \pi & \\ & \mathbb{C}P^{n-1} & \end{array}$$

be the *universal line bundle* over $\mathbb{C}P^{n-1}$. We may identify Γ as the subset of $\mathbb{C}P^{n-1} \times \mathbb{C}^n$ given by

$$\Gamma = \{(l, \mathbf{z}) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n : \mathbf{z} \in l\}.$$

By composing the inclusion with the projection on the second factor we obtain the projection $\varphi: \Gamma \rightarrow \mathbb{C}^n$. Then [7, p. 28]

(i) $\varphi^{-1}(\mathbf{0}) \approx \mathbb{C}P^{n-1}$;

(ii) $\varphi: \Gamma - \varphi^{-1}(\mathbf{0}) \rightarrow \mathbb{C}^n$ is a diffeomorphism.

Γ is called the *blow-up* of $\mathbb{C}^n$ at $\mathbf{0}$; $\dim_{\mathbb{R}} \Gamma = 2n$.

Theorem 3.1. *The complex line bundle Γ is associated to the Hopf fibration (1.9).*

Proof. Consider $S^{2n-1} \times_{S^1} \mathbb{C}$, where the S^1 -action is given by

$$(\eta, (z, w)) \rightarrow (\eta z, \bar{\eta} w).$$

Define a map $S^{2n-1} \times_{S^1} \mathbb{C} \rightarrow \Gamma$ by

$$[z, w] \rightarrow ([\mathbf{z}], w\mathbf{z}),$$

where we view $[\mathbf{z}]$ as a line in $\mathbb{C}^n$ and $w\mathbf{z}$ as a point on $[\mathbf{z}]$. A straightforward check shows that this is a bundle isomorphism. *q.e.d.*

Now let $\mathbf{l}$ be homogeneous coordinates on $\mathbb{C}P^{n-1}$. Then Γ is defined in $\mathbb{C}P^{n-1} \times \mathbb{C}^n$ by the equations

$$(3.1) \quad \mathbf{l} \wedge \mathbf{z} = 0.$$

For later purposes, it is convenient to work this out in coordinates $(v_1, \dots, v_{n-1}, \mathbf{z})$ on $\mathbb{C}P^{n-1} \times \mathbb{C}^n$, where the $v_i = l_i/l_n$, $i = 1, \dots, n-1$, are affine coordinates on $\mathbb{C}P^{n-1}$. Then (3.1) becomes

$$(3.2) \quad z_j = v_j z_n, \quad j = 1, \dots, n-1, \quad v_i z_j - v_j z_i = 0, \quad i, j \neq n.$$

Note that, using the first of these relationships, the second reduces to an identity. In view of (3.2) we may use $(\mathbf{v}, z) := (v_1, \dots, v_{n-1}, z_n)$ as coordinates on Γ . The induced projection $\varphi: \Gamma \rightarrow \mathbb{C}^n$ is now

$$(3.3) \quad \varphi(\mathbf{v}, z) = (z\mathbf{v}, z).$$

B. Blow-ups. We define the *blow-up*, X , of $\mathcal{J}^{-1}(0)$ in Γ to be

$$X = \overline{\varphi^{-1}(\mathcal{J}^{-1}(0) - \{0\})},$$

where the bar denotes Zariski closure (over $\mathbb{R}$). It follows from (3.5) below that X is an $(n + 1)$ -manifold. Note from (ii) above that

$$\psi: X - \varphi^{-1}(0) \rightarrow \mathcal{J}^{-1}(0) - \{0\}$$

is a diffeomorphism, where $\psi = \varphi|_X$. We view $X \subset \Gamma$ as the nonsingular model for $\mathcal{J}^{-1}(0) \subset \mathbb{C}^n$.

However, even though Γ is algebraic in $\mathbb{C}P^{n-1} \times \mathbb{C}^n$, X is *not* algebraic in either Γ or $\mathbb{C}P^{n-1} \times \mathbb{C}^n$. Indeed, consider

$$\mathcal{X} = \mathcal{J} \circ \varphi: \Gamma \rightarrow \mathbb{R}^{n(n-1)/2}.$$

Clearly $X \subset \mathcal{X}^{-1}(0)$ but $\mathcal{X}^{-1}(0)$ contains in addition all of the “exceptional divisor” $\varphi^{-1}(0)$. In charts, $\mathcal{X}$ is given by

$$(3.4) \quad \mathcal{X}_{ij}(\mathbf{v}, z) = \begin{cases} (i/2)|z|^2(v_i\bar{v}_j - \bar{v}_i v_j), & i, j \neq n, \\ (i/2)|z|^2(v_i - \bar{v}_i), & j = n \neq i. \end{cases}$$

The problem is the factor $|z|^2$. From the definition of X , however, we see that, *locally*, X is algebraic: it is determined in Γ by the equations

$$(3.5) \quad v_i = \bar{v}_i, \quad i = 1, \dots, n-1.$$

Note that, in the determination of X , the first equations of (3.4) are redundant.

These observations allow us to explicitly identify X in Γ . Since the defining equations (3.5) for X are reality conditions, it follows that X is *just the restriction of Γ to $\mathbb{R}P^{n-1}$ in $\mathbb{C}P^{n-1}$* . The diagram

$$\begin{array}{ccccc} \mathbb{C}^n & \xleftarrow{\varphi} & \Gamma & \xrightarrow{\pi} & \mathbb{C}P^{n-1} \\ \uparrow & & \uparrow & & \uparrow \iota \\ \mathcal{J}^{-1}(0) & \xleftarrow{\psi} & X & \xrightarrow{\pi} & \mathbb{R}P^{n-1} \end{array}$$

summarizes our results, where the horizontal arrows on the left are all blow-up projections, those on the right are bundle projections, and the vertical arrows are inclusions.

The question remains, however, as to which complex line bundle over $\mathbb{R}P^{n-1}$ the blow-up X corresponds. Recall that the set of (isomorphism classes of) all possible complex line bundles over a manifold M is parametrized by $H^2(M, \mathbb{Z})$. On $\mathbb{C}P^{n-1}$, Γ is characterized by its Chern class $c(\Gamma) = -1$ in $H^2(\mathbb{C}P^{n-1}, \mathbb{Z}) \approx \mathbb{Z}$. Since $X = \iota^*\Gamma$ we have that $c(X) = \iota^*(-1)$, where ι^* is the induced map on second cohomology. Now $H^2(\mathbb{R}P^{n-1}, \mathbb{Z}) \approx \mathbb{Z}_2$ for $n > 2$, so that there are only two line bundles on $\mathbb{R}P^{n-1}$. Thus, X is trivial iff ι^* is trivial on second cohomology. But in the Topological Appendix we show that ι^* is surjective. X is therefore the *nontrivial* complex line bundle over $\mathbb{R}P^{n-1}$. When $n = 2$, on the other hand, X is necessarily trivial.

Using the results of §1 and §3.A, we are now ready to construct X . Since

- (i) Γ is associated to S^{2n-1} ,
- (ii) X is the pullback of Γ to $\mathbb{R}P^{n-1}$, and
- (iii) $S^1 \times_{\mathbb{Z}_2} S^{n-1}$ is the pullback of S^{2n-1} to $\mathbb{R}P^{n-1}$,

we expect that X is associated to $S^1 \times_{\mathbb{Z}_2} S^{n-1}$. In other words, the diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\text{associate}} & \Gamma \\ \downarrow \text{pullback} & & \downarrow \text{pullback} \\ S^1 \times_{\mathbb{Z}_2} S^{n-1} & \xrightarrow{\text{associate}} & X \end{array}$$

should commute. Using [13, §10] we see that this is indeed the case.

Thus from general principles we have that

$$X = (S^1 \times_{\mathbb{Z}_2} S^{n-1}) \times_{S^1} \mathbb{C},$$

where the S^1 -action is given by

$$(\eta, ([\lambda, y], w)) \rightarrow ([\eta\lambda, y], \bar{\eta}w)$$

(cf. (1.7)).

Theorem 3.2. $X \approx S^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}$.

Here the $\mathbb{Z}_2$ -action is $(\mu, (y, w)) \rightarrow (\mu y, \mu w)$ with $\mu = \pm 1$.

Proof. The map $[[\lambda, y], w] \rightarrow [y, \lambda w]$ defines the required isomorphism $(S^1 \times_{\mathbb{Z}_2} S^{n-1}) \times_{S^1} \mathbb{C} \rightarrow S^{n-1} \times_{\mathbb{Z}_2} \mathbb{C}$. q.e.d.

The blow-up fibration is thus

$$\begin{array}{c} \mathbb{C} \rightarrow S^{n-1} \times_{\mathbb{Z}_2} \mathbb{C} \\ \downarrow \\ \mathbb{R}P^{n-1} \end{array}$$

where the projection is $[y, w] \rightarrow [y]$; compare (1.8).

C. The Representation Theorem revisited. As noted earlier, the representation $C^\infty(\mathbb{C}^n)/\mathfrak{S} \approx W^\infty(\mathcal{J}^{-1}(0))$ is not suitable for our purposes. We use the blow-up X of $\mathcal{J}^{-1}(0)$ to represent $C^\infty(\mathbb{C}^n)/\mathfrak{S}$ by its more tractable isomorph

$\psi^*W^\infty(\mathcal{J}^{-1}(0)) \subset C^\infty(X)$. A precise characterization of this kind of subalgebra may be found in [3], but this is not actually needed here.

4. The reduced Poisson algebra

We are finally ready to reduce the Poisson algebra $(C^\infty(\mathbb{C}^n), \{\cdot, \cdot\})$. Rewritten in terms of z and $\bar{z}$, the bracket (1.2) becomes

$$(4.1) \quad \{f, g\} = 2i \sum_{k=1}^n [f, g]_{\bar{z}_k, z_k}.$$

We must first identify the $SO(n)$ -invariant elements of $C^\infty(\mathbb{C}^n)/\mathfrak{S}$. Abstractly, these are the classes $[f]$ such that $\{f, \mathcal{J}_{ij}\} \mid \mathcal{J}^{-1}(0) = 0$ for all $1 \leq i, j \leq n$. Equivalently,

$$(4.2) \quad \{f, \mathcal{J}_{ij}\} \circ \psi = 0.$$

Let $F = f \circ \psi \in \psi^*W^\infty(\mathcal{J}^{-1}(0))$; we will write out these equations explicitly in terms of F on X .

Before doing so we change coordinates on Γ . Since $X \subset \Gamma$ is defined in the (v, z) chart by (3.5), it is convenient to set

$$u = (v + \bar{v})/2, \quad w = (v - \bar{v})/2.$$

The $X \subset \Gamma$ is given by $w = 0$, and we may therefore use (u, z) as coordinates on X . Then (3.3) yields

$$(4.3) \quad \psi(u, z) = (zu, z).$$

From (4.3) we have

$$(4.4) \quad \begin{aligned} \partial F / \partial u_i &= z(\partial f / \partial z_i \circ \psi) + \bar{z}(\partial f / \partial \bar{z}_i \circ \psi), \quad i = 1, \dots, n-1, \\ \partial F / \partial z &= \sum_{i=1}^{n-1} u_i(\partial f / \partial z_i \circ \psi) + \partial f / \partial z_n \circ \psi, \\ \partial F / \partial \bar{z} &= \sum_{i=1}^{n-1} u_i(\partial f / \partial \bar{z}_i \circ \psi) + \partial f / \partial \bar{z}_n \circ \psi. \end{aligned}$$

Also, substituting (1.4) into (4.1) we obtain

$$(4.5) \quad \{f, \mathcal{J}_{ij}\} = \{\bar{z}_i \partial f / \partial \bar{z}_j + z_i \partial f / \partial z_j - (\bar{z}_j \partial f / \partial \bar{z}_i + z_j \partial f / \partial z_i)\}.$$

Combining (4.5) and (4.4), the invariance conditions (4.2) become

$$(4.6) \quad \begin{aligned} u_i \partial F / \partial u_j - u_j \partial F / \partial u_i &= 0, \quad i, j < n, \\ u_i (\bar{z} \partial F / \partial \bar{z} + z \partial F / \partial z) - \partial F / \partial u_i - u_i \sum_{k=1}^{n-1} u_k \partial F / \partial u_k &= 0, \\ i < n, \quad j &= n. \end{aligned}$$

This is a system of $n(n-1)/2$ linear homogeneous partial differential equations on the $(n+1)$ -manifold X . Denote the corresponding distribution by $\mathfrak{D}$; we claim that $\dim_{\mathbf{R}} \mathfrak{D} = n-1$. Indeed, consider the matrix of $\mathfrak{D}$ with respect to $\{\partial_z, \partial_{\bar{z}}, \partial_{u_1}, \dots, \partial_{u_{n-1}}\}$. It is readily verified that the kernel of this matrix is spanned by the vectors

$$(1 + |\mathbf{u}|^2, 0, zu_1, \dots, zu_{n-1})^t \quad \text{and} \quad (0, 1 + |\mathbf{u}|^2, \bar{z}u_1, \dots, \bar{z}u_{n-1})^t.$$

Hence the dimension of the kernel is two and the rank of the matrix is, therefore, $n+1-2 = n-1$. Since $\dim_{\mathbf{R}} X = n+1$, this system of equations has at most two functionally independent first integrals. In fact, there are exactly two: $z(1 + |\mathbf{u}|^2)^{1/2}$ and $\bar{z}(1 + |\mathbf{u}|^2)^{1/2}$.

Let $\mathfrak{F} \subseteq \psi^* W^\infty(\mathcal{J}^{-1}(0))$ denote the subspace of rotationally invariant observables. It corresponds under ψ to those functions in $W^\infty(\mathcal{J}^{-1}(0))$ which are constant along the orbits of $\text{SO}(n)$ on $\mathcal{J}^{-1}(0)$. We have proven:

Theorem 4.1. *Let $F \in \psi^* W^\infty(\mathcal{J}^{-1}(0))$. Then $F \in \mathfrak{F}$ iff*

$$(4.7) \quad F = F(z(1 + |\mathbf{u}|^2)^{1/2}, \bar{z}(1 + |\mathbf{u}|^2)^{1/2}).$$

This characterization of $\mathfrak{F}$ can be substantially simplified. Our main result is

Theorem 4.2. *Let $F \in C^\infty(X)$. Then $F \in \mathfrak{F}$ iff*

$$(4.8) \quad F = F(z^2(1 + |\mathbf{u}|^2), |z|^2(1 + |\mathbf{u}|^2), \bar{z}^2(1 + |\mathbf{u}|^2)).$$

Proof. Fix $F(\mathbf{u}, z) \in \mathfrak{F}$. We first claim that F is even in z . Now $F \in \psi^* W^\infty(\mathcal{J}^{-1}(0))$ so that there exists an $f \in C^\infty(\mathbb{C}^n)$ such that $F = f \circ \psi$. Since F is rotationally invariant

$$(4.9) \quad f(A\mathbf{z}) = f(\mathbf{z})$$

for all $A \in \text{SO}(n)$ and $\mathbf{z} \in \mathcal{J}^{-1}(0)$. In particular, for each $\mathbf{z}$ we may choose A in such a way that $A\mathbf{z} = -\mathbf{z}$. Then (4.9) reduces to $f(-\mathbf{z}) = f(\mathbf{z})$, i.e., $f|_{\mathcal{J}^{-1}(0)}$ must be even. Since ψ maps X onto $\mathcal{J}^{-1}(0)$, this and (4.3) imply that

$$F(\mathbf{u}, -z) = F(\mathbf{u}, z)$$

as was to be shown. But then [9, p. 144] and (4.7) yield (4.8).

For the converse, first observe that

$$(4.10) \quad \begin{aligned} z^2(1 + |\mathbf{u}|^2) &= \psi^*(\mathbf{z} \cdot \mathbf{z}), \\ |\mathbf{z}|^2(1 + |\mathbf{u}|^2) &= \psi^*(\mathbf{z} \cdot \bar{\mathbf{z}}), \\ \bar{z}^2(1 + |\mathbf{u}|^2) &= \psi^*(\bar{\mathbf{z}} \cdot \bar{\mathbf{z}}). \end{aligned}$$

Then it is obvious that any $F \in C^\infty(X)$ of the form (4.8) smoothly factors through ψ . The result now follows from Theorem 4.1. q.e.d.

Thus the requirement that $F \in \psi^*W^\infty(\mathcal{J}^{-1}(0))$ in Theorem 4.1 is superfluous. Theorem 4.2 enables us to directly identify $\mathfrak{F}$ as a subspace of $C^\infty(X)$ without having to pass through the intermediary $\psi^*W^\infty(\mathcal{J}^{-1}(0))$. This, ultimately, is why we resolved the singularity in the constraint set.

It remains to compute the reduced Poisson bracket on $\mathfrak{F}$. Since $\mathcal{J}$ is equivariant, the Poisson bracket $\{\cdot, \cdot\}$ on $C^\infty(\mathbb{C}^n)$ descends to a bracket $[\cdot, \cdot]$ on $\mathfrak{F} \subset C^\infty(\mathbb{C}^n)/\mathfrak{N}$ given abstractly by

$$[[f], [g]] = [\{f, g\}].$$

On X this translates into

$$(4.11) \quad [F, G] = \{f, g\} \circ \psi,$$

where $f, g \in C^\infty(\mathbb{C}^n)$ are such that $F = f \circ \psi$ and $G = g \circ \psi$. Taking (4.8) into account, we calculate

$$\begin{aligned} (\partial f / \partial z_i) \circ \psi &= u_i / (1 + |\mathbf{u}|^2) \partial F / \partial z, \quad i = 1, \dots, n-1, \\ (\partial f / \partial z_n) \circ \psi &= 1 / (1 + |\mathbf{u}|^2) \partial F / \partial z \end{aligned}$$

along with the corresponding complex conjugate equations. Substituting these into (4.1), (4.11) yields

$$[F, G] = 2i / (1 + |\mathbf{u}|^2) [F, G]_{\bar{z}, z}.$$

It is immediate that $[\cdot, \cdot]$ is nondegenerate.

We may further simplify matters by again changing coordinates $(\mathbf{u}, z) \rightarrow (\mathbf{u}, \gamma = z(1 + |\mathbf{u}|^2)^{1/2})$. Then we have

$$(4.12) \quad \mathfrak{F} = \{F \in C^\infty(X) : F = F(\gamma^2, |\gamma|^2, \bar{\gamma}^2)\}$$

and the bracket becomes

$$(4.13) \quad [F, G] = 2i [F, G]_{\bar{\gamma}, \gamma}.$$

The pair $(\mathfrak{F}, [\cdot, \cdot])$ is the *reduced Poisson algebra* of $\text{SO}(n)$ -invariant observables for our particle with zero angular momentum. From this structure one may recover the entire reduced canonical formalism, albeit on the algebraic level (i.e., invariant *observables*) rather than that of manifolds (i.e., invariant *states*).

5. The structure of the orbit space $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$

To ascertain the geometric significance of the reduced Poisson algebra it is useful to also perform a group-theoretic reduction of the system. Here we compute the “Poisson algebras” of the orbit space $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$, and in the next section we compare them to $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket)$.

We first observe that the nontrivial orbits of $\mathrm{SO}(n)$ on $\mathcal{J}^{-1}(0)$ are all diffeomorphic to S^{n-1} .

Theorem 5.1. $\mathcal{J}^{-1}(0)/\mathrm{SO}(n) \approx \mathbb{C}/\mathbb{Z}_2$.

Proof. Fix $\mathbf{q} \in S^{n-1}$ and consider the map $Q: \mathbb{C} \rightarrow \mathbb{C}^n$ given by

$$(5.1) \quad Q(\lambda) = \lambda \mathbf{q}.$$

Clearly, $Q(\lambda) \in \mathcal{J}^{-1}(0) \cap S_{|\lambda|}^{2n-1}$. We claim that as λ varies with $|\lambda|$ fixed, $Q(\lambda)$ hits every orbit in $\mathcal{J}^{-1}(0) \cap S_{|\lambda|}^{2n-1}$. Indeed, every element of $\mathcal{J}^{-1}(0) \cap S_{|\lambda|}^{2n-1}$ may be written in the form $\eta \mathbf{y}$ for some complex $\eta \in S^1$ and some real $\mathbf{y} \in S_{|\lambda|}^{n-1}$. Then, since $\mathrm{SO}(n)$ acts transitively on S^{n-1} , we have

$$\eta \mathbf{y} = \eta |\mathbf{y}| (\mathbf{y}/|\mathbf{y}|) = \eta |\mathbf{y}| A \mathbf{q} = A(Q(\eta \mathbf{y}))$$

for some $A \in \mathrm{SO}(n)$. Furthermore, $Q(\lambda)$ and $Q(\xi)$ both lie on the same orbit, i.e., $\lambda \mathbf{q} = \xi A \mathbf{q}$ for some $A \in \mathrm{SO}(n)$, iff $\xi = \pm \lambda$. It follows that Q induces an isomorphism $\mathcal{J}^{-1}(0)/\mathrm{SO}(n) \approx \mathbb{C}/\mathbb{Z}_2$, where the latter denotes the identification $\lambda \sim -\lambda$. $\square$

Aside. Since every $\mathbf{z} \in \mathcal{J}^{-1}(0)$ may be expressed as $\mathbf{z} = \lambda A \mathbf{q}$ for some $\lambda \in \mathbb{C}$, $A \in \mathrm{SO}(n)$, and fixed $\mathbf{q} \in S^{n-1}$, we may explicitly realize the projection $p: \mathcal{J}^{-1}(0) \rightarrow \mathbb{C}/\mathbb{Z}_2$ as follows:

$$p(\lambda A \mathbf{q}) = [\lambda].$$

Now consider $\mathbb{R}P^1 \subset \mathbb{C}/\mathbb{Z}_2$. Then

$$p^{-1}(\mathbb{R}P^1) = \{\lambda A \mathbf{q} : \lambda \in S^1\} = \mathcal{J}^{-1}(0) \cap S^{2n-1}$$

and we therefore obtain the orbit fibration:

$$\begin{array}{ccc} S^{n-1} & \rightarrow & \mathcal{J}^{-1}(0) \cap S^{2n-1} \\ & & \downarrow p \\ & & \mathbb{R}P^1 \end{array}$$

This gives additional insight into the structure of $\mathcal{J}^{-1}(0)$; compare (1.8) and (G.2).

Thus the orbit space $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ is actually a V -manifold [14]. Coincidentally, it is possible to view $\mathbb{C}/\mathbb{Z}_2$ as a genuine C^∞ -manifold, diffeomorphic to $\mathbb{C}$, by endowing it with the unique C^∞ structure which makes the projection

$\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}_2$ differentiable. We may therefore proceed in either of two ways, depending upon which manifold structure we take on $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$.

Now use Q to pull the symplectic form (1.1), which on $\mathbb{C}^n$ is

$$\omega = (1/2i) \sum_{k=1}^n dz_k \wedge d\bar{z}_k,$$

back to $\mathbb{C}$. Since $|\mathbf{q}| = 1$ we obtain

$$Q^*\omega = (1/2i) d\lambda \wedge d\bar{\lambda}$$

which is just the standard symplectic structure on $\mathbb{C}$. Since $Q^*\omega$ is invariant under the reflection $\lambda \rightarrow -\lambda$, we may obviously regard $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ as a *symplectic V -manifold* [14]. If, however, we take the C^∞ structure on $\mathbb{C}/\mathbb{Z}_2$, then $Q^*\omega$ projects to a *singular* symplectic form on $\mathbb{C}/\mathbb{Z}_2$. We may then regard $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ as a *singular symplectic manifold*.

The corresponding “Poisson algebras” of $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ can be described as follows. In either of the above cases every smooth function $\hat{f}$ on $\mathbb{C}/\mathbb{Z}_2$ may be uniquely represented by an even function $f \in C^\infty(\mathbb{C})$. According to [9, p. 144], such a function must be quadratic in $\lambda, \bar{\lambda}$. Thus, if we regard $\mathbb{C}/\mathbb{Z}_2$ as a V -manifold, then the algebra $\mathfrak{B}$ of smooth functions on $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ has the explicit representation

$$(5.2) \quad \mathfrak{B} \approx \{f \in C^\infty(\mathbb{C}) : f = f(\lambda^2, |\lambda|^2, \bar{\lambda}^2)\}.$$

On the other hand, if we regard $\mathbb{C}/\mathbb{Z}_2$ as a C^∞ -manifold, then such an f must also smoothly factor through the projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}_2$ so that $f = f(\lambda^2, \bar{\lambda}^2)$ only. Thus, denoting $C^\infty(\mathcal{J}^{-1}(0)/\mathrm{SO}(n))$ by $\mathfrak{C}$, we have

$$(5.3) \quad \mathfrak{C} \approx \{f \in C^\infty(\mathbb{C}) : f = f(\lambda^2, \bar{\lambda}^2)\}.$$

In either case, the Poisson bracket of two smooth functions $\hat{f}, \hat{g}$ on $\mathbb{C}/\mathbb{Z}_2$ is represented by

$$(5.4) \quad 2i[f, g]_{\bar{\lambda}, \lambda}.$$

There are consequently two “Poisson algebras” ($\mathfrak{B}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda}$) and ($\mathfrak{C}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda}$) associated to $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$. The first is the usual nondegenerate Poisson V -algebra defined by a symplectic V -structure. The second, although nondegenerate, is *singular* in the sense that $\mathfrak{C}$ is *not* closed under the bracket (5.4). The fact that neither of these Poisson algebras is the Poisson algebra of a symplectic manifold is a reflection of the presence of the singularity in $\mathcal{J}^{-1}(0)$.

6. Comparison of algebraic and group-theoretic reductions

Despite the fact that the group-theoretic reduction of §5 is singular, it yields results which are surprisingly similar to those obtained algebraically in §4. We now show that, when we regard $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ as a symplectic V -manifold, the Poisson V -algebra $(\mathfrak{B}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda})$ of $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ is isomorphic to $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket)$. Furthermore, when we regard $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ as a singular symplectic manifold we find that $(\mathfrak{C}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda})$ and $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket)$, although not isomorphic, are closely related.

Theorem 6.1. $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket) \approx (\mathfrak{B}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda})$.

Proof. Fix $\mathbf{q} \in S^{n-1}$ as before, and use Q to lift elements of $\mathfrak{B}$ to $\mathrm{SO}(n)$ -invariant functions on $\mathbb{C}^n$. From (5.1) and (5.2) we obtain

$$(6.1) \quad \mathfrak{B} \approx \{f \in C^\infty(\mathbb{C}^n) : f = f(\mathbf{z} \cdot \mathbf{z}, \mathbf{z} \cdot \bar{\mathbf{z}}, \bar{\mathbf{z}} \cdot \bar{\mathbf{z}})\}.$$

Now, according to (4.12), the reduced Poisson algebra $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket)$ consists of functions $F \in C^\infty(X)$ of the form

$$F = F(\gamma^2, |\gamma|^2, \bar{\gamma}^2).$$

Since each such F must smoothly factor through ψ , it follows from (4.10) that $F \in \mathfrak{F}$ iff $F = f \circ \psi$ for some $f \in C^\infty(\mathbb{C}^n)$ of the form (6.1). Thus $\mathfrak{B} \approx \mathfrak{F}$. Clearly, from (4.3) and (5.4),

$$\llbracket F, G \rrbracket \sim 2i[f, g]_{\bar{\lambda}, \lambda}$$

and the result follows. q.e.d.

Thus, when we regard $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ as a symplectic V -manifold, the algebraic and group-theoretic reductions coincide. On the other hand, when we regard $\mathcal{J}^{-1}(0)/\mathrm{SO}(n)$ as a singular symplectic manifold, (5.3) and Theorem 6.1 imply that $\mathfrak{C}$ is *strictly* a subspace of $\mathfrak{F}$, so that $(\mathfrak{C}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda})$ can be identified with a *singular* subalgebra of $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket)$. However, it is remarkable that $(\mathfrak{F}, \llbracket \cdot, \cdot \rrbracket)$ is in fact the *closure* of $(\mathfrak{C}, 2i[\cdot, \cdot]_{\bar{\lambda}, \lambda})$ in the following sense:

Theorem 6.2. $\mathfrak{F} = \bar{\mathfrak{C}} := \mathfrak{C} + [\mathfrak{C}, \mathfrak{C}]_{\bar{\lambda}, \lambda} + [\mathfrak{C}, [\mathfrak{C}, \mathfrak{C}]_{\bar{\lambda}, \lambda}]_{\bar{\lambda}, \lambda} + \cdots$.

Proof. We first claim that the above infinite series truncates with the result that

$$(6.2) \quad \bar{\mathfrak{C}} = \mathfrak{C} + [\mathfrak{C}, \mathfrak{C}]_{\bar{\lambda}, \lambda}.$$

To this end, let $f, g \in \mathfrak{C}$; then a computation reveals that

$$(6.3) \quad [f, g]_{\bar{\lambda}, \lambda} = 4|\lambda|^2 [f, g]_{X^2, X^2} \in |\lambda|^2 \mathfrak{C}.$$

Using this, we have

$$[f, [g, h]_{\bar{\lambda}, \lambda}]_{\bar{\lambda}, \lambda} = [f, |\lambda|^2 k]_{\bar{\lambda}, \lambda}$$

for some $k \in \mathfrak{C}$. Then another computation yields

$$[f, [g, h]_{\bar{\lambda}, \lambda}]_{\bar{\lambda}, \lambda} = 2k(\bar{\lambda}^2(\partial f/\partial \bar{\lambda}^2) - \lambda^2(\partial f/\partial \lambda^2)) + 4|\lambda|^4[f, k]_{\bar{\lambda}^2, \lambda^2}$$

which, as each term is in $\mathfrak{C}$, is also in $\mathfrak{C}$. The claim follows by iterating this result.

Next, we establish that $\bar{\mathfrak{C}} = \mathfrak{C} + |\lambda|^2\mathfrak{C}$. The inclusion $\bar{\mathfrak{C}} \subseteq \mathfrak{C} + |\lambda|^2\mathfrak{C}$ follows immediately from (6.2) and (6.3). For the reverse inclusion we need only show that $|\lambda|^2\mathfrak{C} \subseteq [\mathfrak{C}, \mathfrak{C}]_{\bar{\lambda}, \lambda}$ and thus by (6.3) that $\mathfrak{C} \subseteq [\mathfrak{C}, \mathfrak{C}]_{\bar{\lambda}^2, \lambda^2}$. But for $f \in \mathfrak{C}$, let $h(\lambda^2, \bar{\lambda}^2) = \int f(\lambda^2, \bar{\lambda}^2) d\lambda^2 \in \mathfrak{C}$. Then

$$[\bar{\lambda}^2, h]_{\bar{\lambda}^2, \lambda^2} = \partial h/\partial \lambda^2 = f$$

and we are done.

We now show that $\mathfrak{F} \approx \mathfrak{C} + |\lambda|^2\mathfrak{C}$ thereby completing the proof of the theorem. For this we use Theorem 6.1 to identify $\mathfrak{F}$ with $\mathfrak{B}$; then only the inclusion $\mathfrak{B} \subseteq \mathfrak{C} + |\lambda|^2\mathfrak{C}$ is nontrivial. Reverting to real notation, set $\lambda = a + ib$ and let $F \in \mathfrak{B}$. Then by (5.2) $F = F(a^2 - b^2, ab, a^2 + b^2)$, i.e., there exists an $f \in C^\infty(\mathbb{R}^3)$ such that $F(a, b) = f(a^2 - b^2, ab, a^2 + b^2)$. Let $P(a, b, c) = c^2 - (a^2 + 4b^2)$. Applying the Mather Division Theorem [6, p. 95] with distinguished variable c to f yields

$$f(a, b, c) = g(a, b) + ch(a, b) + P(a, b, c)k(a, b)$$

for some $g, h, k \in C^\infty(\mathbb{R}^2)$. Thus

$$\begin{aligned} F(a, b) &= f(a^2 - b^2, ab, a^2 + b^2) \\ &= g(a^2 - b^2, ab) + (a^2 + b^2)h(a^2 - b^2, ab) + 0, \end{aligned}$$

which, upon going back to complex notation, proves that $F \in \mathfrak{C} + |\lambda|^2\mathfrak{C}$. q.e.d.

Thus for a particle in $\mathbb{R}^n$ with zero angular momentum, one may construct the reduced canonical formalism in either of two ways: algebraically or group-theoretically. The group-theoretic reduction yields two reduced Poisson algebras, one regular and the other singular. The fact that the regular Poisson algebra is isomorphic to that constructed algebraically indicates that the V -manifold approach is both the correct and most natural one [J. Arms, private communication]. Moreover, an examination of other examples shows that in general it is not possible to consider the orbit space $\mathcal{J}^{-1}(0)/G$ as a smooth manifold with a singular symplectic structure. Thus the existence of the second Poisson algebra is peculiar to this system. Regardless, this singular algebra is effectively “repaired” by the algebraic reduction procedure which closes it into the regular Poisson algebra $(\mathfrak{F}, [\cdot, \cdot])$ of all $\text{SO}(n)$ -invariant observables for the particle.

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Algebraic appendix

We follow the conventions of §2.

Let $\mathfrak{S}$ be the ideal in $\mathbb{R}[q]$ generated by the polynomials

$$\mathcal{J}_{ij} = q_{1i}q_{2j} - q_{1j}q_{2i}$$

and set

$$\mathfrak{A} = \left\{ P = \sum_i a_i q^i : \sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i = 0 \text{ for all } \mathbf{r} \in \mathbb{R}^2 \text{ and } \mathbf{k} \in \mathbb{R}^n \right\}.$$

Our goal is to prove

Theorem A.1. $I(V(\mathfrak{S})) = \mathfrak{A}$.

Proof. Consider $P \in I(V(\mathfrak{S}))$. Let $g: \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ be given by

$$g(u_1, \dots, u_{n+2}) = P(u_{n+1}\mathbf{u}, u_{n+2}\mathbf{u}),$$

where $\mathbf{u} = (u_1, \dots, u_n)$. Since $u_{n+1}\mathbf{u}$ and $u_{n+2}\mathbf{u}$ are linearly dependent and P is identically zero on $V(\mathfrak{S})$ we have that g is identically zero on $\mathbb{R}^{n+2}$. But if $P = \sum_i a_i q^i$, then

$$g(u_1, \dots, u_{n+2}) = \sum_i a_i (u_{n+1}, u_{n+2})^{\rho(i)} \mathbf{u}^{\chi(i)}$$

and hence, for any fixed $\rho(i) = \mathbf{r}$ and $\chi(i) = \mathbf{k}'$,

$$\sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i = 0.$$

Thus $I(V(\mathfrak{S})) \subseteq \mathfrak{A}$. We now proceed to show that $\mathfrak{A} \subseteq \mathfrak{S} \subseteq I(V(\mathfrak{S}))$.

Suppose that now $P \in \mathfrak{A}$. Without loss of generality we may take P to be of the form

$$P = \sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i q^i.$$

Let j be such that $\rho(j) = \mathbf{r}$ and $\chi(j) = \mathbf{k}'$. As $P \in \mathfrak{A}$,

$$P = \sum_{\substack{\rho(i)=\mathbf{r} \\ \chi(i)=\mathbf{k}'}} a_i (q^i - q^j).$$

It therefore suffices to show that $q^i - q^j \in \mathfrak{S}$ when $\rho(i) = \rho(j)$ and $\chi(i) = \chi(j)$.

By the division of factors common to q^i and q^j , we need only consider pairs i and j where

$$i_{st} = 0 \quad \text{if } j_{st} > 0 \quad \text{and} \quad j_{st} = 0 \quad \text{if } i_{st} > 0,$$

and then by symmetry we may assume that

$$i = \begin{bmatrix} i_{11} & \cdots & i_{1m} & 0 & \cdots & 0 \\ * & \cdots & & & \cdots & * \end{bmatrix},$$

$$j = \begin{bmatrix} 0 & \cdots & 0 & j_{1,m+1} & \cdots & j_{1M} & 0 & \cdots & 0 \\ * & \cdots & & & \cdots & & & \cdots & * \end{bmatrix}.$$

But the column sums $\chi(i)$ and $\chi(j)$ are equal, hence

$$i = \begin{bmatrix} i_{11} & \cdots & i_{1m} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & j_{1,m+1} & \cdots & j_{1M} & 0 & \cdots & 0 \end{bmatrix},$$

$$j = \begin{bmatrix} 0 & \cdots & 0 & j_{1,m+1} & \cdots & j_{1M} & 0 & \cdots & 0 \\ i_{11} & \cdots & i_{1m} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with, as $\rho(i) = \rho(j)$, $i_{11} + \cdots + i_{1m} = j_{1,m+1} + \cdots + j_{1M}$.

Then it is not difficult to see that, using terms of the form

$$i(s, t) = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

$\begin{matrix} & & & \uparrow & & & & \uparrow & & & \\ & & & s & & & & t & & & \end{matrix}$

we can write a telescoping series

$$\begin{aligned} & i - (i - i_1) \\ & + (i - i_1) - (i - i_1 - i_2) \\ & + (i - i_1 - i_2) - (i - i_1 - i_2 - i_3) \\ & + \cdots \\ & + (i - i_1 - \cdots - i_{N-1}) - (i - i_1 - \cdots - i_N) = i - j, \end{aligned}$$

where each of the bracketed terms has *positive* entries and each $i_k = i(s_k, t_k)$ for some s_k and t_k . Defining $l_k = i - i_1 - \cdots - i_k$, the corresponding telescoping series for powers of q yields

$$q^i - q^j = \sum_{k=0}^{N-1} (q^{l_k} - q^{l_{k+1}}),$$

where $l_k - l_{k+1} = i_{k+1}$.

It is easy to verify that if $l_k - l_{k+1} = i(s_{k+1}, t_{k+1})$, then $q^{l_k} - q^{l_{k+1}}$ is a polynomial multiple of $\mathcal{J}_{s_{k+1}, t_{k+1}}$ and hence a member of $\mathfrak{S}$. As $\mathfrak{S}$ is clearly a subset of $I(V(\mathfrak{S}))$, the result follows. q.e.d.

The proof of the theorem actually establishes

Corollary A.2. $I(V(\mathfrak{S})) = \mathfrak{S}$.

Geometric appendix

Our analysis in §1 centered about the fact that $\mathcal{J}^{-1}(0)$ is a *complex* cone over $\mathbb{R}P^{n-1} \subset \mathbb{C}P^{n-1}$. But it is apparent from (1.3) that $\mathcal{J}^{-1}(0)$ is also a *real* cone over some subvariety of $\mathbb{R}P^{2n-1}$. Here we study the structure of the constraint set from the real point of view.

Theorem G.1. $\mathcal{J}^{-1}(0)$ is a real cone over $\mathbb{R}P^1 \times \mathbb{R}P^{n-1}$, i.e.,

$$\mathcal{J}^{-1}(0) = C_{\mathbf{R}}(\mathbb{R}P^1 \times \mathbb{R}P^{n-1}).$$

Proof. Consider the real analog of (1.5), viz.

$$(G.1) \quad \begin{array}{c} \dot{\mathbb{R}} \rightarrow \dot{\mathbb{R}}^{2n} \\ \downarrow \pi_{\mathbf{R}} \\ \mathbb{R}P^{2n-1}. \end{array}$$

View $(\mathbf{x}, \mathbf{p})$ as homogeneous coordinates on $\mathbb{R}P^{2n-1}$ and label them collectively by q_{ik} as in (2.1). Also let λ_i , $i = 1, 2$, and y_k , $k = 1, \dots, n$, be homogeneous coordinates on $\mathbb{R}P^1$ and $\mathbb{R}P^{n-1}$, respectively.

Now recall that the *Segre embedding*

$$S: \mathbb{R}P^1 \times \mathbb{R}P^{n-1} \rightarrow \mathbb{R}P^{2n-1}$$

is defined by $q_{ik} = \lambda_i y_k$. Denote by $\mathfrak{S}$ the homogeneous ideal in $\mathbb{R}[q]$ generated by the polynomials $q_{ik}q_{jl} - q_{il}q_{jk}$. Then [11, Proposition 2.12] shows that

$$S(\mathbb{R}P^1 \times \mathbb{R}P^{n-1}) = V(\mathfrak{S})$$

in $\mathbb{R}P^{2n-1}$. But from (1.3) and (2.1) it follows that

$$V(\mathfrak{S}) = \pi_{\mathbf{R}}(\mathcal{J}^{-1}(0) - \{0\})$$

and the theorem is proven. q.e.d.

Mimicking the analysis in §1, we now study the double covering

$$(G.2) \quad \begin{array}{ccc} \mathbb{Z}_2 & \rightarrow & S^1 \times_{\mathbb{Z}_2} S^{n-1} \\ & & \downarrow \text{pr}_{\mathbb{R}} \\ & & \mathbb{R}P^1 \times \mathbb{R}P^{n-1} \end{array}$$

where $\text{pr}_{\mathbb{R}}([\lambda, y]) = ([\lambda], [y])$ and the $\mathbb{Z}_2$ -action is given by (1.7) for $\eta = \pm 1$. The real analog of (1.9) is the double covering

$$(G.3) \quad \begin{array}{ccc} \mathbb{Z}_2 & \rightarrow & S^{2n-1} \\ & & \downarrow \pi_{\mathbb{R}} \\ & & \mathbb{R}P^{2n-1} \end{array}$$

associated to (G.1).

Theorem G.2. *The fibration (G.2) is the pullback to $\mathbb{R}P^1 \times \mathbb{R}P^{n-1}$ of the fibration (G.3) by the Segre embedding S .*

Proof. First note that the diagram

$$(G.4) \quad \begin{array}{ccc} S^1 \times_{\mathbb{Z}_2} S^{n-1} & \xrightarrow{\tilde{S}} & S^{2n-1} \\ \downarrow \text{pr}_{\mathbb{R}} & & \downarrow \pi_{\mathbb{R}} \\ \mathbb{R}P^1 \times \mathbb{R}P^{n-1} & \xrightarrow{S} & \mathbb{R}P^{2n-1} \end{array}$$

commutes, where, in real notation (cf. (1.10)), $\tilde{S}$ is given by $[\lambda, y] \rightarrow (\lambda_1 y, \lambda_2 y)$. Since $\tilde{S}$ is equivariant with respect to the $\mathbb{Z}_2$ -actions in (G.2) and (G.3), the desired result follows from [13, §10].

Remark. Diagram (G.4) shows that $\tilde{S}$ doubly covers the Segre embedding S .

Topological appendix

Let $\iota: \mathbb{R}P^{n-1} \rightarrow \mathbb{C}P^{n-1}$ be the natural inclusion $[y] \rightarrow [y + 0i]$. We show that the induced map $\iota^*: H^2(\mathbb{C}P^{n-1}, \mathbb{Z}) \rightarrow H^2(\mathbb{R}P^{n-1}, \mathbb{Z})$ is surjective when $n > 2$.

Consider the short exact sequence

$$(T.1) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$$

and the induced cohomology diagram:

$$(T.2) \quad \begin{array}{ccc} H^2(\mathbb{C}P^{n-1}, \mathbb{Z}_2) & \xrightarrow{i^*} & H^2(\mathbb{R}P^{n-1}, \mathbb{Z}_2) \\ \uparrow & & \uparrow \\ H^2(\mathbb{C}P^{n-1}, \mathbb{Z}) & \xrightarrow{i^*} & H^2(\mathbb{R}P^{n-1}, \mathbb{Z}) \end{array}$$

First observe that both vertical arrows are surjections; this follows from the long exact cohomology sequences associated to (T.1) and the facts that both $H^3(\mathbb{C}P^{n-1}, \mathbb{Z})$ and $\partial^*: H^2(\mathbb{R}P^{n-1}, \mathbb{Z}_2) \rightarrow H^3(\mathbb{R}P^{n-1}, \mathbb{Z})$ are always trivial. Furthermore, recall that both $H^2(\mathbb{C}P^{n-1}, \mathbb{Z})$ and $H^2(\mathbb{R}P^{n-1}, \mathbb{Z}_2)$ are isomorphic to $\mathbb{Z}_2$ for $n > 2$. These results imply that the bottom arrow in (T.2) is surjective iff the top arrow is.

In fact, the top arrow is an isomorphism. To see this, factor

$$i = c \circ j_{n-1} \circ \cdots \circ j_1,$$

where $j_k: \mathbb{R}P^{n+k-1} \rightarrow \mathbb{R}P^{n+k}$ is given by

$$j_k(\llbracket (y_1, \dots, y_{n+k}) \rrbracket) = \llbracket (y_1, \dots, y_{2k-1}, 0, y_{2k+1}, \dots, y_{n+k}) \rrbracket$$

for $1 \leq k \leq n$ and $c: \mathbb{R}P^{2n-1} \rightarrow \mathbb{C}P^{n-1}$ is defined by

$$c(\llbracket (y_1, \dots, y_{2n}) \rrbracket) = \llbracket (y_1 + iy_2, \dots, y_{2n-1} + iy_{2n}) \rrbracket.$$

Since each j_k is essentially an inclusion, we have from [10, §8.5.10(a)] that $j_k^*: H^2(\mathbb{R}P^{n+k}, \mathbb{Z}_2) \approx H^2(\mathbb{R}P^{n+k-1}, \mathbb{Z}_2)$. Moreover, [10, §8.5.10(b)] establishes that $c^*: H^2(\mathbb{C}P^{n-1}, \mathbb{Z}_2) \approx H^2(\mathbb{R}P^{2n-1}, \mathbb{Z}_2)$. Thus $i^*: H^2(\mathbb{C}P^{n-1}, \mathbb{Z}_2) \approx H^2(\mathbb{R}P^{n-1}, \mathbb{Z}_2)$.

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UNITED STATES NAVAL ACADEMY
UNIVERSITY OF CALGARY, CANADA

