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Reduction of homogeneous Yang-Mills fields

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Abstract. *The structure of the reduced phase space for a homogeneous Yang-Mills field on a spatially compactified* $(n+1)$ *-dimensional Minkowski spacetime is studied. Using the theory developed in [AGJ], various reductions of this system are considered and are shown to agree. Moreover, the reduced phase space is realized as semi-algebraic set which carries a nondegenerate Poisson algebra. For the gauge groups* $SU(2)$ *or* $SO(3)$ *it is shown that this system is equivalent to that of* n *interacting particles moving in* R^3 *with zero total angular momentum. The particular cases* $n = 1$ *and* 2 *are discussed in detail.*

I. INTRODUCTION

The Yang-Mills equations provide one of the most important examples of a singular constrained system. Through the work of Arms, Marsden and Moncrief one now has a detailed understanding of the structure of the Yang-Mills constraint set. (1) One knows, among other things, where the singularities are, what they look like and how they are related to gauge symmetries.

Less well understood is the behaviour of the reduced phase space for the Yang-Mills equations. Although certain subspaces of its smooth sector have been studied in detail [Mi], one has at present only a local description of its singularity structure [Al, Mon]. One expects from general principles that the reduced space will be a stratified manifold

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(1) See [Al], [Mon] and references cited therein for Yang-Mills theory per se. The general formalism regarding singularities of momentum maps is contained in [AMM]. Both [A 2] and [Ma 2] are useful surveys.

(«cones over cones»), but global phenomena may distort this characterization. There are other fundamental problems as well: To what extent is the reduced space symplectic? How does the dynamics project to the reduced space and in what sense is it Hamiltonian?

In this paper I examine these issues in the context of a *homogeneous* Yang-Mills field propagating on a spatially compactified $(n+1)$-dimensional Minkowski spacetime. The assumption of homogeneity allows me to dispense with infinite-dimensional technicalities while concentrating on the essence of the problem, viz., the presence of singularities. These singularities tremendously complicate the reduction process. In fact, it is not at all clear precisely what one *means* by «reduction» in the singular case: there is no longer a unique, much less preferred, way to reduce the system [AGJ]. Which possibilities «work»? If various reductions differ, how do they differ and what is the physical significance of this? Do these reductions yield symplectic structures – or at least Poisson brackets – on the reduced space? These are all global questions, and their study requires techniques altogether different than the usual ones, which are based on properties of slices for the gauge group action [AMM].

The foundations for such a study have already been laid in [AGJ], and involve ideas from C^∞ real algebraic geometry. Applying this theory to the homogenized Yang-Mills system, I find that the three main methods of reduction – à la Dirac, geometric and group theoretical – are not only applicable, but in fact agree, provided the gauge group is compact. I then show that the resulting reduced phase space is a stratified symplectic manifold which carries a nondegenerate Poisson algebra, and that it can be realized as a semialgebraic set. Moreover, the reduced space coincides with the moduli space of homogeneous Yang-Mills fields, a result which is certainly expected on physical grounds but need not always be true.

To obtain sharper results, I furthermore suppose that the gauge group is either $SO(3)$ or $SU(2)$. Then the system takes the form of n interacting particles moving in R^3 constrained to have zero total angular momentum. In this case I am able to explicitly construct the reduced space and its associated Poisson algebra using classical invariant theory.

These findings indicate that despite being singular, the Yang-Mills system is relatively well-behaved. Thus it furnishes a useful «laboratory» for discussing questions relating to the reduction of singular systems. More importantly, this approach should provide some insight into the infinite-dimensional Yang-Mills case as well as other singular constrained field theories.

II. THE HAMILTONIAN STRUCTURE OF THE YANG-MILLS EQUATIONS

I first briefly sketch the standard Hamiltonian formulation of the vacuum Yang-Mills equations following [BFS]. See also [Al], [Mi], [Mon] and [Sn2] for background and further details.

Let G be a Lie group whose Lie algebra g carries an adjoint-invariant inner product $\langle\ \rangle$, and consider a Yang-Mills theory based on a trivial principal G-bundle P over an $(n+1)$-dimensional spacetime X.

Fix a compact Cauchy surface $S \subset X$, and denote the restriction of P to S by P_S. Then, relative to S, the space + time decomposed configuration space for the Yang-Mills system is the connection bundle A_S of P_S. The corresponding phase space is the L^2-cotangent bundle T^*A_S with its canonical symplectic structure. Elements $(A, E) \in T^*A_S$ consist of a connection A on P_S (viewed as a g-valued 1-form on S) and its canonically conjugate «electric» field E (also viewed as a g-valued 1-form on S). The pairing between A and E is given by

$$\int_S \langle A \wedge \star E\rangle \tag{2.1}$$

where $\star$ is the Hodge star operator of the induced metric on S, and the symplectic form is

$$\omega((a,e),(a',e')) = \int_S (\langle a \wedge \star e'\rangle - \langle a' \wedge \star e\rangle).$$

Finally, let

$$F = \mathrm{d}A + [A \wedge A] \tag{2.2}$$

be the curvature of A, $[\ ]$ being the bracket on g. Then the Hamiltonian density is

$$H(A,E) = \frac{1}{2} \star \langle E \wedge \star E\rangle + \frac{1}{4} \star \langle F \wedge \star F\rangle. \tag{2.3}$$

Consider the group $\mathcal{G}_S$ of automorphisms of P_S which cover the identity on S, thought of as maps $\varphi : S \to G$. Its Lie algebra $\mathfrak{g}_S$ can be identified with the g-valued functions on S. Now $\mathcal{G}_S$ acts on A_S by gauge transformations:

$$(\varphi, A) \to \varphi^{-1}A\varphi + \varphi^{-1}\mathrm{d}\varphi \tag{2.4}$$

where G is represented as a matrix group on g. The induced action on the phase space T^*A_S is the cotangent action with momentum map $J : T^*A_S \to \mathfrak{g}_S^*$ given by

$$J(A,E) = \delta E + \star[A \wedge \star E]. \tag{2.5}$$

In this expression δ is the metric codifferential and a pairing analogous to (2.1) has been used to identify $\mathfrak{g}_S^*$ with $\mathfrak{g}_S$.

The condition $J = 0$ arising from the gauge invariance of the theory is an initial value constraint. That is, only those pairs $(A, E) \in T^*A_S$ satisfying $\delta E + \star[A \wedge \star E] = 0$ constitute (formally) admissible initial data for the Yang-Mills equations.

III. HOMOGENEOUS YANG-MILLS FIELDS AND A MECHANICAL ANALOGY

Now suppose that the spacetime is Minkowskian, $X = T^n \times R$, and that the Yang-Mills field is spatially homogenous. Then A and E are *constant* on $S = T^n$, (2) and (2.2) reduces to

$$F = [A \wedge A]. \tag{3.1}$$

Upon fixing a base point in $\mathcal{A}_S$, the connection bundle may then be identified with $R^{n*} \otimes g \approx \mathcal{L}(R^n, g)$ and the phase space becomes $T^*\mathcal{L}(R^n, g)$ with symplectic form

$$\omega = \star\langle \mathrm{d}A \wedge \star \mathrm{d}E \rangle. \tag{3.2}$$

Furthermore, from (2.4) the group $\mathcal{G}_S$ acts on $\mathcal{L}(R^n, g)$ by

$$(\varphi, A) \to \varphi^{-1} A \varphi \tag{3.3}$$

where now φ is also *constant.* Thus $\mathcal{G}_S$ (and $\mathcal{g}_S$) may be identified with G (and g), so that (3.3) is the adjoint representation of G on the second factor of $R^{n*} \otimes g$. Expression (2.5) for the momentum map reduces to

$$J(A, E) = \star[A \wedge \star E]. \tag{3.4}$$

All this data, when combined with the Hamiltonian (2.3) and the constraint $J = 0$, realizes homogenized Yang-Mills theory as a finite-dimensional constrained dynamical system with symmetry.

I now additionally presume that the gauge group $G = SU(2)$ or $SO(3)$. Under the usual identifications of $(g, [\ ])$ with $(R^3, \times)$ and g with g^*, (3) (3.4) takes the suggestive form

$$J(A, E) = \sum_{i=1}^{n} (A_i \times E_i).$$

In fact, if one views the components $A_i \in R^3$ and $E_i \in R^3$ of A and E as being the position x_i and momentum p_i vectors of a particle, then *this homogeneous* $SU(2)$ *or*

(2) Technically, this presupposes that the trivialization of P_S has been chosen to be T^n-invariant. Note that there is some controversy in general as to whether «homogeneity» requires the gauge potential A or rather just the field strength F to be spatially constant [Mol].

(3) Since the adjoint representations of $SU(2)$ and $SO(3)$ are the same, I will henceforth not distinguish between these two possibilities and will refer solely to the rotation group.

$SO(3)$ Yang-Mills system becomes formally identical to that of a swarm of n interacting particles moving in R^3 with zero total angular momentum: ([4])

$$J(x,p) = \sum_{i=1}^{n}(x_i \times p_i) = 0. \tag{3.5}$$

The configuration space of the system should now be regarded as $\mathcal{L}(R^n, R^3) \approx R^{3n}$, in which context the action (3.3) becomes simply the diagonal representation of the rotation group on n copies of R^3. The phase space is then $R^{3n} \times R^{3n} = (R^3 \times R^3)^n = R^{6n}$, the symplectic structure (3.2) assumes the usual form

$$\omega = \sum_{i=1}^{n} \mathrm{d}\, x_i \wedge \mathrm{d}\, p_i \tag{3.6}$$

and, using (2.3) and (3.1), the Hamiltonian becomes

$$H(x,p) = \frac{1}{2}\sum_{i=1}^{n}\left(\|p_i\|^2 + \sum_{j>i}\|x_i \times x_j\|^2 \right). \tag{3.7}$$

This result is reminiscent of the relations between the KdV equation and the Toda lattice or the Calogero system [Mal]. It is significant in that it enables one to think of the Yang-Mills system in more familiar mechanical terms, and thereby draw upon various results whose applicability would otherwise have remained obscured. ([5]) In particular, the reduction of this system is already completely understood when there is only one particle [GB]; it corresponds to a homogeneous Yang-Mills theory on a $(1+1)$-dimensional spacetime. Now I analyze this system in detail.

IV. THE CONSTRAINT SET

First consider the constraint set $C = J^{-1}(0)$ in the general context of §III, assuming henceforth that G is *compact*. From [AMM] one knows that C will have singularities exactly at those points admitting nontrivial (but nonminimal) isotropy groups. Clearly, in view of (3.4), these singularities will be conical. Moreover, C stratifies according to symmetry type:

$$C = \bigcup_{K \subset G} C^K, \tag{4.1}$$

([4]) A similar observation, in a slightly different context, was made by Patrick [P].

([5]) This is not to say that a system of particles with vanishing total angular momentum is uninteresting in itself. In fact, this system appears in several contexts, notably celestial mechanics (cf. [AM]).

where C^K consists of all points with isotropy groups conjugate to the subgroup K of G. Each stratum individually is a smooth manifold (conceivably with several components of differing dimension). Let $\Sigma(C)$ be the subvariety of singular points of C and let $S(C) = C\backslash\Sigma(C)$ be the open subset of C consisting of smooth points.

Since the action of G on $T^*\mathcal{L}(R^n, g)$ is lifted from the base, the main result of [AGW] yields, for any G whatsover,

THEOREM 1. *C is coisotropic.* (6)

Now specialize to the case when G is the rotation group and suppose that $n > 1$. (The case $n = 1$ is special and will be discussed separately in § VII.) Then, viewing $J : R^{6n} \to R^3$, a computation of the rank of J shows that $C_n = J^{-1}(0)$ is a $(6n-3)$-dimensional variety. The isotropy group of a point $(x, p) \in C_n$ can be either 0-, 1- or 3-dimensional and the constraint set decomposes correspondingly:

$$C_n = C_n^0 \cup C_n^1 \cup C_n^3. \tag{4.2}$$

The origin is the most singular point, as it is invariant under the entire group. Thus $C_n^3 = \{(0,0)\}$. The only other points with nontrivial isotropy groups are of the form

$$(x, p) = (s_1\hat{e}, \ldots, s_n\hat{e}, t_1\hat{e}, \ldots, t_n\hat{e})$$

for any constants s_i, t_i not all zero, where $\hat{e}$ is a unit vector. Such points are invariant under an $SO(2)$ subgroup of rotations about the axis $\hat{e}$. Consequently C_n^1 is diffeomorphic to the antipodal identification $\dot{R}^{2n} \times_{Z_2} S^2$ and fibers over RP^2 with fiber $\dot{R}^{2n}$, where the projection $C_n^1 \to RP^2$ is

$$[((s_1, \ldots, s_n, t_1, \ldots, t_n), \hat{e})] \to [\hat{e}]$$

(brackets denote equivalence classes).

Thus for $n > 1$

$$\Sigma(C_n) = C_n^1 \cup C_n^3$$

can be realized as the space formed by collapsing the zero section of the vector bundle

$$\begin{array}{ccc} R^{2n} & \to & R^{2n} \times_{Z_2} S^2 \\ & & \downarrow \\ & & RP^2 \end{array}$$

(6) This result is not entirely obvious. Certainly the components of the momentum map are themselves first class constraints, but it does *not* necessarily follow that *all* constraints are first class. This phenomenon, which of course not occur in the regular case, is discussed in both [AGW] and §V of [AGJ].

to a point. The smooth sector $S(C_n)$ is more difficult to describe (the case $n = 2$ is treated in § VIII). In the remainder of this section I prove that C_n is «as nice as possible».

THEOREM 2. *C_n is irreducible.*

Proof. It suffices to show that $S(C_n)$ is arc connected. (7) For $n = 2$, this can be checked using the description of $S(C_2)$ given in § VIII. I now proceed by induction on n, assuming the result is true for $S(C_{n-1})$. Viewing $C_{n-1} \subset C_n$ in the obvious way (i.e. by setting $x_n = 0, p_n = 0$), I will show that every point $(x,p) \in S(C_n)$ can be joined by an arc in $S(C_n)$ to a point of $S(C_{n-1})$ ($\subset S(C_n)$ for $n > 2$). As $S(C_{n-1})$ is arc connected, so then is $S(C_n)$.

First note that $(x,p) \in S(C_n)$ iff (3.5) is satisfied and at least two of the x_i and p_i are lineary independent.

There are two cases to consider, depending upon whether $x_n \times p_n = 0$. Begin by supposing that it is not, so that $(x_n, p_n, e = x_n \times p_n)$ is a frame for R^3. For $i < n$, write

$$x_i = a_i x_n + b_i p_n + c_i e, \; p_i = r_i x_n + s_i p_n + t_i e.$$

Then the line segment obtained by simultaneously scaling each c_i and t_i to zero (while leaving all other components unchanged) lies entirely within $S(C_n)$ and connects (x,p) with a point whose e components vanish.

For such a point write

$$J(x,p) = \left(\sum_{i=1}^{n-1} k_i + 1 \right) e, \tag{4.3}$$

where $x_i \times p_i = k_i e$ for $i < n$. Now fix a j for which $k_j \neq 0$; by perturbing (x,p), if necessary, one may assume that $k_j \neq -1$. For $\sigma \in [0,1]$ consider the curve given by

$$x_j(\sigma) = (1 + \sigma/k_j) x_j,$$
$$x_n(\sigma) = \sqrt{(1-\sigma)}\, x_n, \; p_n(\sigma) = \sqrt{(1-\sigma)}\, p_n,$$

where all other x_i and p_i remain unchanged. From (4.3) J vanishes along this curve. Furthermore, since $k_j \neq -1$ and both $x_n(1)$ and $p_n(1)$ are zero, this curve lies in $S(C_n)$ and its endpoint is in $S(C_{n-1})$.

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(7) See, e.g., the Proposition on p. 21 of [GH]. Although the desired result and its proof are stated in the context of analytic varieties, they remain valid in the real case (but not the converse!). Recall also that, as the $S(C_n)$ are manifolds, connectedness is equivalent to arc connectedness.

Now consider the case when $x_n \times p_n = 0$. If any two of the x_i and p_i for $i < n$ are linearly independent, the desired result follows simply by scaling both $x_n, p_n \to 0$. If not, the point in question must look like

$$x_i = c_i e,\ p_i = t_i e,\ x_n = c_n f,\ p_n = t_n f$$

for all $i < n$, where $e \times f \neq 0$. Again, one may homotope such a point within $S(C_n)$ to one of the form

$$x_{n-1} = 0,\ p_{n-1} = e,\ x_n = f,\ p_n = 0 \tag{4.4}$$

with all other x_i and p_i vanishing. Then the line segment

$$x_1(\sigma) = \sigma f,\ x_n(\sigma) = (1-\sigma) f$$

(leaving all other vectors unchanged) connects the point (4.4) with one in $S(C_{n-1})$. ■

V. THE REDUCED PHASE SPACE

There are (at least) three ways of constructing a reduced phase space $\hat{C}$ for the Yang-Mills system: à la Dirac, geometrically and group theoretically ([AGJ], [Snl], [MW]; see also [ACG]). Which method one chooses depends upon what aspect of the formalism one considers primary. For example, the Dirac approach centers around the concept of «observable», the geometric method is more symplectic in nature and the group theoretical reduction is closely tied to the notion of gauge equivalence. These types of reduction will usually not agree (although they coincide when the system is regular). In view of Theorem 3 below, it is not necessary to go into the details of these procedures here; see instead [AGJ] for a comprehensive exposition along with numerous examples. The local theory is discussed in [Al] and [Mon]; cf. also § VII (c) of [AMM].

The main result of this section is that all these reductions are «equivalent», provided the gauge group is compact.

THEOREM 3. *Consider a homogeneous Yang-Mills theory with compact connected gauge group G. Then the reduced phase spaces resulting from the three above-mentioned methods are all isomorphic.*

This follows directly from Prop. 5.5 of [AGJ]. Roughly speaking, it means that the notions of observable and gauge equivalence are tied to the symplectic structure and each other in the «correct» – and expected – way, despite the presence of singularities.

Since the group theoretically reduced space is just the orbit space of C, one has that

$$\hat{C} \approx C/G. \tag{5.1}$$

This implies that the reduced space is also a stratified manifold. Indeed, as G is compact and each stratum C^K of C consists of points with the *same* symmetry type, $\hat{C}^K = C^K/G$ is a Hausdorff manifold (cf. exercise 4.1M in [AM]). Then the decomposition (4.1) yields

$$\hat{C} = \bigcup_{K \subset G} \hat{C}^K. \tag{5.2}$$

In addition the local theory (cf. § VII (c) of [AMM] guarantees that each stratum $\hat{C}^K$ of $\hat{C}$ is actually *symplectic*. Thus

THEOREM 4. *$\hat{C}$ is a stratified symplectic manifold.*

The identification (5.1) has three further consequences.

First, from general principles the geometrically reduced space can be interpreted as the set of gauge equivalence classes of solutions of the homogeneous Yang-Mills equations. (8), (9) Combining this observation with (5.1), one sees that *the reduced space coincides with the moduli space of homogeneous Yang-Mills fields.*

Second, as the Dirac reduced space is completely regular in the topological sense [Sn1], it guarantees that the orbit space is reasonably well behaved.

Third, using a result of Schwarz [Sc] and Prop. 5.6 of [AGJ], it enables one to model the reduced space as a *semialgebraic variety*:

THEOREM 5. *There exists a map $\rho : C \to R^N$ for some N such that $\hat{C} \approx \rho(C)$.*

When G is the rotation group this realization may be made completely explicit using classical invariant theory. The standard reference for what follows is [W], especially § IX and § XVII of Chapter 2. It is helpful to introduce some notation. Let u_α, $1 \leq \alpha \leq 2n$, be defined by $u_i = x_i$, $u_{n+i} = p_i$, and set

$$\begin{vmatrix} u_{\alpha_1} & \dots & u_{\alpha_k} \\ u_{\beta_1} & \dots & u_{\beta_k} \end{vmatrix} = \det \begin{pmatrix} u_{\alpha_1} \cdot u_{\beta_1} & \dots & u_{\alpha_1} \cdot u_{\beta_k} \\ \vdots & & \vdots \\ u_{\alpha_k} \cdot u_{\beta_1} & \dots & u_{\alpha_k} \cdot u_{\beta_k} \end{pmatrix}.$$

(8) Here «gauge» is to be understood in the context of the Dirac theory of contraints, not in the Yang-Mills sense.

(9) Strictly speaking, the geometrically reduced space consists of gauge equivalence classes of admissible *initial data* for the Yang-Mills equations. Since the evolution generated by the Hamiltonian (1.3) is complete (cf. p. 232 of [AM]), this space may be identified with the set of gauge equivalence classes of *solutions* to the Yang-Mills equations.

For the diagonal action of the rotation group on $R^{6n} = (R^3)^{2n}$, the basic scalar invariants are the dot products $u_\alpha \cdot u_\beta$ and the triple scalar products $[u_\alpha, u_\beta, u_\gamma]$. These invariants are the components of the map ρ of Theorem 5. They are not all independent, but are subject to three types of relations:

(R1) $$\begin{vmatrix} u_{\alpha_1} & \cdots & u_{\alpha_4} \\ u_{\beta_1} & \cdots & u_{\beta_4} \end{vmatrix} = 0,$$

(R2) $$[u_\alpha, u_\beta, u_\gamma]\,[u_\lambda, u_\mu, u_\upsilon] - \begin{vmatrix} u_\alpha & u_\beta & u_\gamma \\ u_\lambda & u_\mu & u_\upsilon \end{vmatrix} = 0,$$

and

(R3) $$\sum_{(\alpha,\beta,\gamma,\delta)} [u_\alpha, u_\beta, u_\gamma]\,(u_\delta \cdot u_\lambda) = 0,$$

where the last sum is cyclic. These relations also are not all independent. For later use, observe that together (R1) and (R2) imply that

(R4) $$\begin{vmatrix} u_\alpha & u_\beta & u_\gamma \\ u_\alpha & u_\beta & u_\delta \end{vmatrix}^2 = \begin{vmatrix} u_\alpha & u_\beta & u_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} \quad \begin{vmatrix} u_\alpha & u_\beta & u_\delta \\ u_\alpha & u_\beta & u_\delta \end{vmatrix}.$$

Furthermore, there are various inequalities that must be satisfied; these are

(I1) $$||u_\alpha||^2 \geq 0$$

and

(I2) $$||u_\alpha||^2\, ||u_\beta||^2 - (u_\alpha \cdot u_\beta)^2 \geq 0.$$

Now the constraint $J = 0$ gives rise to further scalar conditions of the form

(J1) $$u_\alpha \cdot J = 0$$

and

(J2) $$[u_\alpha, u_\beta, J] = 0.$$

These constraints are not independent either among themselves or in conjunction with (R1)-(R3).

All told, then, if N is the total number of invariants listed above, the reduced space may be identified with the semialgebraic set in R^N determined by the above relations, inequalities and constraints. Since the dimension of a generic orbit on C_n is three, (5.1) implies that the dimension of the reduced space will be $6n-6$.

This characterization is complicated and unwieldy, but it does enable one to construct the reduced space without having to compute a quotient. Of course, substantial simplifications can be made for specific small n. The details for $n = 1$ and 2 will be given in § VII and § VIII, respectively.

For the rotation group, (4.2) and (5.2) yeld the symplectic stratification

$$\hat{C}_n = \hat{C}_n^0 \cup \hat{C}_n^1 \cup \hat{C}_n^3, \tag{5.3}$$

where $\hat{C}_n^3$ is the vertex of $\hat{C}_n^1 \approx \dot{R}^{2n}/Z_2$. These two strata fit together to form a half-cone over which $\hat{C}_n^0$ lies. This description clearly indicates the «cones over cones» singularity structure of the reduced space. In the Yang-Mills context [A1], the «0», «1», and «3» strata represent gauge equivalence classes of irreducible $SU(2)$ or $SO(3)$ solutions, «electromagnetic» $U(1)$ or $SO(2)$ solutions and the trivial solution, respectively.

VI. THE REDUCED POISSON ALGEBRA

Due to singularities in both the constraint set and the reduced space, one cannot expect the symplectic structure on phase space to project directly to $\hat{C}$ although, as Theorem 4 shows, it does project to each stratum $\hat{C}^K$ of $\hat{C}$. Instead, one should try to reduce the symplectic structure *algebraically,* that is, use the Poisson bracket $\{\ \}$ associated to (3.2) to induce a Poisson bracket on $\hat{C}$.

This may or may not be possible with the reduction techniques discussed above and, even when it is, the resulting reduced Poisson algebras need not be isomorphic and/or nondegenerate [AGJ]. Fortunately, in the case considered here, such pathologies do not arise.

THEOREM 6. *Consider a homogeneous Yang-Mills theory with compact connected gauge group G. Then reduction gives rise to a nondegenerate Poisson bracket on $\hat{C}^\infty(\hat{C})$.*

Here, the «smooth» structure on $\hat{C}$ is defined as follows. A function $\hat{f}$ on $\hat{C}$ is *smooth,* $\hat{f} \in \hat{C}^\infty(\hat{C})$, if it is the projection of a Whitney smooth function on C, i.e., $\hat{f} \circ \pi = f$ for some $f \in W^\infty(C)$, where $\pi : C \to \hat{C}$ is the reduction. (10)

(10) A function f on C is *Whitney smooth,* $f \in W^\infty(C)$, provided f is the restriction to C of a smooth function on the ambient manifold.

Proof. I will show that the three reduction techniques considered in §V define the same Poisson bracket on the reduced space. All references are to [AGJ] unless otherwise noted.

The geometrically reduced bracket exists by Prop. 5.7. Since by Theorem 1 above C is coisotropic, Prop. 3.1 guarantees that the Dirac reduced bracket is well-defined. These two reduced Poisson algebras then coincide by Cor. 5.8. On the other hand, Lemma 5.4 shows that the group theoretical and Dirac reductions are the same.

Finally, nondegeneracy is a consequence of Prop. 5.9. ∎

The Poisson bracket on $\hat{C}^\infty(\hat{C})$ effectively glues the individual symplectic forms on the strata $\hat{C}^K$ together into a global geometric structure on the reduced space. Because the reduced Poisson algebra by construction incorporates the «compatibility» conditions which arise when strata join, it is actually more appropriate to regard $\hat{C}$ as a space with a Poisson bracket than as a stratified symplectic manifold. (11)

It is possible to explicitly realize the reduced Poisson algebra as follows. Recall from Theorem 3 that $\hat{C} \approx C/G$. Then $\hat{C}^\infty(\hat{C}) \approx W^\infty(C)^G$, the G-invariant Whitney smooth functions on C. For $\hat{f}, \hat{g} \in W^\infty(C)^G$, the reduced Poisson bracket is

$$[\hat{f}, \hat{g}] = \{f, g\}|C, \tag{6.1}$$

where f, g are any extensions of $\hat{f}, \hat{g}$ to all of $T^*\mathcal{L}(R^n, g)$. Thus one may also construct the reduced Poisson algebra without having to compute any quotients. Using Theorem 5 and the results of Schwarz [Sc] this may be pushed one step further, yielding

$$\hat{C}^\infty(\hat{C}^k) \approx W^\infty(\rho(C)).$$

It should be mentioned that there are (at least) two other ways of reducing a singular constrained system with symmetry which are purely algebraic in character. These procedures construct reduced Poisson *algebras* but not reduced *spaces*. I now briefly consider these, showing that they give results equivalent to those obtained above, at least when G is the rotation group.

(11) One reason for this is the following. The bracket associated to the reduced symplectic structure on the *manifold* $\hat{C}^K$ is defined on all of $C^\infty(\hat{C}^K)$, whereas the bracket induced on $\hat{C}^K$ by that on $\hat{C}$ is defined only on $\hat{C}^\infty(\hat{C}^K) \subseteq C^\infty(\hat{C}^K)$. (This is because $W^\infty(C^K) \subseteq C^\infty(C^K)$; the inclusions are usually strict as C^K is not necessarily closed.) The «compatibility» conditions referred to in the text comprise exactly this cutting down from $C^\infty(\hat{C}^K)$ to $\hat{C}^\infty(\hat{C}^K)$. For an example see [G]. As a consequence, it is probably best *not to* think of $\hat{C}^K$ as a symplectic manifold as is often done [AMM, Mon], since this appears to lead to incorrect results . (This can be seen even in the $n = 1$ example in §VII; see also [GB].) These matters will be discussed in more detail elsewhere.

The first algebraic reduction procedure, due to Sniatycki and Weinstein [SW], provides a Poisson bracket on the function space $[C^\infty(R^{6n})/\mathcal{J}]^G$, where $\mathcal{J}$ is the ideal in $C^\infty(R^{6n})$ generated by the components of the momentum map J. The other, which is discussed in [AGJ], centers instead on $C^\infty(R^{6n})^G/\mathcal{J}^G$. By virtue of Prop. 5.12 of [AGJ], these two procedures coincide when G is compact, so I restrict attention to the former.

THEOREM 7. *When G is the rotation group, the Sniatycki-Weinstein reduced Poisson algebra is isomorphic to that given by Theorem 6.*

Proof. Since $W^\infty(C)^G = [C^\infty(R^{6n})/I(C_n)]^G$, it sufficies to establish that $I(C_n) = \mathcal{J}$, where $I(C_n)$ is the ideal of C_n (cf. Thm. 6.1 of [AGJ]). As $\mathcal{J}$ is generated by polynomials, Thm. 6.3 of [AGJ] reduces the problem to showing that $\mathcal{J}$ is a real ideal of the polynomial ring $R[x,p]$. For this it is useful to complexify.

Let $\mathcal{J}^C = \mathcal{J} \otimes_R C$, an ideal in $C[x,p]$, and let $C_n^C \subset C^{6n}$ be the variety determined by $\mathcal{J}^C$. Clearly $dim_C C_n^C = dim_R C_n$. Now Theorem 2 – with obvious modifications – holds in the complex case as well, with the result that C_n^C is irreducible; therefore $\mathcal{J}^C$ is prime. By a theorem in commutative algebra (6.5 in [AGJ]), $\mathcal{J}$ is real. ∎

Finally, a brief word about dynamics. Since it is gauge invariant, the Hamiltonian (2.3) projects to a function $\hat{H}$ on $\hat{C}$. In terms of this reduced Hamiltonian and the reduced Poisson bracket, an observable $\hat{f} \in \hat{C}^\infty(\hat{C})$ evolves in the usual way:

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} = [\hat{f}, \hat{H}].$$

In this sense the evolution on the reduced space is Hamiltonian.

VII. THE CASE $n = 1$

Consider now the special case of one particle with zero angular momentum. This system is simpler than the others and does not fit the pattern established for $n > 1$. The basic reference here is [AGJ]; a different approach is pursued in [GB].

The main difference between one and several particles is that C_1^0 is empty. Consequently, points of C_1^1 are *nonsingular* and $\Sigma(C_1) = \{(0,0)\}$. Now $S(C_1) = C_1^1$ can be identified with $\dot{R}^2 \times_{Z_2} S^2$, and from this description it is obvious that $S(C_1)$ is connected. Therefore C_1 is irreducible. Moreover C_1 in its entirety forms a complex cone over RP^2 [GB].

I next construct the reduced space via invariant theory. The basic list of invariants is (dropping the subscripts «1»):

$$||x||^2, x \cdot p, ||p||^2.$$

Thus $N = 3$. There are no relations of types (R1)-(R3) in this instance, and the only nontrivial constraint is of type (J2), viz.,

$$[x, p, J] = 0.$$

As $J = x \times p$, this reduces to

$$||x||^2 \, ||p||^2 - (x \cdot p)^2 = 0.$$

Taking this into account, the inequality (I2) becomes vacuous, and the remaining inequalities (I1) are

$$||x||^2 \geq 0, \; ||p||^2 \geq 0.$$

Defining $\rho : C_1 \to R^3$ by

$$\rho(x, p) = \left(||x||^2, \; x \cdot p, \; ||p||^2 \right),$$

one then has the reduced space $\hat{C}_1$ realized as the half-cone

$$\rho(C_1) = \{\rho \in R^3 | \, (\rho_2)^2 = \rho_1 \rho_3 \text{ and } \rho_1, \rho_3 \geq 0\}.$$

Topologically $\hat{C}_1 \approx R^2/Z_2$.

The reduced Poisson bracket (6.1) on $W^\infty(\rho(C_1))$ works out to be

$$[\hat{f}, \hat{g}] = 2\{f, g\}_{1,2}\rho_1 + 4\{f, g\}_{1,3}\, \rho_2 + 2\{f, g\}_{2,3}\, \rho_3,$$

where

$$\{f, g\}_{a,b} := (\partial f/\partial \rho_a)\, (\partial g/\partial \rho_b) - (\partial g/\partial \rho_a)\, (\partial f/\partial \rho_b),$$

and the reduced Hamiltonian (3.7) is $\hat{H}(\rho) = \frac{1}{2}\rho_3$.

VIII. THE CASE $n = 2$

First I give an explicit description of $S(C_2)$. Let $(x_1, p_1, x_2, p_2) \in C_2$, in which case these four vectors must be coplanar by (3.5). If $(x_1, p_1, x_2, p_2) \in S(C_2)$ then, since at least two – and hence exactly two – of these vectors are linearly independent, this plane is uniquely determined. Let e and f be a basis for this plane. Expanding the x_i and the p_i in this basis, one has in matrix terms that

$$(x_1 \, p_1 \, x_2 \, p_2) = (e \; f)(E|F),$$

where E and F are 2×2 matrices such that

(i) $$rk(E|F) = 2$$

and

(ii) $$\det E + \det F = 0.$$

Denote by $G_2(R^3)$ the Grassmann manifold of 2-planes in R^3. Then the above shows that $S(C_2)$ is a bundle over $G_2(R^3)$ with 7-dimensional fiber M consisting of all 2×4 matrices $(E|F)$ satisfying (i) and (ii). Using this description, it is straightforward to check that $S(C_2)$ is arc connected.

Now consider the two-particle reduced space. There are 14 basic invariants: 10 dot products and 4 triple scalar products. In view of (3.5), the four constraints of type (J1) force all the triple scalar products to vanish; this in turn vacates the relations (R3) altogether. The ten relations (R2) then collapse to

(8.1) $$\begin{vmatrix} u_{\alpha_1} & u_{\alpha_2} & u_{\alpha_3} \\ u_{\beta_1} & u_{\beta_2} & u_{\beta_3} \end{vmatrix} = 0.$$

However, a calculation reveals that

$$\begin{vmatrix} u_\alpha & u_\beta & u_\gamma \\ u_\alpha & u_\beta & u_\gamma \end{vmatrix} = [u_\alpha, u_\beta, J](u_\gamma \cdot u_\gamma) + [u_\gamma, u_\alpha, J](u_\beta \cdot u_\gamma) + [u_\beta, u_\gamma, J](u_\alpha \cdot u_\gamma).$$

The vanishing of these particular determinants is therefore an automatic consequence of the constraints (J2) and so, taking the subsidiary relations (R4) into account, all the relations (8.1) are redundant. Finally, there is only one nontrivial relation of type (R1); it is

(8.2) $$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ u_1 & u_2 & u_3 & u_4 \end{vmatrix} = 0.$$

A cofactor expansion of this 4×4 determinant shows that (8.2) is automatically satisfied by virtue of (8.1), these 3×3 determinants being the minors of the 4×4.

Thus there are just the ten dot product invariants left, subject only to the constraints (J2). Of these six constraints another computation reveals that only four are independent. It follows that the reduced space is the 6-dimensional semialgebraic set in R^{10} determined by the conditions $[u_\alpha, u_\beta, J] = 0$ and the inequalities (I1) and (I2).

The two-particle reduced space stratifies as in (5.3). From the descriptions of $\hat{C}_2^1$ and $\hat{C}_2^3$ given in §V it follows that $\Sigma(\hat{C}_2) \approx R^4/Z_2$. Using the characterization of C_2^0 as a bundle over $G_2(R^3)$, $S(\hat{C}_2) = \hat{C}_2^0$ can be realized topologically as $M/SO(2)$, where $SO(2)$ is the subgroup of $SO(3)$ which covers the identity on $G_2(R^3)$. To be precise, the action is

$$(\theta,(E|F)) \to (ER(\theta)|FR(\theta))$$

where the matrices $R(\theta)$ represent rotations about the axis $e \times f$ (the «rest» of the rotation group acts transitively on the base).

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