Poisson reduction and quantization for the $n+1$ photon

Mark J. Gotay
Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada

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For a dynamical system in which the constraints are given by the vanishing of a singular momentum map J, reduction in the usual group-theoretic sense may not be possible. Nonetheless, one may still "reduce" $J^{-1}(0)$, at least on the level of Poisson algebras. An example of such a singular constrained system is the "$n+1$ photon," that is, a massless, spinless particle in $(n+1)$-dimensional Minkowski space-time. We apply the generalized reduction procedure to the $n+1$ photon, explicitly constructing the Poisson algebra of gauge invariant observables. This technique also enables us to completely analyze the effects of the singularities in $J^{-1}(0)$ on the system. We then quantize, obtaining results which are in agreement with a quantization of the extended phase space and the subsequent imposition of the constraint.

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I. INTRODUCTION

Let (X,ω) be a symplectic manifold and let G be a connected Lie group with Lie algebra g. Assume that there is a Hamiltonian action of G on (X,ω) with a G-equivariant momentum map $J: X \to g^*$. If $0 \in g^*$ is a regular value of J and if the action of G on $J^{-1}(0)$ is sufficiently nice, then the Marsden–Weinstein reduced space $J^{-1}(0)/G$ will be a symplectic manifold.[1]

These constructs are particularly relevant to physics. In this context, (X,ω) represents the extended phase space of a dynamical system, G is the gauge group, and, typically, the constraints are given by $J=0$.[2] The reduced phase space $J^{-1}(0)/G$ is then interpreted as the space of gauge invariant states of the system.

In many interesting situations, however, this group-theoretical reduction procedure does not work. For instance, it may happen that 0 is not a regular value of J as in gravity and Yang–Mills theory. Moreover, even if $J^{-1}(0)$ is smooth, $J^{-1}(0)/G$ need not exist as a symplectic manifold. In either case J is said to be "singular."

For systems with singular momentum maps, then, reduction in the usual sense often cannot be carried out. Nonetheless, Śniatycki and Weinstein[3] have recently pointed out that it is still possible to "reduce" $J^{-1}(0)$, at least on the level of Poisson algebras. This generalized reduction procedure allows one to determine the effects of the singularities of J on the structure of the system as well as uncover certain dynamical features which would otherwise remain inaccessible. In particular, it identifies the gauge-invariant observables and equips them with the structure of a Poisson algebra. This is very useful when quantizing such a system.

Under sufficiently regular conditions, one may quantize a constrained system in two equivalent ways. The first is to quantize the extended phase space (X,ω) and then impose the constraints $J=0$ on the quantum wave functions; this ensures that the physically admissible states are gauge invariant.[4,5] Alternatively, one may quantize the reduced phase space $J^{-1}(0)/G$,[5,6] in which case gauge invariance is directly incorporated. When J is singular the latter technique is, of course, no longer applicable. But then the reduction procedure of Śniatycki and Weinstein enables one to do the next best thing, viz., to quantize the Poisson algebra of gauge-invariant observables.

Probably the simplest physically interesting example of a singular constrained system is that of a massless, spinless relativistic particle in $(n+1)$-dimensional Minkowski space-time, which we refer to as the "$n+1$ photon." The extended phase space is $\mathbb{R}^{2n+2}$ with coordinates $(\mathbf{p},p_t,\mathbf{x},t)$ and symplectic form

$$\omega = dp_t \wedge dt + \sum_{i=1}^{n} dp_i \wedge dx_i .$$

The gauge group is $\mathbb{R}$ with momentum map

$$J(\mathbf{p},p_t,\mathbf{x},t) = p_t^2 - \|\mathbf{p}\|^2.$$

Since the particle is massless, J must vanish. The constraint set is thus

$$J^{-1}(0) = C^n \times \mathbb{R}^{n+1},$$

where C^n is the null cone in $\mathbb{R}^{n+1}$. In this paper we reduce $J^{-1}(0)$ on the Poisson algebra level and then quantize, obtaining results which are in exact agreement with the quantization of the extended phase space $(\mathbb{R}^{2n+2},\omega)$ and the subsequent imposition of the constraint $J=0$.

This example serves three purposes: First, it illustrates the usefulness and essential correctness, at least in this instance, of the generalized reduction procedure. Second, it is simple enough that we can both identify and completely analyze the effects of the singularities in $J^{-1}(0)$ on this system. In this regard, our presentation seems to be the first which treats the singularities seriously (compare with standard discussions of the $3+1$ photon, e.g., that given in Ref. 7). Finally, Arms, Marsden, and Moncrief[8] have shown that singular momentum mappings typically have quadratic singularities so that $J^{-1}(0)$ is always a "cone." Since the $n+1$ photon is an elementary, and in some sense canonical, example of this phenomenon, its elucidation is essential for further progress in understanding the structure of singular constrained systems.

In the next section we briefly recall the basic features of the Śniatycki–Weinstein reduction procedure. The details for the $1+1$ photon are then worked out in Sec. III. The $n=1$ case is done separately, since it is rather "special" and technically much easier than the $n>1$ case, which is elaborated upon in Sec. IV. The physical interpretation of these results is discussed in the last section.

II. POISSON ALGEBRAS, REDUCTION AND QUANTIZATION

Let $\mathscr{F}$ be a commutative algebra over $\mathbb{R}$. If $[\cdot,\cdot]$ is a bracket operation on $\mathscr{F}$ such that (i) the pair $(\mathscr{F},[\cdot,\cdot])$ is a Lie algebra and (ii) the Leibniz rule

$$[f,f_1f_2] = [f,f_1]f_2 + [f,f_2]f_1$$

holds, then $(\mathscr{F},[\cdot,\cdot])$ is called a *Poisson algebra*. The basic example of a Poisson algebra is $C^\infty(X)$, where (X,ω) is symplectic and the Poisson bracket is given by

$$\{f,g\} = -\omega(\xi_f,\xi_g).$$

Here ξ_f, the Hamiltonian vector field of f, is defined via

$$i_{\xi_f}\omega = -df.$$

Now let (X,ω), G, and J be as in the Introduction. For each $a\in\mathfrak{g}$ define the function J_a on X by $J_a(x) = \langle J(x),a\rangle$, and denote by $\mathscr{J}$ the ideal (relative to the associative algebra structure) in $C^\infty(X)$ generated by the J_a. Since J is G-equivariant, the action of G on $C^\infty(X)$ induces an action of G on $C^\infty(X)/\mathscr{J}$ in such a way that the projection homomorphism j: $C^\infty(X)\to C^\infty(X)/\mathscr{J}$ is G-equivariant. Let $\mathscr{F}$ be the space of G-invariant elements of $C^\infty(X)/\mathscr{J}$, that is, the collection of all equivalence classes jf for which $j(\{f,\mathscr{J}\}) = 0$. Again by equivariance, the Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(X)$ descends to a bracket $[\cdot,\cdot]$ on $\mathscr{F}$ given by

$$[jf,jg] = j(\{f,g\}). \tag{2.1}$$

The pair $(\mathscr{F},[\cdot,\cdot])$ is the *reduced Poisson algebra* of the constrained system under consideration.

If 0 is a regular value of J, then $C^\infty(X)/\mathscr{J} = C^\infty(J^{-1}(0))$. Furthermore, if $J^{-1}(0)/G$ is a quotient manifold of $J^{-1}(0)$, then the reduced Poisson algebra $\mathscr{F}$ is canonically isomorphic to the Poisson algebra of the reduced symplectic space $J^{-1}(0)/G$. Under sufficiently regular conditions, then, this generalized reduction procedure is consistent with the Marsden–Weinstein technique, and we may therefore interpret $(\mathscr{F},[\cdot,\cdot])$ as the Poisson algebra of gauge-invariant observables. It is important to note, however, that in the singular case $\mathscr{F}$ need not be the Poisson algebra of any symplectic manifold nor must it be nondegenerate (in the sense that the only elements of $\mathscr{F}$ which Poisson commute with everything are "constant"[9]).

We close this section with some remarks concerning the quantization of a Poisson algebra $(\mathscr{F},[\cdot,\cdot])$. The problem is to construct the quantum state space from a knowledge of this Poisson algebra. This is fairly straightforward, using the techniques of geometric quantization theory,[7] when $\mathscr{F}$ is associated with a symplectic manifold. In the singular case it is necessary to proceed by analogy; briefly, this works as follows.[3]

Let $\Gamma = \mathscr{F}\otimes\mathbb{C}$ be the complexification of $\mathscr{F}$; elements $\sigma\in\Gamma$ are the algebraic counterparts of sections of the prequantization line bundle (which we take to be trivial). Given a derivation ξ of $\mathscr{F}$, we may compute the "covariant derivative" $\nabla_\xi\sigma$ of a section σ once a connection ∇ on Γ has been specified. A *polarization* $\mathscr{P}$ is a maximal commuting subalgebra of $(\mathscr{F},[\cdot,\cdot])$. A section $\sigma\in\Gamma$ is said to be "polarized" provided $\nabla_{\xi_f}\sigma = 0$ for all $f\in\mathscr{P}$, where ξ_f is the derivation $g\to[g,f]$ corresponding to the Hamiltonian vector field of f. The quantum state space relative to this data is then defined to be the set of all linear functionals on the space of polarized sections in Γ.

For our purposes we may choose a connection ∇ such that

$$\nabla_{\xi_f}\sigma = [\sigma,f]$$

for all $f\in\mathscr{P}$. Then the space of polarized sections in Γ is precisely $\mathscr{P}\otimes\mathbb{C}$, and the quantum wave functions are elements of the dual $(\mathscr{P}\otimes\mathbb{C})'$.

Turning now to the example, we compute the reduced Poisson algebra for the $n+1$ photon and quantize it.

III. THE 1 + 1 PHOTON

The analysis of the $n+1$ photon is considerably easier when $n=1$, for then the constraint $J=0$ factors. This circumstance simplifies the algebraic computations required for the construction of the reduced Poisson algebra as well as its presentation. This simplicity is also reflected in the structure of the constraint set $J^{-1}(0) = C^n\times\mathbb{R}^{n+1}$, which is essentially trivial when $n=1$.

We begin by changing to null coordinates

$$u = t - x,\quad v = t + x,$$

and their corresponding momenta

$$\mu = p_t - p_x,\quad \nu = p_t + p_x.$$

The symplectic form on $\mathbb{R}^4$ is then

$$\omega = \tfrac{1}{2}(d\mu\wedge du + d\nu\wedge dv)$$

and the momentum map becomes

$$J(\mu,\nu,u,v) = \mu\nu.$$

The ideal $\mathscr{J}$ of $C^\infty(\mathbb{R}^4)$ is thus generated by the product $\mu\nu$. Define j: $C^\infty(\mathbb{R}^4)\to C^\infty(\mathbb{R}^3)\times C^\infty(\mathbb{R}^3)$ by

$$jf = (f(\mu,0,u,v), f(0,\nu,u,v)). \tag{3.1}$$

Proposition 3.1: The quotient $C^\infty(\mathbb{R}^4)/\mathscr{J}$ may be identified with the image of $C^\infty(\mathbb{R}^4)$ in $C^\infty(\mathbb{R}^3)\times C^\infty(\mathbb{R}^3)$ under j.

Proof: If $f\in\mathscr{J}$, then clearly $jf=0$. On the other hand, suppose that $jf=0$. Then $f(\mu,0,u,v)=0$ which, by Hadamard's lemma, implies that f is divisible by ν. Thus $f=\nu h$ for some smooth h. Then $f(0,\nu,u,v)=0$ yields $h(0,\nu,u,v)=0$, which similarly implies that h is divisible by μ and so $f\in\mathscr{J}$. Thus ker $j=\mathscr{J}$ and the claim follows. Q.E.D.

Now $jf\in\mathscr{F}$ iff $j(\{f,J\})=0$. From (3.1) this will be the case iff

$$\frac{\partial f}{\partial v}(\mu,0,u,v) = 0 = \frac{\partial f}{\partial u}(0,\nu,u,v),$$

so that the invariant elements of $C^\infty(\mathbb{R}^4)/\mathscr{J}$ are of the form

$$(f(\mu,0,u,0), f(0,\nu,0,v))$$

with $f(0,0,u,v)$ constant. We may thus regard $\mathscr{F}$ as consisting of pairs of functions

$$(\psi(\mu,u),\ \phi(\nu,v))\in C^\infty(\mathbb{R}^2)\times C^\infty(\mathbb{R}^2)$$

subject to the *compatibility conditions*

$$\psi(0,u) = \phi(0,v)\quad (=\text{const}). \tag{3.2}$$

In these terms, a direct calculation shows that the induced Poisson bracket (2.1) on $\mathscr{F}$ is given by

$$[(\psi_1,\phi_1),(\psi_2,\phi_2)] = (2[\psi_1,\psi_2]_{u,\mu}, 2[\phi_1,\phi_2]_{v,\nu}), \qquad (3.3)$$

where

$$[\psi_1,\psi_2]_{u,\mu} = \frac{\partial\psi_1}{\partial u}\frac{\partial\psi_2}{\partial\mu} - \frac{\partial\psi_1}{\partial\mu}\frac{\partial\psi_2}{\partial u}$$

denotes the ordinary Poisson bracket with respect to the pair u,μ etc. It is straightforward to check that $[\cdot,\cdot]$ is nondegenerate.

In view of (3.3), the reduced Poisson algebra $\mathscr{F}$ is closely related to the Poisson algebra $C^\infty(\mathbb{R}^2)\times C^\infty(\mathbb{R}^2)$ of the symplectic manifold consisting of two disjoint copies of $\mathbb{R}^2$. Due to the compatibility conditions (3.2), however, $\mathscr{F}$ is strictly a subalgebra of this Poisson algebra, and so is not the Poisson algebra of any symplectic manifold. These conditions therefore express the influence of the singularities in $J^{-1}(0)$ upon the system. In fact, a correlation between these two Poisson algebras might have been expected from a consideration of the case when the photon has a mass m. Then the constraint set $J^{-1}(m^2)$ is nonsingular, but disconnected, and the reduced phase space is symplectomorphic to $\mathbb{R}^2\cup\mathbb{R}^2$. It follows that the reduced Poisson algebra for a massive particle is exactly $C^\infty(\mathbb{R}^2)\times C^\infty(\mathbb{R}^2)$. The effect of letting $m\to 0$ is thus to reduce the number of gauge-invariant observables. We shall have more to say about the physical interpretation of this phenomenon, and its relationship to the singular space $J^{-1}(0)/\mathbb{R}$, in Sec. V.

To construct the quantum state space, we must choose a polarization $\mathscr{P}$ of $\mathscr{F}$. Noting that the horizontal polarization P on $\mathbb{R}^4$ spanned by the vector fields ξ_μ and ξ_ν projects onto $J^{-1}(0)$, a natural choice for $\mathscr{P}$ is

$$\mathscr{P} = \{(\psi(\mu),\phi(\nu)) \mid \psi(0)=\phi(0)\}. \qquad (3.4)$$

According to general considerations, then, the quantum wave functions are elements of $(\mathscr{P}\otimes\mathbb{C})'$.

To represent these states, we need the following result: Consider $\mathbb{R}^2$ with coordinates μ and ν, and let $\hat{\mathscr{J}}$ be the ideal in $C^\infty(\mathbb{R}^2,\mathbb{C})$ generated by the product $\mu\nu$.

Lemma: $C^\infty(\mathbb{R}^2,\mathbb{C})/\hat{\mathscr{J}} = \mathscr{P}\otimes\mathbb{C}$.

Proof: Mimicking the proof of Proposition 3.1, we have that $C^\infty(\mathbb{R}^2)/\hat{\mathscr{J}}$ may be identified with the image of $C^\infty(\mathbb{R}^2)$ in $C^\infty(\mathbb{R})\times C^\infty(\mathbb{R})$ under the map $f\to(f(\mu,0),f(0,\nu))$. Comparison with (3.4) and complexification then yields the desired result. Q.E.D.

With this in hand, we now establish:

Proposition 3.2: $(\mathscr{P}\otimes\mathbb{C})'$ is isomorphic to the space of all complex-valued distributions Φ on $\mathbb{R}^2$ satisfying

$$\mu\nu\Phi = 0. \qquad (3.5)$$

Proof: Let Φ be such a distribution, in which case Φ annihilates all functions which are divisible by $\mu\nu$. Then Φ induces a linear functional $\hat{\Phi}$ on $C^\infty(\mathbb{R}^2,\mathbb{C})/\hat{\mathscr{J}}$ so that, by the Lemma, $\hat{\Phi}\in(\mathscr{P}\otimes\mathbb{C})'$. Conversely, every linear functional on $\mathscr{P}\otimes\mathbb{C} = C^\infty(\mathbb{R}^2,\mathbb{C})/\hat{\mathscr{J}}$ can be lifted to a distribution on $\mathbb{R}^2$ satisfying (3.5). Q.E.D.

These distributions Φ take the form

$$\Phi(\mu,\nu) = \lambda(\mu)\otimes\delta(\nu) + \delta(\mu)\otimes\chi(\nu),$$

where λ and χ are distributions on $\mathbb{R}$. Then for $f\in C^\infty(\mathbb{R}^2,\mathbb{C})$,

$$\langle\Phi,f\rangle = \langle\lambda(\mu),f(\mu,0)\rangle + \langle\chi(\nu),f(0,\nu)\rangle,$$

from which we obtain the explicit representation

$$\hat{\Phi}(\mu,\nu) = (\lambda(\mu),\chi(\nu))$$

of $\hat{\Phi}$ as a linear functional on $\mathscr{P}\otimes\mathbb{C}$.

Proposition 3.2 is the main result of this section. Not surprisingly, it shows that the gauge invariant wave functions must satisfy the $1+1$ wave equation, which is just the Fourier transform of (3.5). It also guarantees that this quantization is equivalent to that of the extended phase space $(\mathbb{R}^4,\omega)$. In fact, quantizing in the momentum representation defined by the polarization P, we find that the quantum Hilbert space is $L^2(\mathbb{R}^2)$ and that the quantum operator $\mathscr{Q}J$ corresponding to J is given by

$$\mathscr{Q}J[\Phi] = \mu\nu\Phi.$$

Thus, from this point of view as well, the physically admissible photon states must coincide with the distributional solutions of (3.5).

Finally, note the crucial role of the compatibility conditions (3.2), in the guise of (3.4), in Proposition 3.2. Without them (3.5) would not follow and the correlation with the wave equation would be lost.

IV. THE $n+1$ PHOTON

For the $1+1$ photon the constraint set consists simply of two intersecting hyperplanes in $\mathbb{R}^4$. This enabled us to compute directly on $J^{-1}(0)$; in effect, we worked on each of the two hyperplanes and then "glued" along their intersection by means of the compatibility conditions. For $n>1$, $J^{-1}(0)$ is more complicated and we can no longer proceed in this straightforward manner. In particular, it is now necessary to "resolve" the singularity.

Our first task is to construct the quotient $C^\infty(\mathbb{R}^{2n+2})/\mathscr{J}$. The following result is the higher-dimensional analog of Proposition 3.1. Let $f\in C^\infty(\mathbb{R}^{2n+2})$.

Proposition 4.1: $f\in\mathscr{J}$ iff $f|J^{-1}(0)=0$.

Proof: The obverse is apparent. For the converse, it is clear from the structure of the constraint set $J^{-1}(0)=C^n\times\mathbb{R}^{n+1}$ that the configuration variables $(\mathbf{x},t)$ are largely irrelevant and may accordingly be factored out. We are thus effectively reduced to proving that if $g:\mathbb{R}^{n+1}\to\mathbb{R}$ is such that $g|C^n=0$, then g is globally divisible by $p_t^2-\|\mathbf{p}\|^2$.

There is no problem off C^n. On either of the regular components of C^n, this follows from the inverse function theorem and Hadamard's lemma. It remains only to demonstrate that g is divisible by $p_t^2-\|\mathbf{p}\|^2$ at the vertex of C^n, and for this it suffices (Ref. 10, p. 72) to show that the formal Taylor series of g at the origin is divisible by $p_t^2-\|\mathbf{p}\|^2$. We now establish this for $n=2$; this case is prototypical, and the generalization to arbitrary n is immediate.

Thus let

$$T_0^r g = \sum_{i+j+k=r}\frac{1}{i!j!k!}g_{ij}^k p_x^i p_y^j p_t^k \qquad (4.1)$$

be the homogeneous part of the rth Taylor polynomial of g at the origin of $\mathbb{R}^3$, where

$$g_{ij}^k = \frac{\partial^{i+j+k}g}{\partial p_x^i\,\partial p_y^j\,\partial p_t^k}(0,0,0).$$

In (4.1) view all variables other than p_t as parameters. Then to say that $T_0^r g$ is divisible by $p_t^2-(p_x^2+p_y^2)$ is equivalent to requiring that both $p_t=\pm(p_x^2+p_y^2)^{1/2}$ be roots of $T_0^r g$. Substituting these values for p_t into (4.1), decomposing the sum into even and odd powers of $(p_x^2 + p_y^2)^{1/2}$, expanding these powers in a binomial series and reorganizing gives

$$\left(\sum_{m+n=r} a_{mn} p_x^m p_y^n\right) \pm (p_x^2 + p_y^2)^{1/2} \left(\sum_{m+n=r-1} b_{mn} p_x^m p_y^n\right), \tag{4.2}$$

where

$$a_{mn} = \sum_{l=0}^{[m/2]} \sum_{k=l}^{[n/2+l]} \binom{k}{l} \times \frac{1}{(m-2l)!(n-2k+2l)!(2k)!} g^{2k}_{m-2l,n-2k+2l}, \tag{4.3}$$

$$b_{mn} = \sum_{l=0}^{[m/2]} \sum_{k=l}^{[n/2+l]} \binom{k}{l} \times \frac{1}{(m-2l)!(n-2k+2l)!(2k+1)!} g^{2k+1}_{m-2l,n-2k+2l}, \tag{4.4}$$

and $[k]$ denotes the greatest integer less than or equal to k. From (4.2) it follows that $p_t = \pm (p_x^2 + p_y^2)^{1/2}$ will be roots of $T_0^r g$ iff the coefficients a_{mn} and b_{mn} vanish.

Now let $\mathbf{v}$ be a vector at the origin which points along a generator of the cone, and consider the rth derivative of g in the direction $\mathbf{v}$:

$$D_{\mathbf{v}}^r g(0,0,0) = \left[\left(\mathbf{v}_x \frac{\partial}{\partial p_x} + \mathbf{v}_y \frac{\partial}{\partial p_y} + \mathbf{v}_t \frac{\partial}{\partial p_t}\right)^r g\right](0,0,0).$$

Another lengthy calculation, consisting of expanding this expression out, separating into even and odd powers of $\mathbf{v}_t$, and then using the fact that $\mathbf{v}_t^2 = \mathbf{v}_x^2 + \mathbf{v}_y^2$, yields

$$D_{\mathbf{v}}^r g(0,0,0) = r!\left(\sum_{m+n=r} a_{mn} \mathbf{v}_x^m \mathbf{v}_y^n\right) \pm (r-1)!(\mathbf{v}_x^2 + \mathbf{v}_y^2)^{1/2}\left(\sum_{m+n=r-1} b_{mn} \mathbf{v}_x^m \mathbf{v}_y^n\right),$$

where a_{mn} and b_{mn} are given by (4.3) and (4.4), respectively. But by assumption $g|C^n = 0$ so that $D_{\mathbf{v}}^r g(0,0,0) = 0$ for all such $\mathbf{v}$. This implies that $a_{mn} = 0$ and $b_{mn} = 0$, and we are finished. Q.E.D.

This proposition shows that

$$C^\infty(\mathbb{R}^{2n+2})/\mathscr{J} = C^\infty(J^{-1}(0)),$$

the smooth functions on $J^{-1}(0)$ in the sense of Whitney.[11] Unfortunately, $C^\infty(J^{-1}(0))$ is rather difficult to handle. To obtain a more tractable representation of $C^\infty(\mathbb{R}^{2n+2})/\mathscr{J}$, we "resolve" the singularity by means of the map $\tilde{\phi}: \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ given by

$$\tilde{\phi}(\boldsymbol{\pi}, p_t, \mathbf{x}, t) = (p_t \boldsymbol{\pi}, p_t, \mathbf{x}, t).$$

Note that now the physical momenta are given by p_t and $\mathbf{p} = p_t \boldsymbol{\pi}$. If we define $K: \mathbb{R}^{2n+2} \to \mathbb{R}$ via

$$K(\boldsymbol{\pi}, p_t, \mathbf{x}, t) = 1 - ||\boldsymbol{\pi}||^2,$$

then $K^{-1}(0) = (S^{n-1} \times \mathbb{R}) \times \mathbb{R}^{n+1}$ and $\tilde{\phi}(K^{-1}(0)) = J^{-1}(0)$. Let ϕ be the restriction of $\tilde{\phi}$ to $K^{-1}(0)$. Note that ϕ is a local diffeomorphism away from the "equator" $p_t = 0$ and collapses the equator $(S^{n-1} \times \{0\}) \times \mathbb{R}^{n+1}$ onto the *singular set* $S = \{(\mathbf{0}, 0)\} \times \mathbb{R}^{n+1}$ in $J^{-1}(0)$.

We think of $K^{-1}(0)$ as being a "covering manifold" of the singular space $J^{-1}(0)$; using ϕ, we pull the entire formalism on $J^{-1}(0)$ back to $K^{-1}(0)$. The advantages of this procedure are (i) $K^{-1}(0)$ is a manifold and (ii) we can dispense with $C^\infty(J^{-1}(0))$ directly and work instead with its more manageable isomorph $\phi^* C^\infty(J^{-1}(0)) \subset C^\infty(K^{-1}(0))$. The key fact which makes this possible is that $\phi^* C^\infty(J^{-1}(0))$ admits a relatively simple characterization in $C^\infty(K^{-1}(0))$ in terms of formal Taylor series.[12]

Proposition 4.2: Let $F \in C^\infty(K^{-1}(0))$. Then $F \in \phi^* C^\infty(J^{-1}(0))$ iff for each $s \in S$ there exists a formal power series $\mathscr{f}_s$ at s such that

$$T_q F = \mathscr{f}_s \circ T_q \phi \tag{4.5}$$

for all $q \in \phi^{-1}(s)$.

Proof: Suppose that $F = f \circ \phi$ for some $f \in C^\infty(J^{-1}(0))$. Let $\tilde{f}$ be any extension of f to $\mathbb{R}^{2n+2}$; then $\mathscr{f}_s = T_s \tilde{f}$ will do in (4.5). The reverse implication follows from the inverse function theorem and Theorem 3.2 of Ref. 12. Q.E.D.

Note that (4.5) is a very strong condition: for a smooth function F on $K^{-1}(0)$ to lie in $\phi^* C^\infty(J^{-1}(0))$, it does *not* suffice for F simply to factor through ϕ. Rather, (4.5) requires that F *and* all its formal Taylor series $T_q F$ factor through ϕ.

In summary, we henceforth work on $K^{-1}(0)$ and identify

$$C^\infty(\mathbb{R}^{2n+2})/\mathscr{J} = \phi^* C^\infty(J^{-1}(0)).$$

From this standpoint, the conditions (4.5) reflect the presence of the singularities in $J^{-1}(0)$.[13] With these considerations out of the way, we are now ready to construct the reduced Poisson algebra.

Let $F \in \phi^* C^\infty(J^{-1}(0))$ so that there exists a smooth function $\tilde{f}$ on $\mathbb{R}^{2n+2}$ with $F = \tilde{f} \circ \phi$. Then F will be invariant provided $\{\tilde{f}, J\} \circ \phi = 0$ which, on $K^{-1}(0)$, translates into

$$\frac{\partial F}{\partial t} - \sum_{i=1}^n \pi_i \frac{\partial F}{\partial x_i} = 0.$$

Setting $\mathbf{w} = \mathbf{x} + \boldsymbol{\pi} t$, this implies that $F = F(\boldsymbol{\pi}, p_t, \mathbf{w})$ only. Since F must also factor through ϕ, it follows (with a slight abuse of notation) that

$$\mathscr{F} = \{F \in \phi^* C^\infty(J^{-1}(0)) | F = F(p_t \boldsymbol{\pi}, p_t, p_t \mathbf{w})\}. \tag{4.6}$$

Now if F and G are two elements of $\mathscr{F}$ with $F = \tilde{f} \circ \phi$ and $G = \tilde{g} \circ \phi$, then the induced Poisson bracket (2.1) on $\mathscr{F}$ is $[F,G] = \{\tilde{f}, \tilde{g}\} \circ \phi$. After making the coordinate change $(\boldsymbol{\pi}, p_t, \mathbf{x}, t) \mapsto (\boldsymbol{\pi}, p_t, \mathbf{w}, t)$ on $K^{-1}(0)$, a straightforward computation yields

$$[F,G] = \sum_{i=1}^n [F,G]_{w_i, p_t} \pi_i + \frac{1}{p_t} \sum_{i,j=1}^n [F,G]_{\omega_i, \pi_j} (\delta_{ij} - \pi_i \pi_j). \tag{4.7}$$

Although this expression would appear to be singular when $p_t = 0$, in fact it is not because of (4.6).

We show that (4.7) is nondegenerate. Indeed, suppose that $[F,G] = 0$ for all G in $\mathscr{F}$. Take $G = p_t w_k$. Then $[F, p_t w_k] = 0$ reduces to

$$\pi_k \left(p_t \frac{\partial F}{\partial p_t} - \sum_{i=1}^n \pi_i \frac{\partial F}{\partial \pi_i}\right) + \frac{\partial F}{\partial \pi_k} = 0.$$

Multiply this by π_k and sum; since $||\boldsymbol{\pi}||^2 = 1$, it follows that $\partial F / \partial p_t = 0$. But then, by (4.6), $F(p_t \boldsymbol{\pi}, p_t, p_t \mathbf{w}) = F(\mathbf{0}, 0, \mathbf{0})$ is constant and nondegeneracy is proven.

The quantization of the $n+1$ photon is patterned after that of the $1+1$ photon given in Sec. III. The analog of the horizontal polarization P on $\mathbb{R}^{2n+2}$ spanned by the vector fields ξ_{p_t} and ξ_{p_i}, $i = 1,...,n$, is the maximal commuting subalgebra

$$\mathscr{P} = \{F \in \mathscr{F} \mid F = F(p_t \pi, p_t)\} \tag{4.8}$$

of $\mathscr{F}$. We now construct the quantum state space $(\mathscr{P} \otimes \mathbb{C})'$.

Let $\hat{J}$ and $\hat{K}$ be the restrictions of J and K to the first factor of $\mathbb{R}^{n+1}$ in $\mathbb{R}^{2n+2}$, and denote by $\hat{\mathscr{J}}$ the ideal in $C^\infty(\mathbb{R}^{n+1})$ generated by $\hat{J}$. From the proof of Proposition 4.1 we see that

$$C^\infty(\mathbb{R}^{n+1})/\hat{\mathscr{J}} = C^\infty(C^n).$$

Letting $\hat{\phi}$ be the restriction of $\tilde{\phi}$ to $\hat{K}^{-1}(0)$, we may then identify $C^\infty(\mathbb{R}^{n+1})/\hat{\mathscr{J}}$ with the subalgebra $\hat{\phi}^* C^\infty(C^n)$ of $C^\infty(S^{n-1} \times \mathbb{R})$. From (4.8), (4.5), and the analog of Proposition 4.2 applied to $\hat{\phi}^* C^\infty(C^n) \subset C^\infty(S^{n-1} \times \mathbb{R})$, it follows that $\hat{\phi}^* C^\infty(C^n)$ is isomorphic to $\mathscr{P}$. Upon complexifying, we finally obtain

$$C^\infty(\mathbb{R}^{n+1}, \mathbb{C})/\hat{\mathscr{J}} = \mathscr{P} \otimes \mathbb{C}.$$

Imitating the proof of Proposition 3.2, this last result yields:

Proposition 4.3: $(\mathscr{P} \otimes \mathbb{C})'$ is isomorphic to the space of all complex-valued distributions Φ on $\mathbb{R}^{n+1}$ satisfying

$$(p_t^2 - \|\mathbf{p}\|^2)\Phi = 0.$$

Thus, as before, the physically admissible photon states must satisfy the Fourier transformed $n + 1$ wave equation. As expected, this is consistent with the quantization of the extended phase space $(\mathbb{R}^{2n+2}, \omega)$ in the polarization P. Indeed, we compute

$$\mathscr{Q}J[\Phi] = (p_t^2 - \|\mathbf{p}\|^2)\Phi$$

on $L^2(\mathbb{R}^{n+1})$ and gauge invariance demands $\mathscr{Q}J[\Phi] = 0$.

V. DISCUSSION

We spend a moment correlating our results with the structure of the singular reduced space $J^{-1}(0)/\mathbb{R}$. This will incidentally help clarify the physical significance of the compatibility conditions (3.2) and their higher-dimensional analogs (4.6) which arise both from the presence of singularities and the requirements of gauge invariance.

The action of the gauge group $\mathbb{R}$ on $\mathbb{R}^{2n+2}$ is given by

$$(\lambda; \mathbf{p}, p_t, \mathbf{x}, t) \mapsto (\mathbf{p}, p_t, \mathbf{x} - 2\lambda\mathbf{p}, t + 2\lambda p_t).$$

On $J^{-1}(0) = C^n \times \mathbb{R}^{n+1}$ this action fixes every point of the singular set S and is otherwise free. We may therefore schematically represent $J^{-1}(0)/\mathbb{R}$ as shown in Fig. 1. The trouble with $J^{-1}(0)/\mathbb{R}$, aside from the expected conical singularity, stems from the anomalous factor of $\mathbb{R}^{n+1}$ associated with the vertex. This is actually a remnant of a slight defect in the extended phase space description of the $n + 1$ photon concerning the physical interpretation of states in the singular set $S \subset J^{-1}(0)$. Such a state $(\mathbf{0}, 0, \mathbf{x}, t)$ represents a photon with vanishing momentum located at $(\mathbf{x}, t)$, that is, a vacuum state. But presumably there is only a *single* vacuum state, not one located at every space-time point. It is this $(n + 1)$-dimensional array of unphysical vacua which contributes to the pathology in $J^{-1}(0)/\mathbb{R}$ and prevents the latter from being construed as the space of all gauge-invariant states.

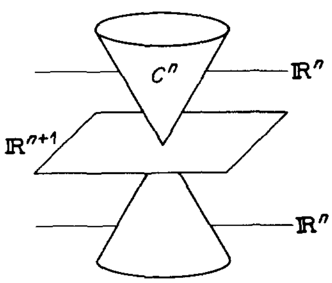

FIG. 1. The singular reduced space $J^{-1}(0)/\mathbb{R}$.

On the other hand, a physical observable should be unable to distinguish between these spurious vacua. The topology of the reduced space indicates that this will be the case: since $J^{-1}(0)/\mathbb{R}$ fails to be Hausdorff along this $\mathbb{R}^{n+1}$, continuous functions cannot separate these states. This observation is substantiated by our analysis above, and here is where both gauge invariance and the compatibility conditions enter. For $n = 1$, (3.2) guarantees that a physical observable is constant on S. Similarly, for $n > 1$, the form (4.6) of a gauge invariant function ensures that it is constant along the equator $\phi^{-1}(S)$ and hence also cannot differentiate between these states. Consequently, the generalized reduction process "corrects" the flaws in both the original description of the system and the reduced phase space, at least to the extent that it guarantees that the gauge invariant observables "detect" but a single vacuum state, as required.

Our analysis of the $n + 1$ photon thus demonstrates the utility of the Poisson algebra approach: even though a system may be singular, one can still construct the essential components of the reduced canonical formalism. Moreover, subsequent quantization yields results in exact correspondence with those obtained by standard methods. We hope that this example will encourage further study of the structure of singular constrained systems. Techniques for resolving singularities and, in particular, the work of Bierstone and Milman[12] on composite differentiable functions (of which Proposition 4.2 is a special case) should prove to be quite valuable in this regard.

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[9]Here is an example of a degenerate reduced Poisson algebra. Lift the irrational flow on the torus T^2 to T^*T^2 and let J be the usual momentum map for this cotangent action. Then $J^{-1}(0) \approx T^2 \times \mathbb{R}$, $\mathscr{F} \approx C^\infty(\mathbb{R})$, and $[\cdot,\cdot] = 0$. However, it is still possible that nondegeneracy will hold whenever the group action is proper.

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[11] That is, a function f on $J^{-1}(0)$ belongs to $C^\infty(J^{-1}(0))$ iff f extends to a smooth function on an open set in $\mathbf{R}^{2n+2}$ containing $J^{-1}(0)$.

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[13] Insofar as $\phi^* C^\infty(J^{-1}(0))$ is strictly a subspace of $C^\infty(K^{-1}(0))$. Note that our method of resolving the singularity (using the map $\tilde{\phi}$) yields the cylinder $K^{-1}(0)$ as the "nonsingular model" for the cone $J^{-1}(0)$ rather than a two-component hyperboloid as might be expected on physical grounds. In fact, it does not seem possible to resolve $J^{-1}(0)$ as $J^{-1}(m^2)$ for any mass m; this indicates that the $m \to 0$ limit is in some sense highly singular.