On quantizing semisimple basic algebras, I: sl(2,R)

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Dedicated to Jerry Marsden on the occasion of his 60th birthday

ABSTRACT We show that there is a consistent polynomial quantization of the coordinate ring of a basic nilpotent coadjoint orbit of a semisimple Lie group. We also show, at least in the case of a nilpotent orbit in $\mathrm{sl}(2, \mathbf{R})^*$, that any such quantization is essentially trivial. Furthermore, we prove that there is no consistent polynomial quantization of the coordinate ring of a basic semisimple orbit in $\mathrm{sl}(2, \mathbf{R})^*$.

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1 Introduction

We continue our study of Groenewold-Van Hove obstructions to quantization. Let M be a symplectic manifold, and suppose that $\mathfrak{b}$ is a finite-dimensional "basic algebra" of observables on M. Given a Lie subalgebra $\mathcal{O}$ of the Poisson algebra $C^\infty(M)$ containing $\mathfrak{b}$, we are interested in determining whether $\mathcal{O}$ can be "quantized." (See §§2–3 and Gotay [2000] for the precise definitions.) Already we know that such obstructions exist in many circumstances: In Gotay and Grundling [1999] we proved that there are no finite-dimensional quantizations of $(\mathcal{O}, \mathfrak{b})$ on a noncompact symplectic manifold, for any such Lie subalgebra $\mathcal{O}$. Based on the work of Avez [1974] or Ginzburg and Montgomery [2000], it is straightforward to show that there are no quantizations of $(C^\infty(M), \mathfrak{b})$ for any compact symplectic manifold M and basic algebra $\mathfrak{b}$. Furthermore, in Gotay, Grabowski, and Grundling [2000] we proved that there are no quantizations of the pair $(P(M), \mathfrak{b})$ on a compact symplectic manifold, where $P(M)$ is the Poisson algebra of polynomials on M generated by $\mathfrak{b}$.

It remains to understand the case when M is noncompact and the quantizations are infinite-dimensional, which is naturally the most interesting and difficult one. Here one has little control over either the types of basic algebras that can appear (in examples they range from nilpotent to simple), their representations, or the structure of the polynomial algebras they generate. However, in this context it is known from Gotay and Grabowski [2001] that there is an obstruction to quantizing $P(M)$ when $\mathfrak{b}$ is nilpotent, but that there is no universal obstruction when $\mathfrak{b}$ is merely solvable.

In this paper we consider the problem of quantizing $(P(M), \mathfrak{b})$ in the other extreme case, viz. when the basic algebra is semisimple. To begin, we recall from Gotay [2000] that if a symplectic manifold M admits $\mathfrak{b}$ as a basic algebra, then M must be a coadjoint orbit in $\mathfrak{b}^*$. Unfortunately, it is difficult to determine exactly which orbits $M \subset \mathfrak{b}^*$ are "basic," i.e. admit $\mathfrak{b}$ as a basic algebra (cf. §2). Nonetheless, we are able to give conditions which guarantee that various types of orbits will be basic (Proposition 2.1). In particular, principal nilpotent orbits in $\mathfrak{b}^*$ are basic.

We then prove in §3 that there *do* exist polynomial quantizations of certain basic orbits, specifically the nilpotent ones:

1.1 Theorem. *Let $\mathfrak{b}$ be a finite-dimensional semisimple Lie algebra, and M a basic nilpotent coadjoint orbit in $\mathfrak{b}^*$. Then there exists a polynomial quantization of $(P(M), \mathfrak{b})$.*

The crucial structural feature underlying Theorem 1.1 is that nilpotent orbits $M \subset \mathfrak{b}^*$ are conical, so that the (polynomial) ideal $I(M)$ of M is homogeneous. This allows us to split the coordinate ring of M as a semidirect product

$$P(M) = (\mathbf{R} \oplus \mathfrak{b}) \ltimes P_{(2)}(M),$$

where $P_{(2)}(M)$ is the ideal of polynomials all of whose terms are at least quadratic. The quantization constructed in the proof of Theorem 1.1 has the property that it is zero on $P_{(2)}(M)$, and so is "essentially trivial." We then show that *any* polynomial quantization of a nilpotent orbit in $\mathrm{sl}(2,\mathbf{R})^*$ must be essentially trivial (Proposition 3.3). Thus, while polynomial quantizations of basic nilpotent orbits do exist, this example indicates that they are likely to be uninteresting.

If $I(M)$ is not homogeneous, then one might expect that there is an obstruction to quantizing $P(M)$, cf. Gotay [2000]. We show in §3 that this is indeed the case when $\mathfrak{b} = \mathrm{sl}(2,\mathbf{R})$. Thus polynomial quantizations are forced to be trivial for nilpotent orbits in $\mathrm{sl}(2,\mathbf{R})^*$, and are genuinely obstructed for all other basic orbits.

2 Semisimple Basic Algebras

A key ingredient in the quantization process is the choice of a *basic algebra of observables* in the Poisson algebra $C^\infty(M)$. This is a (real) Lie subalgebra $\mathfrak{b}$ of $C^\infty(M)$ such that:

(B1) $\mathfrak{b}$ is finitely generated,

(B2) the Hamiltonian vector fields $X_b, b \in \mathfrak{b}$, are complete,

(B3) $\mathfrak{b}$ is transitive and separating, and

(B4) $\mathfrak{b}$ is a minimal Lie algebra satisfying these requirements.

A subset $\mathfrak{b} \subset C^\infty(M)$ is "transitive" if $\{X_b(m) \mid b \in \mathfrak{b}\}$ spans T_mM at every point. It is "separating" provided its elements globally separate points of M. Throughout this paper we assume that $\mathfrak{b}$ is finite-dimensional and semisimple, and we routinely use the Killing form to identify $\mathfrak{b}$ with $\mathfrak{b}^*$.

As previously noted, if the symplectic manifold M admits $\mathfrak{b}$ as a basic algebra, then M must be a coadjoint orbit of the adjoint group B of $\mathfrak{b}$. It is of interest to determine those orbits $M \subset \mathfrak{b}^*$ which admit $\mathfrak{b}$ as a basic algebra. Unfortunately, this is not a straightforward matter. For instance, let $\mathfrak{b} = \mathrm{sl}(2,\mathbf{R})$, so that the nonzero orbits are either open half-cones, hyperboloids of one sheet, or components of hyperboloids of two sheets. One can verify that the first two types of orbits are basic for $\mathrm{sl}(2,\mathbf{R})$, but that the third type is not. (Instead, the components of hyperboloids of two sheets are are basic for subalgebras of triangular matrices.) Note that these orbits are all principal (i.e. have maximal dimension) in $\mathrm{sl}(2,\mathbf{R})^*$.

The instances in which $M \subset \mathfrak{b}^*$ is guaranteed to be basic are listed below.

2.1 Proposition. *Let $\mathfrak{b}$ be a finite-dimensional semisimple Lie algebra, and $M \subset \mathfrak{b}^*$ a nonzero coadjoint orbit. If either:*

(*i*) $\mathfrak{b}$ *is compact and* M *is principal,*

(*ii*) $\mathfrak{b}$ *is compact and simple, and* M *is arbitrary, or*

(*iii*) M *is nilpotent and principal,*

then M *admits* $\mathfrak{b}$ *as a basic algebra.*

Before giving the proof, we make some remarks and recall several important facts. As the $\mathrm{sl}(2,\mathbf{R})$ example shows, neither (*i*) nor (*ii*) remain valid when $\mathfrak{b}$ is noncompact. It also shows that (*iii*) fails if "nilpotent" is replaced by "semisimple." It is easy to see that (*iii*) no longer holds if "principal" is deleted: Let O be a nilpotent half cone in $\mathrm{sl}(2,\mathbf{R})$. Then the nilpotent orbit $O \times \{0\} \subset \mathrm{sl}(2,\mathbf{R}) \oplus \mathrm{sl}(2,\mathbf{R})$ has $\mathrm{sl}(2,\mathbf{R})$ as a basic algebra, not $\mathrm{sl}(2,\mathbf{R}) \oplus \mathrm{sl}(2,\mathbf{R})$. Similarly (*ii*) fails if "simple" is deleted. Finally, regarding (*iii*), observe that if there is a nonzero nilpotent orbit in $\mathfrak{b}^*$, then $\mathfrak{b}$ is necessarily noncompact.

Given a (noncompact) semisimple Lie algebra $\mathfrak{b}$, recall that a "standard triple" is a trio $\{h, e_+, e_-\}$ of elements of $\mathfrak{b}$ satisfying the commutation relations

$$[h, e_\pm] = \pm 2 e_\pm \quad \text{and} \quad [e_+, e_-] = h.$$

Thus $\{h, e_+, e_-\}$ spans a subalgebra of $\mathfrak{b}$ isomorphic to $\mathrm{sl}(2,\mathbf{R})$. The neutral element h is semisimple, while $e_\pm$ are nilpotent. Given a nilpotent element $e \in \mathfrak{b}$, the Jacobsen-Morozov theorem (Thm. 9.2.1 in Collingwood-McGovern [1993]) asserts that there exists a standard triple $\{h, e_+, e_-\}$ in $\mathfrak{b}$ with nilpositive element $e_+ = e$.

Proof of Proposition 2.1. Parts (*i*) and (*ii*) are proven in §4 of Gotay, Grabowski, and Grundling [2000], so here we consider only the remaining case (*iii*), the proof of which has been kindly supplied by R. Brylinski.

Clearly conditions (B1)–(B3) are satisfied, so we need only check the minimality condition (B4). Suppose $\mathfrak{a} \subset \mathfrak{b}$ is transitive on M, so that

$$\mathfrak{b} = \mathfrak{a} + \mathfrak{b}^e \tag{2.1}$$

for every $e \in M$, where $\mathfrak{b}^e$ denotes the centralizer of e.

Fix a principal nilpotent $e_+ \in M$. We first show that e_+ is contained in a Borel subalgebra ("BSA") of $\mathfrak{b}$. Let $\{h, e_+, e_-\}$ be a standard triple in $\mathfrak{b}$ with nilpositive element e_+. From the representation theory of $\mathrm{sl}(2,\mathbf{R})$ we see that the eigenvalues of ad_h are integral; we may therefore decompose

$$\mathfrak{b} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{b}_i \tag{2.2}$$

where $\mathfrak{b}_i$ is the eigenspace of ad_h corresponding to the eigenvalue i. Since e_+ is principal, the neutral element h is generic, so its centralizer $\mathfrak{h} = \mathfrak{b}_0$

is a Cartan subalgebra ("CSA") of $\mathfrak{b}$. Since furthermore $[\mathfrak{b}_i, \mathfrak{b}_j] \subset \mathfrak{b}_{i+j}$, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{n}$ is a BSA, where $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{b}_i$. Finally, as $[h, e_+] = 2e_+ \in \mathfrak{b}_2$, it follows that $\mathfrak{k}$ is the desired BSA.

From the proof of Thm. 5 in Kostant [1963] we know that $\mathfrak{b}^{e_+} \subset \mathfrak{n}$, which together with (2.1) implies that $\mathfrak{b} = \mathfrak{a} + \mathfrak{m}$ for every B-conjugate $\mathfrak{m}$ of $\mathfrak{n}$. We will prove this forces $\mathfrak{a} = \mathfrak{b}$.

Since $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$, we may write $h = h' + n$ where $h' \in \mathfrak{a}$ and $n \in \mathfrak{n}$. So

$$h' = h - n$$

lies in $\mathfrak{a}$ and is generic (since h and h' have the same characteristic polynomial). Thus the centralizer $\mathfrak{h}'$ of h' is also a CSA of $\mathfrak{b}$. A calculation based on the decomposition (2.2) shows that $\mathfrak{h}' \subset \mathfrak{k}$. This gives rise to the Levi decomposition

$$\mathfrak{k} = \mathfrak{h}' \oplus \mathfrak{n}.$$

We next claim that $\mathfrak{a}$ contains $\mathfrak{h}'$. Indeed, using $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$ again, we see that each element $x' \in \mathfrak{h}'$ gives rise to an element $x = x' - n_{x'}$ of $\mathfrak{a}$, where $n_{x'} \in \mathfrak{n}$. Since $\mathfrak{a}$ is stable under $ad_{h'}$, it follows that both x' and $n_{x'}$ lie in $\mathfrak{a}$. (The reason is that $\mathfrak{h}'_{\mathbf{C}}$ is the zero eigenspace of $ad_{h'}$ in $\mathfrak{b}_{\mathbf{C}}$ and $\mathfrak{n}_{\mathbf{C}}$ is the sum of nonzero eigenspaces. So both x' and $n_{x'}$ lie in $\mathfrak{a}_{\mathbf{C}}$. As both x' and $n_{x'}$ are real they must belong to $\mathfrak{a}$.) In particular $\mathfrak{a}$ contains $\mathfrak{h}'$.

We can now finish the proof. We have the triangular decomposition

$$\mathfrak{b} = \mathfrak{m} \oplus \mathfrak{h}' \oplus \mathfrak{n}$$

where $\mathfrak{m}$ is the unique $ad_{h'}$-stable complement to $\mathfrak{k}$ in $\mathfrak{b}$. By a result of Borel and Tits [1965], the two Borel subalgebras $\mathfrak{h}' \oplus \mathfrak{n}$ and $\mathfrak{m} \oplus \mathfrak{h}'$ are B-conjugate, whence their nilradicals $\mathfrak{n}$ and $\mathfrak{m}$ are as well. Since $\mathfrak{a}$ contains $\mathfrak{h}'$, $\mathfrak{a}_{\mathbf{C}}$ is the direct sum of $\mathfrak{h}'_{\mathbf{C}}$ and some of its root spaces. Using $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$, we see that $\mathfrak{a}_{\mathbf{C}}$ contains $\mathfrak{m}_{\mathbf{C}}$. Similarly, using $\mathfrak{b} = \mathfrak{a} + \mathfrak{m}$, we see that $\mathfrak{a}_{\mathbf{C}}$ contains $\mathfrak{n}_{\mathbf{C}}$. Thus $\mathfrak{a}_{\mathbf{C}} = \mathfrak{b}_{\mathbf{C}}$ and so $\mathfrak{a} = \mathfrak{b}$. ■

Let $\mathfrak{b}$ be a Lie algebra and M a coadjoint orbit in $\mathfrak{b}^*$. Consider the symmetric algebra $S(\mathfrak{b})$, regarded as the ring of polynomials on $\mathfrak{b}^*$. The Lie bracket on $\mathfrak{b}$ may be extended via the Leibniz rule to a Poisson bracket on $S(\mathfrak{b})$, so that the latter becomes a Poisson algebra. Let $I(M)$ be the associative ideal in $S(\mathfrak{b})$ consisting of all polynomials which vanish on M and set $P(M) = S(\mathfrak{b})/I(M)$. Since M is an orbit $I(M)$ is also a Lie ideal, hence a Poisson ideal, so the coordinate ring $P(M)$ of M inherits the structure of a Poisson algebra from $S(\mathfrak{b})$. We denote the Poisson bracket on $P(M)$ by $\{\cdot, \cdot\}$.

Let $P^k(M)$ denote the subspace of polynomials of degree at most k. (When $I(M) \neq \{0\}$, $P(M)$ is not freely generated as an associative algebra by the elements of $\mathfrak{b}$. Consequently, the notion of "homogeneous polynomial" is not necessarily well-defined, but that of "degree" is.) In the cases when it does make sense, we let $P_l(M)$ denote the subspace of homogeneous polynomials of degree l, so that $P^k(M) = \bigoplus_{l=0}^{k} P_l(M)$. We then also introduce $P_{(k)}(M) = \bigoplus_{l \geq k} P_l(M)$. Notice that when $\mathfrak{b}$ is semisimple, $P_1(M) = \mathfrak{b}$ and $P^1(M) = \mathbf{R} \oplus \mathfrak{b}$.

3 Quantization

Fix a basic algebra $\mathfrak{b}$ on M, and let $\mathcal{O}$ be any Lie subalgebra of $C^\infty(M)$ containing 1 and $\mathfrak{b}$. By a *quantization* of $(\mathcal{O}, \mathfrak{b})$ we mean a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the linear space $\mathrm{Op}(D)$ of symmetric operators which preserve a fixed dense domain D in some separable Hilbert space $\mathcal{H}$, such that for all $f, g \in \mathcal{O}$,

(Q1) $\mathcal{Q}(\{f, g\}) = i[\mathcal{Q}(f), \mathcal{Q}(g)]$,

(Q2) $\mathcal{Q}(1) = I$,

(Q3) if the Hamiltonian vector field X_f of f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D,

(Q4) $\mathcal{Q}$ represents $\mathfrak{b}$ irreducibly,

(Q5) D contains a dense set of separately analytic vectors for some set of Lie generators of $\mathcal{Q}(\mathfrak{b})$, and

(Q6) $\mathcal{Q}$ represents $\mathfrak{b}$ faithfully.

We refer the reader to Gotay [2000] for an extensive discussion of these definitions. We take Planck's reduced constant to be 1. Here we are interested in the case when $\mathcal{O} = P(M)$.

Let $\mathcal{A}$ be the associative algebra over $\mathbf{C}$ generated by I along with $\{\mathcal{Q}(b) \mid b \in \mathfrak{b}\}$, and let $\mathcal{A}^k$ denote the subspace of polynomials of degree at most k in the $\mathcal{Q}(b)$. We say that a quantization $\mathcal{Q}$ of $P(M)$ is *polynomial* if it is valued in $\mathcal{A}$. That "polynomials quantize to polynomials" can be regarded as a generalized "Von Neumann rule," cf. Gotay [2000].

Proof of Theorem 1.1. Let M be a basic nilpotent orbit. Since each nilpotent orbit is conical (Brylinski [1998]), it follows that we may choose a set of generators for $I(M)$ which are homogeneous. As a consequence, the gradation of $S(\mathfrak{b})$ by degree passes to the quotient $P(M)$. Thus the notion of homogeneous polynomial *does* make sense in $P(M)$. Furthermore, by virtue of the commutation relations of $\mathfrak{b}$, for each $l \geq 0$ the subspaces $P_l(M)$ are *ad*-invariant: $\{P_1(M), P_l(M)\} \subset P_l(M)$. In view of this, $\{P_k(M), P_l(M)\} \subset P_{k+l-1}(M)$, whence each $P_{(l)}(M)$ is a Lie ideal. We thus have the semidirect sum decomposition

$$P(M) = P^1(M) \ltimes P_{(2)}(M). \tag{3.1}$$

Because of (3.1), we can obtain a polynomial quantization $\mathcal{Q}$ of *all* of P simply by finding an appropriate representation of $P^1(M) = \mathbf{R} \oplus \mathfrak{b}$ and setting $\mathcal{Q}(P_{(2)}(M)) = \{0\}$! To this end, let $\tilde{B}$ be the connected, simply connected Lie group with Lie algebra $\mathfrak{b}$, and let Π be a faithful irreducible unitary representation of $\tilde{B}$ on a Hilbert space $\mathcal{H}$. (For instance, we may take Π to be a generic irreducible component of the left regular representation of $\tilde{B}$ on $L^2(\tilde{B})$, cf. §5.6 in Barut and Rączka [1986].) Let $D \subset \mathcal{H}$ be the dense set of analytic vectors for Π, and define $\pi = -i\, d\,\Pi \restriction D$, cf. §11.4 *ibid.* Extend π to $P^1(M)$ by setting $\pi(1) = I$. Now take $\mathcal{Q} = \pi \oplus 0$ (recall (3.1)); then it is straightforward to verify that $\mathcal{Q}$ satisfies (Q1)–(Q6) and so is the required quantization of $(P(M), \mathfrak{b})$. ■

Note that the quantization constructed above is infinite-dimensional. Indeed, there can be no finite-dimensional quantizations of a noncompact basic algebra (Gotay and Grundling [1999]); this is a reflection of the fact that semisimple Lie groups of noncompact type have no faithful finite-dimensional unitary representations. Furthermore, since $\mathcal{Q}(P_{(2)}(M)) = \{0\}$, this quantization is essentially trivial. When $\mathfrak{b} = \mathrm{sl}(2, \mathbf{R})$ it turns out that *any* polynomial quantization is essentially trivial, as we show after some preliminaries.

Henceforth take $\mathfrak{b} = \mathrm{sl}(2, \mathbf{R})$ and let M be an arbitrary coadjoint orbit. It is convenient to complexify. Define

$$h = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad e_\pm = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right].$$

Then $\{h, e_+, e_-\}$ is a standard triple in $\mathfrak{b}_{\mathbf{C}} = \mathrm{sl}(2, \mathbf{C})$. Note that $h^2 + 4e_+e_-$ is the Casimir element for $\mathfrak{b}_{\mathbf{C}}$; consequently

$$h^2 + 4e_+e_- = c$$

is constant on M.

Suppose $\mathcal{Q}$ were a polynomial quantization of $(P(M), \mathfrak{b})$ on a dense invariant domain D in an infinite-dimensional Hilbert space $\mathcal{H}$. By requiring $\mathcal{Q}$ to be complex linear, we can regard it as a "quantization" of $(P(M)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. From now on, we abbreviate $P(M)_{\mathbf{C}} = P$, etc. We set $H = \mathcal{Q}(h)$ and $E_\pm = \mathcal{Q}(e_\pm)$, and let $(\cdot, \cdot)$ denote the anti-commutator. Finally, observe that $H^2 + 4(E_+, E_-)$ is the Casimir element for the representation $\mathcal{Q}$ of $\mathfrak{b}_{\mathbf{C}}$; since by axiom (Q4) this representation is irreducible,

$$H^2 + 4(E_+, E_-) = CI \tag{3.2}$$

for some fixed constant C (cf. Prop. 3 in Gotay and Grabowski [2001]).

We first establish the following technical result.

3.1 Lemma. *For any nonnegative integer r, the set of operators*

$$S_r = \{H^j E_+^{\,l}, H^k E_-^{\,m} \mid j + l \le r,\ k + m \le r\}$$

forms a basis for $\mathcal{A}^r$.

Proof. We proceed by induction on r. The statement is obviously true for $S_0 = \{I\}$. Now assume S_{r-1} is a basis for $\mathcal{A}^{r-1}$.

Any element of $\mathcal{A}^r$ can be written

$$\sum_{k+l+m=r} \alpha^r_{klm} H^k E_+^{\,l} E_-^{\,m} + \text{lower degree terms.}$$

Now observe that

$$E_+E_- = (E_+, E_-) - \frac{i}{2}H.$$

Applying (3.2) we may use this relation to eliminate all factors of E_+E_- in the leading terms of the expression above, thereby obtaining

$$\alpha_r H^r + \sum_{\substack{j+l=r \\ l\geq 1}} \beta^+_{jl} H^j E_+^{\,l} + \sum_{\substack{k+m=r \\ m\geq 1}} \beta^-_{km} H^k E_-^{\,m} + \text{lower degree terms} \tag{3.3}$$

for some coefficients $\alpha_r, \beta^+_{jl}, \beta^-_{km}$. Together with the induction hypothesis, this shows that S_r spans $\mathcal{A}^r$.

Now suppose there exist coefficients $\alpha_r, \beta^+_{jl}, \beta^-_{km}$, not all zero, such that the expression (3.3) vanishes. We claim that without loss of generality we may assume $\alpha_r \neq 0$. For suppose β^+_{JL} were the first nonzero coefficient in this expression. By taking the commutator of the equation (3.3) = 0 with E_- L-times, applying the commutation relations, and simplifying using (3.2), we obtain a condition of the form (3.3) = 0 where now the coefficient of H^r is nonzero. Similarly, if β^-_{KM} were the first nonzero coefficient in (3.3), then taking the commutator with E_+ M-times would lead to the same end.

Now repeatedly take the commutator of the equation (3.3) = 0 with H. This yields further independent conditions of the form (3.3) = 0 but with no terms involving H^r. By Gaussian elimination, we may then remove all terms on the left hand side of (3.3) = 0 of the types $\beta^+_{jl} H^j E_+^{\,l}$ and $\beta^-_{km} H^k E_-^{\,m}$ with $j, k < r$. Thus we end up with

$$\alpha_r H^r + A_{r-1} = 0 \tag{3.4}$$

where $\alpha_r \neq 0$ and $A_{r-1} \in \mathcal{A}^{r-1}$. Taking the commutator of (3.4) with H yields $[A_{r-1}, H] = 0$. Applying the induction hypothesis, it follows that A_{r-1} can only depend upon H. Thus (3.4) reduces to

$$\sum_{k=0}^{r} \alpha_k H^k = 0.$$

Factor this equation over $\mathbf{C}$:

$$\alpha_r (H - \lambda_r) \cdots (H - \lambda_1) = 0. \tag{3.5}$$

As $\alpha_r \neq 0$, (3.5) implies that the range of $T_{r-1} = (H - \lambda_{r-1}) \cdots (H - \lambda_1)$ is contained in the λ_r-eigenspace of H. By the induction hypothesis $T_{r-1} \neq 0$, so there exists $\psi \in D$ such that $\psi_{r-1} = T_{r-1}\psi$ is a (nonzero) eigenvector of H. In view of the irreducibility assumption (Q4), we conclude from sl(2, $\mathbf{R}$) theory (cf. Lang [1975]) that the set $\{E_+^{\,l}\psi_{r-1}, E_-^{\,m}\psi_{r-1} \,|\, l, m \in \mathbf{N}\}$ contains an infinite number of eigenvectors of H, corresponding to distinct eigenvalues λ. Each such λ must satisfy $\sum_{k=0}^{r} \alpha_k \lambda^k = 0$ which is impossible. Thus $\alpha_r = 0$ and so S_r is a linearly independent set. ■

We now determine what $\mathcal{Q}(h^2)$ must be.

3.2 Lemma. $\mathcal{Q}(h^2) = \alpha H^2 + \gamma I$, *where* $\alpha, \gamma \in \mathbf{C}$.

Proof. By assumption $\mathcal{Q}(h^2)$ must be a polynomial of degree r, say, in H, E_+, E_-, which by Lemma 3.1 we may write in the form (3.3). Since H commutes with $\mathcal{Q}(h^2)$, from Lemma 3.1 we see that $\mathcal{Q}(h^2)$ can only depend on H:

$$\mathcal{Q}(h^2) = \sum_{k=0}^{r} \alpha_k H^k. \tag{3.6}$$

Using (Q1) and (Q2) to quantize the classical identity

$$3h^2 - \frac{1}{2}\{\{h^2, e_-\}, e_+\} = c$$

we obtain

$$3\mathcal{Q}(h^2) + \frac{1}{2}[[\mathcal{Q}(h^2), E_-], E_+] = cI. \tag{3.7}$$

Substituting (3.6) into (3.7) and simplifying yields

$$\left(3 - \frac{1}{2}r(r+1)\right)\alpha_r H^r + \text{lower degree terms} = cI.$$

From Lemma 3.1 it follows that $\mathcal{Q}(h^2)$ is at most quadratic in H. Taking (3.6) with $r = 2$, again substituting into (3.7) and simplifying, we obtain the advertised expression for $\mathcal{Q}(h^2)$, where $\alpha = \alpha_2$ is arbitrary and γ satisfies

$$3\gamma = c - \alpha C. \tag{3.8}$$

■

Using (Q1) to quantize the identity

$$he_\pm = \pm\frac{1}{4}\{h^2, e_\pm\},$$

applying Lemma 3.2, and simplifying, we obtain

$$\mathcal{Q}(he_\pm) = \alpha(H, E_\pm).$$

In turn, using this to quantize the identities

$$e_\pm{}^2 = \pm\frac{1}{2}\{he_\pm, e_\pm\},$$

we find that

$$\mathcal{Q}(e_\pm{}^2) = \alpha E_\pm{}^2.$$

Similarly, upon quantizing

$$e_+e_- = \frac{1}{2}\left(h^2 - \{he_+, e_-\}\right)$$

and using the formulæ above, we get

$$\mathcal{Q}(e_+e_-) = \alpha(E_+, E_-) + \frac{\gamma}{2}I.$$

Next use these formulæ to quantize the classical identities

$$2\{e_+{}^2, e_-{}^2\} + \{he_+, he_-\} = ch$$

and

$$\begin{aligned}\big\{(e_+ - e_-)^2, \{e_+{}^2 - e_-{}^2, h(e_+ + e_-)\}\big\} \\ + \tfrac{3}{4}\big\{(e_+ + e_-)^2, \{(e_+ + e_-)^2, h(e_+ - e_-)\}\big\} \\ = 8ch(e_+ - e_-).\end{aligned}$$

After tedious calculations and simplifications, we end up with

$$\alpha^2\,(C+3)\,H = cH \tag{3.9}$$

and

$$\alpha^3\,(C+9)\,(H, E_+ - E_-) = \alpha c(H, E_+ - E_-), \tag{3.10}$$

respectively.

With these formulæ in hand, we are now ready to prove

3.3 Proposition. *Let M be a nilpotent orbit in $\mathrm{sl}(2,\mathbf{R})^*$. Then for any polynomial quantization $\mathcal{Q}$ of $(P(M), \mathrm{sl}(2,\mathbf{R}))$,*

$$\mathcal{Q}(P_{(2)}(M)) = \{0\}.$$

Proof. We first claim that $\mathcal{Q}(P_2) = \{0\}$. To see this, observe that since M is nilpotent, the constant $c = 0$. Since by (Q6) $H \neq 0$, (3.9) implies that either $\alpha = 0$ or $C = -3$ in the given representation. But if $\alpha = 0$, then from (3.8) we conclude that $\mathcal{Q}(h^2) = 0$ which, as we show below, leads to the desired conclusion.

In the event that $C = -3$, we turn to (3.10). Since $(H, E_+ - E_-) \neq 0$ by Lemma 3.1, we must again have $\alpha = 0$. Thus in any eventuality $\mathcal{Q}(h^2) = 0$ and it follows from (Q1) that $\mathcal{Q}(P_2) = \{0\}$, since h^2 is a cyclic vector for the adjoint action of $\mathrm{sl}(2, \mathbf{C})$ on P_2 (i.e., every element of P_2 can be written as a sum of repeated brackets of elements of $\mathrm{sl}(2, \mathbf{C})$ with h^2, as the calculations above show).

Finally, it is straightforward to check that h^l is a cyclic vector for the adjoint representation of $\mathrm{sl}(2, \mathbf{C})$ on P_l. Since for $l \geq 2$

$$h^l = \frac{1}{2l+2}\big\{\{h^2, h^{l-2}e_+\}, e_-\big\}$$

(recall that $c = 0$), $\mathcal{Q}(h^2) = 0$ together with (Q1) imply that $\mathcal{Q}(h^l) = 0$ for $l > 2$. Thus $\mathcal{Q}(P_{(2)}) = \{0\}$. ■

When $M \subset \mathrm{sl}(2, \mathbf{R})^*$ is not nilpotent (in which case it must be semisimple), it turns out that it is not even possible to polynomially quantize $(P(M), \mathfrak{b})$; rather than finding that $\mathcal{Q}(P_{(2)}(M)) = \{0\}$, we get an outright inconsistency.

3.4 Proposition. *If M is a basic semisimple orbit in* $\mathrm{sl}(2, \mathbf{R})^*$*, then there is no polynomial quantization of* $(P(M), \mathfrak{b})$.

Proof. We mimic the proof of Proposition 3.3; the only difference is that c is now nonzero. As before, $H \neq 0$, so by (3.9)

$$\alpha^2 (C + 3) = c.$$

In particular, since $c \neq 0$, $\alpha \neq 0$. Since $(H, E_+ - E_-) \neq 0$, (3.10) then gives

$$\alpha^2 (C + 9) = c,$$

which is the required contradiction. ■

Proposition 3.4 is the noncompact analogue of the results obtained in Gotay, Grundling, and Hurst [1996] for $\mathfrak{b} = \mathrm{su}(2)$, in which context every orbit is semisimple. In fact, the only significant difference between the analyses of semisimple orbits in the $\mathrm{sl}(2, \mathbf{R})$ and $\mathrm{su}(2)$ cases is that the representations for the former are infinite-dimensional, while those for the latter are finite-dimensional. Since moreover the complexifications of these Lie algebras are the same (viz. $\mathrm{sl}(2, \mathbf{C})$), the arguments leading from Lemma 3.2 to Proposition 3.4 don't distinguish between $\mathrm{sl}(2, \mathbf{R})$ and $\mathrm{su}(2)$. The same is true of the results in §2 *ibid.*, which we may therefore immediately carry over to the present context, yielding:

3.5 Proposition. *Let M be a basic semisimple orbit in* $\mathrm{sl}(2,\mathbf{R})^*$*. Then* $P^1(M) = \mathbf{R} \oplus \mathrm{sl}(2,\mathbf{R})$ *is the largest Lie subalgebra of the coordinate ring* $P(M)$ *that can be consistently polynomially quantized.*

Thus the obstruction to quantizing polynomial algebras on semisimple orbits in $\mathrm{sl}(2,\mathbf{R})^*$ is very severe: the *best* one can do is quantize the Lie subalgebra of affine polynomials!

We end this section with a discussion of the assumption that $\mathcal{Q}$ be polynomial. In general, when the basic algebra $\mathfrak{b}$ is compact (or, equivalently, when the coadjoint orbit M is compact) every quantization of $(P(M), \mathfrak{b})$ is polynomial. For then the Hilbert space $\mathcal{H}$ must be finite-dimensional, and the claim follows from a well known property of enveloping algebras, cf. Prop. 2.6.5 in Dixmier [1977]. Furthermore, when $\mathfrak{b}$ is nilpotent, it was proven that $\mathcal{Q}$ must be polynomial in Gotay and Grabowski [2001]. These results are direct consequences of the irreducibility condition (Q4). However, the analogous statement does not seem to hold for noncompact semisimple basic algebras.

To see this, we provide an alternate version of Lemma 3.2, which does not assume that $\mathcal{Q}$ is polynomial *ab initio*. For what follows, we need to be more specific about the domain D. As a consequence of (Q5), $\mathcal{Q} \restriction \mathfrak{b}$ integrates to a unique unitary representation Π of $\tilde{B}$ on $\mathcal{H}$ (Cor. 1 of Flato and Simon [1973]). Naturally associated with Π is the derived representation of $\mathfrak{b}$ on the domain $C^\omega(\Pi)$ consisting of analytic vectors of Π. We shall henceforth assume that $D \supset C^\omega(\Pi)$. Furthermore, for the sake of simplicity, we suppose that the representation Π drops to $\mathrm{SL}(2,\mathbf{R})$ from its double cover $\tilde{B}$.

Then from $\mathrm{sl}(2,\mathbf{R})$ theory (cf. Lang [1975]) we know that (*i*) the spectrum Δ of H consists of certain imaginary integers, (*ii*) in view of (Q4), for each $-in \in \Delta$ the corresponding eigenspaces $\mathcal{H}_n$ are 1-dimensional, and (*iii*) each eigenvector of H is an analytic vector, so that $\mathcal{H}_n \subset D$. Furthermore, the quantizations of $\mathfrak{b}$ are labeled by certain complex numbers s, and that for each $-in \in \Delta$, there is a vector $\psi_n \in \mathcal{H}_n$ such that

$$H\psi_n = -in\psi_n \quad \text{and} \quad E_\pm \psi_n = -\frac{i}{2}(s + 1 \pm n)\psi_{n\pm 2}. \tag{3.11}$$

By (Q1), both H and $\mathcal{Q}(h^2)$ commute. From observations (*ii*) and (*iii*) above, and the fact that $\bigoplus_{n \in i\Delta} \mathcal{H}_n$ is dense in $\mathcal{H}$, it follows that

$$\mathcal{Q}(h^2) = \xi(H) \tag{3.12}$$

for some Borel function ξ on the spectrum of H. We now compute ξ.

Apply the relation (3.7) to ψ_n; from (3.11) and (3.12) we get the recursion relation

$$3\xi_n - \frac{1}{8}\big[\,(s + (1+n))\,(s - (1+n))\,(\xi_n - \xi_{n+2}) \\ - (s + (1-n))\,(s - (1-n))\,(\xi_{n-2} - \xi_n)\big] = c, \tag{3.13}$$

where ξ_n is defined via $\xi(H)\psi_n = \xi_n\psi_n$. It is straightforward to check that any *polynomial* solution of this recursion relation is of the form $\xi_n = \gamma - \alpha n^2$ from which, in view of (3.12) and (3.11), we recover the formula derived previously for $\mathcal{Q}(h^2)$. But there are other solutions of (3.13) which are transcendental: for instance, consider the discrete series representation with $s \geq 1$ an even integer. Then $\Delta = -i\{s+1, s+3, \dots\}$, and with some effort one can show that the general solution of (3.13) is

$$\xi_n = \gamma - \alpha n^2 + \beta\left((s^2 - 3n^2 - 1)\left[F\left(\tfrac{1+n-s}{2}\right) - F\left(\tfrac{1+n+s}{2}\right)\right] - 6ns\right),$$

where α, β are arbitrary, γ is given by (3.8), and the digamma function F is the logarithmic derivative of the gamma function. Similar formulæ hold for other allowable values of s.

Thus in the case of $\mathrm{sl}(2,\mathbf{R})$ irreducibility enables one to determine $\mathcal{Q}(h^2)$ and then, following the template set forth after the proof of Lemma 3.2, all of $\mathcal{Q}(P^2(M))$, and so on. But unlike for su(2), irreducibility alone apparently does *not* suffice to guarantee that $\mathcal{Q}$ is polynomial. While Proposition 3.4 shows that polynomial quantizations of $(P(M), \mathrm{sl}(2,\mathbf{R}))$ for semisimple M cannot exist, it is unclear whether such transcendental quantizations are similarly obstructed.

4 Discussion

The quantization of $(P(M), \mathfrak{b})$ for $M \subset \mathfrak{b}$ nilpotent given above is not the first know example of a consistent quantization: In Gotay [1995] a full quantization of $(C^\infty(T^2)), \mathfrak{t})$ was exhibited, where $\mathfrak{t}$ is the basic algebra of trigonometric polynomials of mean zero; and in Gotay and Grabowski [2001] a polynomial quantization of $P(T^*\mathbf{R}_+)$, with the basic algebra being the affine algebra a(1), was constructed. This last example "works" for exactly the same reason the nilpotent one does, viz. the ideal $I(M)$ is homogeneous. However, in contrast to the case of $\mathrm{sl}(2,\mathbf{R})$ (cf. Proposition 3.3), a polynomial quantization of $P(T^*\mathbf{R}_+)$ with basic algebra a(1) need *not* be zero on $P_{(2)}$.

In fact, a moment's reflection shows that there will exist a polynomial quantization of $(P(M), \mathfrak{b})$ for any basic algebra $\mathfrak{b}$ whenever $I(M)$ is homogeneous, for then one has the crucial splitting (3.1). But this construction will fail whenever $I(M)$ is inhomogeneous so that $P_{(2)}(M)$ is not well-defined. It is tempting to conjecture that an obstruction to quantization exists whenever $I(M)$ is inhomogeneous; this is borne out explicitly here in the case of semisimple orbits in $\mathrm{sl}(2,\mathbf{R})$ by Proposition 3.4. This correlation is also known to hold in all other examples that have been investigated thus far (Gotay [2000]).

The next step is to extend Propositions 3.3 and 3.4 to higher rank semisimple basic algebras. Clearly, this necessitates using more Poisson theoretic techniques, as opposed to the computational approach taken here. These issues are addressed in Gotay [2001].

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