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A Class of Non-Polarizable Symplectic Manifolds

By

Mark J. Gotay, Annapolis, Maryland

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Abstract. A class of compact 4-dimensional symplectic manifolds which admit no polarizations whatsoever is presented. These spaces also provide examples of nonparallelizable manifolds which are symplectic but have no complex, and hence no Kähler, structures.

I. Introduction. The most important element in the geometric quantization of a symplectic manifold is the choice of polarization [9]. Although polarizations usually abound in practice, relatively little is known regarding their existence in general. What hard data one has typically concerns either specific types of polarizations or those which satisfy certain regularity conditions.

For instance, VAISMAN [7] has derived topological obstructions to the existence of "nice" polarizations with non-zero real indices. A nontrivial example is the symplectic product manifold $S^2 \times S^2$ which appears in the Kepler problem: SIMMS [6] has shown that it has no such polarizations. On the other hand, a symplectic manifold carries totally complex (resp. Kähler) polarizations iff it admits compatible complex (resp. Kähler) structures [8]. Thus the bundles E^4 of [2] with $b_1(E^4) = 2$ or 3 have no Kähler polarizations. It should be remarked that almost all symplectic manifolds carry totally complex polarizations. It was not until just recently that examples which admit no polarizations of this type were discovered (these are the bundles E^4 with $b_1(E^4) = 2$).

Despite these negative results, all these symplectic manifolds seem to have at least *one* polarization. In particular, $S^2 \times S^2$ obviously has a Kähler polarization, and it is not hard to see that the E^4 often have real polarizations. (Actually, those E^4 with $b_1(E^4) = 3$ sometimes have totally complex polarizations as well, cf. [2].) Regardless, I know of no example in the literature of a symplectic manifold which cannot be polarized in some fashion.

Here I present *a class of compact 4-dimensional symplectic manifolds* $\tilde{E}^4_k$ *which do not admit polarizations of any type whatsoever.* These spaces are obtained by repeatedly blowing up the bundles E^4 with $b_1(E^4) = 2$. Note that no regularity conditions are imposed. Furthermore, the $\tilde{E}^4_k$ are apparently the first known examples of *non-parallelizable* manifolds which are symplectic but have no complex, and hence no Kähler, structures.

These results have significant implications for conventional geometric quantization theory, since the $\tilde{E}^4_k$ would represent (in principle) the phase spaces of classical systems which have no quantum analogues. Clearly the concept of polarization needs to be broadened to cover such cases.

II. Polarizations [8, 9]. Let (X, ω) be a $2n$-dimensional symplectic manifold. A *polarization* of (X, ω) is a Nirenberg integrable subbundle P of the complexified tangent bundle $T^{\mathbb{C}}X$ which is Lagrangian with respect to $\omega^{\mathbb{C}}$. This means that:

(i) P is an involutive rank n complex subbundle of $T^{\mathbb{C}}X$;

(ii) $\omega^{\mathbb{C}} | (P \times P) = 0$;

(iii) the involutive real distribution L defined by $L^{\mathbb{C}} = P \cap \bar{P}$ has constant dimension; and

(iv) the real distribution K defined by $K^{\mathbb{C}} = P + \bar{P}$ is involutive.

The dimension l of L is the *real index* of P.

I now collect a few facts about particular polarizations which will be useful later.

When $l = n$, $P = \bar{P}$ then P is said to be *real.* In this case $L = K = P \cap TX$. Now, the symplectic structure on X determines a homotopy class of almost complex structures $[J]$ on X. Using any $J \in [J]$, WEINSTEIN [8, § 2] shows that there is a Lagrangian splitting

$$TX = L \oplus JL$$

so that (TX, J), viewed as a complex vector bundle, may be identified with the complexification of L. It follows that the odd real Chern classes of (TX, J) vanish.

At the other extreme, a polarization for which $l = 0$ is called *totally complex*. Then $P \cap \bar{P} = \{0\}$ and $K = TX$. Such a P determines an almost complex structure J on X satisfying $\omega(Ju, Jv) = \omega(u, v)$ for all tangent vectors u and v (again see [8, § 2]). Since P is Nirenberg integrable this J is actually a *complex* structure on X. Together ω and J define a J-hermitian metric $\langle\ ,\ \rangle$ on X according to

$$\langle u, v\rangle = \omega(u, Jv) .$$

If $\langle\ ,\ \rangle$ is positive definite then $(X, J, \langle\ ,\ \rangle)$ is a Kähler manifold, and P is said to be *Kähler*.

III. The Spaces $\tilde{E}^4_k$ [2]. I first recall some facts about the manifolds E^4 of [2] with $b_1(E^4) = 2$. These spaces are nontrivial circle bundles over nontrivial circle bundles over a 2-torus. They are compact symplectic 4-manifolds. Moreover they are parallelizable, whence $b_2(E^4) = 2$, and have signature zero. The key feature of these spaces is that they all have non-vanishing Massey products.

Now blow up these E^4 at k distinct points using the technique of Gromov and McDuff (cf. [5]). The resulting spaces $\tilde{E}^4_k$ are then compact 4-manifolds diffeomorphic to $E^4 \# k\overline{\mathbb{C}P^2}$, where $\overline{\mathbb{C}P^2}$ denotes $\mathbb{C}P^2$ with the reversed orientation. The $\tilde{E}^4_k$ thus have signature $\sigma(\tilde{E}^4_k) = -k$ and Betti numbers $b_1(\tilde{E}^4_k) = 2$ and $b_2(\tilde{E}^4_k) = 2 + k$, so their Euler characteristics are $\chi(\tilde{E}^4_k) = k$.

Proposition. *The manifolds $\tilde{E}^4_k$ have symplectic structures but no complex structures.*

Proof. That the $\tilde{E}^4_k$ are symplectic follows from Proposition 3.7 of [5]. Now, since the E^4 have non-vanishing Massey products, the $\tilde{E}^4_k$ do also. Thus the minimal models of the $\tilde{E}^4_k$ are not formal, and the main result of [1] then implies that the $\tilde{E}^4_k$ cannot be Kählerian. Now suppose the $\tilde{E}^4_k$ had complex structures. Since their first Betti numbers are even, Theorem 25 of [3] would imply that each $\tilde{E}^4_k$ is a deformation of, and hence diffeomorphic to, an algebraic surface. But the $\tilde{E}^4_k$ would then be Kählerian, which is impossible. □

IV. Nonexistence of Polarizations. Here is the main result:

Theorem. *The symplectic manifolds $\tilde{E}^4_k$, $k > 0$, cannot be polarized.*

Proof. There are three cases to consider, depending upon the value of the real index l, $0 \leqslant l \leqslant 2$.

$l = 0$: The $\tilde{E}_k^4$ cannot carry any totally complex polarizations because, according to the proposition, they have no complex structures (cf. Section II).

$l = 1$: In this case L would define a field of line elements on $\tilde{E}_k^4$. But this is impossible since $\chi(\tilde{E}_k^4) \neq 0$.

$l = 2$: When P is real the first real Chern class of $(T\tilde{E}_k^4, J)$ must vanish, as noted in Section II. But by the Ehresmann-Wu theorem (cf. [4]), $c_1(T\tilde{E}_k^4, J)$ must satisfy

$$c_1^2(T\tilde{E}_k^4, J) = 3\sigma(\tilde{E}_k^4) + 2\chi(\tilde{E}_k^4) = -k$$

which is a contradiction. □

Remark. A similar analysis along with the facts that $\sigma(S^2 \times S^2) = 0$ and $\chi(S^2 \times S^2) = 4$ shows that $S^2 \times S^2$ cannot admit *any* polarizations with $l \neq 0$, nice or not.

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M. J. GOTAY
Mathematics Department
United States Naval Academy
Annapolis, MD 21402, U.S.A.