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On Quantizing Nonnilpotent Coadjoint Orbits of Semisimple Lie Groups

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Abstract. We prove that there is no consistent polynomial quantization of the coordinate ring of a nonnilpotent coadjoint orbit of a semisimple Lie group.

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1. Introduction

In a recent paper [1], we showed that there do not exist polynomial quantizations of the coordinate ring $P(M)$ of a semisimple coadjoint orbit $M \subset \mathrm{sl}(2, \mathbb{R})^*$. Here we extend that result to any nonnilpotent coadjoint orbit of a general semisimple Lie group:

THEOREM 1. *Let* $\mathfrak{b}$ *be a finite-dimensional semisimple Lie algebra, and* M *a nonnilpotent coadjoint orbit in* $\mathfrak{b}^*$. *Then there are no polynomial quantizations of the coordinate ring* $P(M)$.

Consider the symmetric algebra $S(\mathfrak{b})$, regarded as the ring of polynomials on $\mathfrak{b}^*$. The Lie bracket on $\mathfrak{b}$ may be extended via the Leibniz rule to a Poisson bracket on $S(\mathfrak{b})$, so that the latter becomes a Poisson algebra. Let $I(M)$ be the associative ideal in $S(\mathfrak{b})$ consisting of all polynomials which vanish on M and set $P(M) = S(\mathfrak{b})/I(M)$. Since M is an orbit $I(M)$ is also a Lie ideal, hence a Poisson ideal, so the coordinate ring $P(M)$ of M inherits the structure of a Poisson algebra from $S(\mathfrak{b})$. We denote the Poisson brackets on both $P(M)$ and $S(\mathfrak{b})$ by $\{\cdot, \cdot\}$.

Here we are interested in quantizing the coordinate ring $P(M)$. By a *quantization* of $P(M)$ we mean a Lie representation $\mathcal{Q}$ thereof by symmetric operators preserving a fixed dense domain D in some separable Hilbert space $\mathcal{H}$, such that $\mathcal{Q} \upharpoonright \mathfrak{b}$ is irreducible, integrable, and faithful. Let $\mathcal{A}$ be the associative operator algebra generated over $\mathbb{C}$ by I and $\{\mathcal{Q}(b) | \, b \in \mathfrak{b}\}$. We say that a quantization $\mathcal{Q}$ of $P(M)$ is *polynomial* if $\mathcal{Q}$ is valued in $\mathcal{A}$. We refer the reader to [2] for a detailed discussion of quantization.

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2. Proof of Theorem 1

Suppose to the contrary that $\mathcal{Q}$ were a polynomial quantization of $P(M)$ in a dense invariant domain D in a Hilbert space $\mathcal{H}$. By extending $\mathcal{Q}$ to be complex linear, we obtain a Lie representation $\mathcal{Q}_{\mathbb{C}}$ of the Poisson algebra $P(M, \mathbb{C})$ of complex-valued polynomials on M in D.

By assumption the representation of $\mathfrak{b}$ in D provided by $\mathcal{Q}$ may be integrated to a strongly continuous unitary representation Π of the 1-connected Lie group B with Lie algebra $\mathfrak{b}$ in $\mathcal{H}$. Let $B_{\mathbb{C}}$ be the universal complexification of B; since B is simply connected, $B_{\mathbb{C}}$ can be identified with the 1-connected semisimple complex analytic group with Lie algebra the complexification $\mathfrak{b}_{\mathbb{C}}$ of $\mathfrak{b}$. (See [3], pp. 256–258 and 400–404 for background on complexifications of Lie groups.) Since B is semisimple, B is a closed subgroup of $B_{\mathbb{C}}$, and so we may use induction to obtain a strongly continuous unitary representation $\Pi_{\mathbb{C}}$ of $B_{\mathbb{C}}$ in a certain infinite-dimensional Hilbert space $\mathcal{K}$.

Now let C be a compact real form of $B_{\mathbb{C}}$, and denote by Γ the restriction of $\Pi_{\mathbb{C}}$ to C. As every strongly continuous unitary representation of a compact Lie group is completely reducible, we may decompose $\mathcal{K} = \widehat{\bigoplus}_{i \in I} \mathcal{K}_i$ for some index set $I \subset \mathbb{Z}$, where the finite-dimensional invariant subspaces $\mathcal{K}_i$ are the carriers of the irreducible constituents Γ_i of Γ. Let $\mathfrak{c}$ be the Lie algebra of C; then for each $i \in I$, we have the derived representation $\mathrm{d}\Gamma_i$ of $\mathfrak{c}$ in $\mathcal{K}_i$. Set $\mathrm{d}\Gamma = \bigoplus_{i \in I} \mathrm{d}\Gamma_i$; this gives a representation of $\mathfrak{c}$ in the dense subspace $D_C = \bigoplus_{i \in I} \mathcal{K}_i$.

Choose a basis $\{c_1, \ldots, c_r\}$ of $\mathfrak{c}$. Since $\mathfrak{c}_{\mathbb{C}} = \mathfrak{b}_{\mathbb{C}}$ and as by assumption $\mathcal{Q}$ is valued in $\mathcal{A}$, for every $f \in P(M, \mathbb{C})$ we may expand

$$\mathcal{Q}_{\mathbb{C}}(f) = \sum_{n_1, \ldots, n_r} a^f_{n_1, \ldots, n_r} \mathcal{Q}_{\mathbb{C}}(c_1)^{n_1} \cdots \mathcal{Q}_{\mathbb{C}}(c_r)^{n_r}$$

for some coefficients $a^f_{n_1, \ldots, n_r}$. By means of this formula we can extend the representation $\mathrm{d}\Gamma$ of $\mathfrak{c}$ to a Lie representation γ of $P(M, \mathbb{C})$ in D_C:

$$\gamma(f) = \sum_{n_1, \ldots, n_r} a^f_{n_1, \ldots, n_r} \mathrm{d}\Gamma(c_1)^{n_1} \cdots \mathrm{d}\Gamma(c_r)^{n_r}$$

with the *same* coefficients. As each subspace $\mathcal{K}_i$ is invariant, γ restricts to a representation γ_i of $P(M, \mathbb{C})$ in $\mathcal{K}_i$. We will show that the existence of these representations γ_i leads to a contradiction.

To this end we recall the following algebraic fact, the proof of which is given in [4].

LEMMA 2. *If L is a finite-codimensional Lie ideal of an infinite-dimensional Poisson algebra P with identity, then either L contains the derived ideal $\{P, P\}$ or there is a maximal finite-codimensional associative ideal J of P such that $\{P, P\} \subset J$.*

We apply Lemma 2 to each $L_i = \ker \gamma_i$ which, as $\mathcal{K}_i$ is finite-dimensional, has finite codimension in $P = P(M, \mathbb{C})$. First suppose there is an i for which $\{P, P\} \not\subset L_i$. Then there must exist a maximal finite-codimensional associative ideal J_i in P with

$\{P, P\} \subset J_i$. If ρ is the projection $S(\mathfrak{h}_{\mathbb{C}}) \to P$, then $I_i = \rho^{-1}(J_i)$ is a maximal finite-codimensional associative ideal in $S(\mathfrak{h}_{\mathbb{C}})$ with $\{S(\mathfrak{h}_{\mathbb{C}}), S(\mathfrak{h}_{\mathbb{C}})\} \subset I_i$. Since by semisimplicity

$$\mathfrak{h}_{\mathbb{C}} = \{\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}\} \subset \{S(\mathfrak{h}_{\mathbb{C}}), S(\mathfrak{h}_{\mathbb{C}})\} \subset I_i,$$

and since $1 \notin I_i$ (as I_i is proper), it follows that I_i is the associative ideal generated by $\mathfrak{h}_{\mathbb{C}}$. (Actually, this shows that $S(\mathfrak{h}_{\mathbb{C}}) = \mathbb{C} \oplus I_i$.)

Since the orbit M is not nilpotent, there is a nonzero Casimir $\Omega \in S(\mathfrak{h}_{\mathbb{C}})$, i.e. $\rho(\Omega) = \omega$ for some constant $\omega \neq 0$. Since $\mathfrak{h}_{\mathbb{C}}$ is semisimple it follows from the above observations that $\Omega \in I_i$. But then $\Omega - \omega \notin I_i$, which is a contradiction since $\Omega - \omega \in \ker \rho \subset I_i$.

Thus for *every* i it must be the case that $\{P, P\} \subset L_i$. Again semisimplicity gives $\mathfrak{h}_{\mathbb{C}} = \{\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}\} \subset L_i$, and so $\gamma \restriction \mathfrak{h}_{\mathbb{C}} = 0$. In particular, then, $d\Gamma = 0$. Since $\mathfrak{k}$ is a compact real form of $\mathfrak{h}_{\mathbb{C}}$, the Cartan decomposition of $\mathfrak{h}_{\mathbb{C}}$ implies that $d\Pi_{\mathbb{C}} = 0$. It follows from the induction construction that the original derived representation $d\Pi$ of $\mathfrak{h}$ in the domain D must be zero as well. But then $\mathcal{Q} \restriction \mathfrak{h} = 0$, which contradicts the requirement that a quantization represent $\mathfrak{h}$ faithfully. This concludes the proof of Theorem 1. □

We remark that Theorem 1 was already known when $\mathfrak{h}$ is compact [4], in which case the proof above simplifies greatly and provides an alternate means of establishing Theorem 2 *ibid*. Notice also that when $\mathfrak{h}$ is compact every quantization of $P(M)$ is necessarily polynomial; this follows from the observation that since $\mathcal{Q} \restriction \mathfrak{h}$ is irreducible the representation space $\mathcal{H}$ must be finite-dimensional together with a well known fact about enveloping algebras (Prop. 2.6.5 in [5]).

3. Discussion

The key observation underlying Theorem 1 is that as $M \subset \mathfrak{h}^*$ is nonnilpotent, its ideal $I(M)$ is nonhomogeneous. If M is a nilpotent orbit, on the other hand, then $I(M)$ is homogeneous, and from Theorem 1.1 in [1] we know that there do exist polynomial quantizations of $P(M)$. (Although it is not clear to what extent these are 'nontrivial' in general.) Taken together, these results serve to establish a conjecture of Gotay [2] when $\mathfrak{h}$ is semisimple: *There exists a consistent polynomial quantization of $P(M)$ if and only if $I(M)$ is homogeneous.*

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