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REDUCED CANONICAL FORMALISM FOR A PARTICLE WITH ZERO ANGULAR MOMENTUM

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1. Introduction

Our aim is to construct the reduced canonical formalism for a nonrelativistic particle moving in $\mathbb{R}^n$ with fixed angular momentum J, where $J:\mathbb{R}^{2n} \to so(n)^*$ is the momentum map for the cotangent action of $SO(n)$ on the phase space $\mathbb{R}^{2n}$. If we set $J = l$, the admissible states for such a particle are constrained to lie on the level set $J^{-1}(l)$. When $l \neq 0$ this is a manifold and the reduced canonical formalism is given by the symplectic structure on the Marsden-Weinstein reduced phase space $J^{-1}(l)/SO(n)$. This case, which is well-understood, is discussed in [1].

Here, we concentrate on the critical case of *zero* angular momentum. Then J is "singular" in the sense that $J^{-1}(0)$ fails to be a manifold, so that the Marsden-Weinstein reduction procedure is no longer applicable. To construct the reduced canonical formalism it is now necessary to use the algebraic reduction technique of Sniatycki and Weinstein [2]. This yields a "reduced Poisson algebra" of $SO(n)$-invariant observables which contains all the essential components of the reduced canonical formalism.

In this report we compute the reduced Poisson algebra when $n = 2$. This case is algebraically and topologically much simpler than the higher-dimensional cases, due to the existence of "magic" coordinates which effectively trivialize everything. We briefly discuss the cases

$n > 2$ in the conclusion. We also correlate our results with the structure of the orbit space $J^{-1}(0)SO(2)$ which surprisingly turns out to be a symplectic V-manifold.

2. Structure of the Constraint Set $J^{-1}(0)$

On R^4 with coordinates (x, y, p_x, p_y) the angular momentum map J is

$$J = xp_y - yp_x \quad .$$

We first introduce coordinates

$$s = \frac{1}{2}(p_x - y) \quad , \qquad u = \frac{1}{2}(p_x + y)$$

$$t = \frac{1}{2}(p_y + z) \quad , \qquad v = \frac{1}{2}(p_y - x)$$

which diagonalize J:

$$J = s^2 + t^2 - u^2 - v^2 \quad . \tag{1}$$

Now view $R^4 = C^2$ via

$$\alpha = s + it \quad , \qquad \beta = u + iv \quad ;$$

then

$$J = |\alpha|^2 - |\beta|^2 \tag{2}$$

and the standard symplectic form on R^4 becomes

$$\Omega = i(d\alpha \wedge d\bar{\alpha} - d\beta \wedge d\bar{\beta}) \quad . \tag{3}$$

From (2) we see that $J^{-1}(0)$ is a 3-manifold everywhere except at the origin; it is also apparent that $J^{-1}(0)$ is a complex cone. If

z denotes an affine coordinate on CP^1, the equation $J = 0$ projects to $|z| = 1$ which defines a circle on CP^1. Thus, *$J^{-1}(0)$ is a complex cone over S^1.*

We may also describe $J^{-1}(0)$ as follows: Let S^3 be given in C^2 by

$$|\alpha|^2 + |\beta|^2 = r^2 \quad .$$

Since $|\alpha| = |\beta|$ on $J^{-1}(0)$, $J^{-1}(0) \cap S^3$ is determined by the equations $|\alpha| = r/\sqrt{2} = |\beta|$ and so

$$J^{-1}(0) \cap S^3 \approx T^2 \quad .$$

These results can be neatly combined by noting that the pullback of the Hopf fibration $S^1 \to S^3 \to CP^1$ to $S^1 \subset CP^1$ yields the trivial bundle

$$S^1 \to J^{-1}(0) \cap S^3 \approx T^2 \to S^1 \quad .$$

3. Resolution of the Singularity

We now apply the algebraic reduction procedure of Sniatycki and Weinstein (for generalities and examples, see [2] and [3]). Let J be the ideal in the ring $C^\infty(R^4)$ generated by J. Our first task is to construct the quotient $C^\infty(R^4)/J$ which — if $J^{-1}(0)$ were a manifold — would simply be $C^\infty(J^{-1}(0))$. Instead, we have the next best thing:

Theorem: $C^\infty(R^4)/J = W^\infty(J^{-1}(0))$, the smooth functions on $J^{-1}(0)$ in the sense of Whitney.

Proof: It suffices to show that $f \in J$ iff $f|J^{-1}(0) = 0$. The obverse is clear; for the converse, the inverse function theorem and Hadamard's Lemma imply that such an f is divisible by J everywhere except possibly at 0.

Now consider f near the origin. Using (1) and applying the Mather division theorem with distinguish variable s we have

$$f(s, t, u, v) = Jg(s, t, u, v) + sh(t, u, v) + k(t, u, v) \quad .$$

It is only necessary to demonstrate that the remainder R is divisible by J at the origin and for this it suffices [4, p.72] to show that its formal Taylor series T_0R at 0 vanishes. If (t, u, v) is such that $t^2 \leq u^2 + v^2$, then $s = \pm\sqrt{(u^2 + v^2 - t^2)}$ is well-defined and $(s, t, u, v) \in J^{-1}(0)$. Since $R(s, t, u, v) = 0$ for both values of s, a straightforward elimination yields $h(t, u, v) = 0 = k(t, u, v)$. Hence the supports of h and k are contained in the conical region $t^2 \geq u^2 + v^2$, so that their formal Taylor series vanish when $t^2 < u^2 + v^2$. But this along with continuity forces $T_0R = 0$. Q.E.D.

$W^\infty(J^{-1}(0))$ is a rather awkward space, so before proceeding with the reduction we obtain a more tractable representation of $C^\infty(\mathrm{R}^4)/\mathrm{J}$. This is accomplished by resolving the singularity in the constraint set. Since $J^{-1}(0)$ is a complex circular cone, we blow it up into a complex circular cylinder via the map $\phi: \mathrm{C}^2 \to \mathrm{C}^2$ given by

$$\phi(\omega, \xi) = (\omega, \omega\xi) \quad . \tag{4}$$

Now define $K: \mathrm{C}^2 \to \mathrm{R}$ via

$$K(\omega, \xi) = 1 - |\xi|^2 \quad .$$

Then $K^{-1}(0) = \mathrm{C} \times S^1$ is the blow-up of $J^{-1}(0)$. Indeed, $J \cdot \phi = |\omega|^2 K$ so that $\phi(K^{-1}(0)) = J^{-1}(0)$ and, moreover, if $\psi = \phi | K^{-1}(0)$, then ψ is a local diffeomorphism away from the equator $\{0\} \times S^1$ which it collapses onto the singularity. We replace $W^\infty(J^{-1}(0))$ with its more tractable isomorph $\psi^* W^\infty(J^{-1}(0)) \subset C^\infty(K^{-1}(0))$, so that finally

$$C^\infty(\mathrm{R}^4)/\mathrm{J} = \psi^* W^\infty(J^{-1}(0)) \quad . \tag{5}$$

4. The Reduced Poisson Algebra

The next step is to identify the subspace F of $C^\infty(\mathrm{R}^4)/\mathrm{J}$ consisting of $SO(2)$-invariant elements. These are classes jf

satisfying

$$j(\{f, J\}) = 0 \quad ,$$

where $j: C^\infty(\mathrm{R}^4) \to C^\infty(\mathrm{R}^4)/\mathrm{J}$ is the projection homomorphism and $\{\ ,\ \}$ is the usual Poisson bracket on R^4. F corresponds to the set of functions in $W^\infty(J^{-1}(0))$ which are constant along the orbits of $SO(2)$ on $J^{-1}(0)$.

Using the representation (5), on $K^{-1}(0)$ this invariance condition translates into

$$\omega \partial F/\partial\omega - \bar{\omega}\partial F/\partial\bar{\omega} = 0 \tag{6}$$

where $F = f\cdot\psi \in \psi^* W^\infty(J^{-1}(0))$. Referring to (4) we thus have that $F \in \psi^* W^\infty(J^{-1}(0))$ is invariant iff $F = F(|\omega|^2\xi, |\omega|^2, |\omega|^2\bar{\xi})$. However, any $F \in C^\infty(K^{-1}(0))$ of this form, since it satisfies the differential equation (6), must smoothly factor through ψ. Thus the requirement that F be in $\psi^* W^\infty(J^{-1}(0))$ is superfluous, and we conclude that

$$\mathrm{F} = \{F \in C^\infty(K^{-1}(0)) \mid F = F(|\omega|^2\xi, |\omega|^2, |\omega|^2\bar{\xi})\} \quad . \tag{7}$$

Since J is equivalent, the Poisson bracket $\{\ ,\ \}$ on $C^\infty(\mathrm{R}^4)$ descends to a bracket $[\ ,\ ]$ on F given by

$$[jf, jg] = j(\{f, g\}) \quad .$$

Using (7) and (4), and recalling that $|\xi| = 1$ on $K^{-1}(0)$, we compute

$$[F, G] = i\left(\frac{\xi}{\omega}\{F, G\}_{\bar{\omega}, \xi} - \frac{\bar{\xi}}{\bar{\omega}}\{F, G\}_{\omega, \bar{\xi}}\right) \tag{8}$$

where $F = f\cdot\psi$ etc., and $\{F, G\}_{\bar{\omega},\xi}$ denotes the ordinary Poisson bracket with respect to the pair $\bar{\omega}$, ξ etc. A straightforward check shows that $[\ ,\ ]$ is nondegenerate.

The pair $(F, [\ ,\ ])$ is the *reduced Poisson algebra* of $SO(2)$-invariant observables for our particle with zero angular momentum. From this structure one may recover the entire reduced canonical formalism, albeit on the algebraic level (i.e., invariant *observables*) rather than that of manifolds (i.e., invariant *states*).

5. The V-Symplectic Orbit Space $J^{-1}(0)/SO(2)$

Despite the fact that $J^{-1}(0)$ is singular, $J^{-1}(0)/SO(2)$ turns out to be a symplectic V-manifold. In this section we prove this assertion and in the next we compare the Poisson algebra of the orbit space $J^{-1}(0)/SO(2)$ with the reduced Poisson algebra of §4.

$$(\alpha, \beta) \to e^{i\theta}(\alpha, \beta) \quad .$$

Consider the map $q: C \to C^2$ given by

$$q(\lambda) = (\lambda, \bar{\lambda}) \quad . \tag{9}$$

From (2) we see that q maps into $J^{-1}(0)$. Furthermore, for every $SO(2)$-orbit $O \subset J^{-1}(0)$ there exists λ such that $q(\lambda) \in O$: if $(re^{i\gamma}, re^{i\delta}) \in O$, then $\lambda = re^{i(\gamma-\delta)/2}$ satisfies

$$q(\lambda) = \exp(-\frac{i}{2}(\gamma+\delta))(re^{i\gamma}, re^{i\delta}) \in O \quad .$$

Note also that $q(\lambda)$ and $q(\eta)$ lies on the same orbit iff $\lambda = \pm\eta$. From all this it follows that q induces an isomorphism

$$J^{-1}(0)/SO(2) \approx C/Z_2 \quad ,$$

where C/Z_2 denotes the identification $\lambda \sim -\lambda$.

Now use q to pull the symplectic form (3) on C^2 back to

$$q^*\Omega = 2i\, d\lambda \wedge d\bar{\lambda} \quad .$$

Since $q^*\Omega$ is invariant under the map $\lambda \to -\lambda$, it projects to a singular symplectic structure $\hat{\Omega}$ on C/Z_2. If follows that the orbit space $J^{-1}(0)/SO(2)$ is actually a symplectic V-manifold [5].

The singular Poisson algebra of $J^{-1}(0)/SO(2)$ can be described as follows: Every $\hat{f} \in C^\infty(C/Z_2)$ may be uniquely represented by an even function $f \in C^\infty(C)$. According to [6, p.144], such a function must be quadratic in λ and $\bar{\lambda}$. Thus,

$$C^\infty(J^{-1}(0)/SO(2)) \approx \{f \in C^\infty(C) \,|\, f = f(\bar{\lambda}^2, \lambda^2)\} \quad . \tag{10}$$

Since $\hat{\Omega}$ is represented by $q^*\Omega$, the Poisson bracket of two functions $\hat{f}$, $\hat{g}$ in $C^\infty(C/Z_2)$ is represented by

$$\frac{i}{2}\{f, g\}_{\lambda, \bar{\lambda}} \quad . \tag{11}$$

Note, however, that $C^\infty(J^{-1}(0)/SO(2))$ is *not* closed under this bracket; this is a reflection of the fact that $\hat{\Omega}$ is singular.

6. Comparison of Algebraic and Group-Theoretical Reductions

To compare the singular Poisson algebra of $(J^{-1}(0)/SO(2), \hat{\Omega})$ with (F, [,]), we first define

$$\begin{aligned} \mathrm{f} &= \{f \in C^\infty(C) \,| \\ f &= f(\lambda^2, |\bar{\lambda}|^2, \lambda^2)\} \quad . \end{aligned} \tag{12}$$

Now use q to lift elements of $C^\infty(J^{-1}(0)/SO(2))$ to $SO(2)$-invariant functions on C^2; from (9) we may take

$$\lambda^2 = q^*(\alpha\bar{\beta}),\ |\lambda|^2 = q^*(|\alpha|^2),\ \bar{\lambda}^2 = q^*(\bar{\alpha}\beta) \quad . \tag{13}$$

Comparing (12) with (7) via (13) and (4) yields an isomorphism $\mathrm{F} \approx \mathrm{f}$ given by

$$F(|\omega|^2\xi, |\omega|^2, |\omega|^2\bar{\xi}) \sim f(\bar{\lambda}^2, |\lambda|^2, \lambda^2) \quad . \tag{14}$$

If we equip f with the Poisson bracket (11) it becomes a regular Poisson algebra. Then computing the Poisson brackets (8) and (11) while taking into account the functional forms (7) and (12), respectively, and applying the isomorphism (14), we find that

$$[F, G] \sim \frac{i}{2}\{f, g\}_{\lambda,\bar{\lambda}} \quad .$$

It follows that the Poisson algebras $(\mathbf{f}, i/2\{\ ,\ \}_{\lambda,\bar{\lambda}})$ and $(\mathrm{F}, [\ ,\ ])$ are isomorphic.

According to (10), however, $C^\infty(J^{-1}(0), SO(2))$ is *strictly* a subspace of $\mathbf{f}$, so that $(C^\infty(J^{-1}(0)/SO(2), i/2\{\ ,\ \}_{\lambda,\bar{\lambda}})$ can be identified with a *singular* subalgebra of $(\mathrm{F}, [\ ,\ ])$. It seems very likely that the reduced Poisson algebra $(\mathrm{F}, [\ ,\ ])$ is in fact the *closure* of the singular Poisson algebra $(C^\infty(J^{-1}(0)/SO(2), i/2\{\ ,\ \}_{\lambda,\bar{\lambda}})$.

Thus, for a planar particle with zero angular momentum, one may construct the reduced canonical formalism in either of two ways: algebraically or group-theoretically, giving nearly isomorphic results. The group theoretical reduction yields a *singular* Poisson algebra $(C^\infty(J^{-1}(0)/SO(2), i/2\{\ ,\ \}_{\lambda,\bar{\lambda}})$, which the algebraic reduction procedure apparently "repairs"; by closing it into the *regular* Poisson algebra $(\mathrm{F}, [\ ,\ ])$ of all $SO(2)$ invariant observables for our particle.

7. Concluding Remarks

The $n = 2$ case is relatively easy because J can be diagonalized. In higher dimensions this is no longer possible and the analysis is correspondingly more complicated. Regardless, the results for $n > 2$, which we now briefly describe, are quite similar to those obtained above.

For arbitrary n, $J^{-1}(0) \subset \mathbb{C}^n$ may be viewed as a complex cone over $\mathbb{R}P^{n-1} \subset \mathbb{C}P^{n-1}$ with $J^{-1}(0) \cap S^{2n-1} \approx (S^1 \times S^{n-1})/Z_2$. The characterization theorem of §3 still holds, but requires entirely different techniques for its proof. The blow-up of $J^{-1}(0)$ is the pullback of the universal line bundle over CP^{n-1} to RP^{n-1}, with total space $(C \times S^{n-1})/Z_2$, and is now *non*trivial. The reduced Poisson algebra (F, [,]) is calculated along the above lines and remains nondegenerate. As in the $n = 2$ case, $J^{-1}(0)/SO(n)$ is V-symplectomorphic to $(C/Z_2, \hat{\Omega})$ for all n, and we conjecture that the closure of its singular Poisson algebra is naturally isomorphic to (F, [,]). These results will appear in a forthcoming paper [7].

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