Obstructions to Quantization

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Abstract

Quantization is not a straightforward proposition, as demonstrated by Groenewold's and Van Hove's discovery, more than fifty years ago, of an "obstruction" to quantization. Their "no-go theorems" assert that it is in principle impossible to consistently quantize every classical polynomial observable on the phase space $\mathbf{R}^{2n}$ in a physically meaningful way. Similar obstructions have been recently found for S^2 and T^*S^1, buttressing the common belief that no-go theorems should hold in some generality. Surprisingly, this is not so—it has just been proven that there are no obstructions to quantizing either T^2 or $T^*\mathbf{R}_+$.

In this paper we work towards delineating the circumstances under which such obstructions will appear, and understanding the mechanisms which produce them. Our objectives are to conjecture—and in some cases prove—generalized Groenewold-Van Hove theorems, and to determine the maximal Lie subalgebras of observables which can be consistently quantized. This requires a study of the structure of Poisson algebras of symplectic manifolds and their representations. To these ends we include an exposition of both prequantization (in an extended sense) and quantization theory, here formulated in terms of "basic algebras of observables." We then review in detail the known results for $\mathbf{R}^{2n}$, S^2, T^*S^1, T^2, and $T^*\mathbf{R}_+$, as well as recent theoretical work. Our discussion is independent of any particular method of quantization; we concentrate on the structural aspects of quantization theory which are common to all Hilbert space-based quantization techniques.

1 Introduction

Quantization—the problem of constructing the quantum formulation of a system from its classical description—has always been one of the great mysteries of mathematical

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physics. It is generally acknowledged that quantization is an ill-defined procedure which cannot be consistently applied to all classical systems. While there is certainly no extant quantization procedure which works well in all circumstances, this assertion nonetheless bears closer scrutiny.

Already from first principles one encounters difficulties. Given that the classical description of a system is an approximation to its quantum description, obtained in a macroscopic limit (when $\hbar \to 0$), one expects that some information is lost in the limit. So quantization should somehow have to compensate for this. But how can a given quantization procedure select, from amongst the myriad of quantum theories all of which have the same classical limit, the physically correct one?

In view of this ambiguity it is not surprising that the many quantization schemes which have been developed over the years—such as the physicists' original "canonical quantization" [Di] (and its modern formulations, such as geometric quantization [Ki, So, Wo]), Weyl quantization [Fo] (and its successor deformation quantization [BFFLS, Ri2, Ri3, Ri4]), path integral quantization [GJ], and the group theoretic approach to quantization [AA, Is], to cite just some—have shortcomings. Rather, is it amazing that they work as well as they do!

But there are deeper, subtler problems, involving the Poisson algebras of classical systems and their representations. In this context the conventional wisdom is that it is impossible to "fully" quantize any given classical system—regardless of the particular method employed—in a way which is consistent with the physicists' Schrödinger quantization of $\mathbf{R}^{2n}$. (We will make this somewhat nebulous statement precise later.) In other words, the assertion is that there exists a universal "obstruction" which forces one to settle for something less than a complete and consistent quantization of *any* system. Each quantization procedure listed above evinces this defect in various examples.

That there are problems in quantizing even simple systems was observed very early on. One difficulty was to identify the analogue of the multiplicative structure of the classical observables in the quantum formalism. For instance, consider the quantization of $\mathbf{R}^{2n}$ with canonical coordinates $\{q^i, p_i \mid i = 1, \dots, n\}$, representing the phase space of a particle moving in $\mathbf{R}^n$. For simple observables the "product $\to$ anti-commutator" rule worked well. But for more complicated observables (say, ones which are quartic polynomials in the positions and momenta), this rule leads to inconsistencies. (See [AB, §4], [Fo, §1.1] and §§5.1 and 6.5 for discussions of these factor-ordering ambiguities.) Of course this, in and by itself, might only indicate the necessity of coming up with some subtler symmetrization rule. But attempts to construct a quantization map also conflicted with Dirac's "Poisson bracket $\to$ commutator" rule. This was implicitly acknowledged by Dirac [Di, p. 87], where he made the now famous hedge:

> *"The strong analogy between the quantum P.B. . . . and the classical P.B. . . . leads us to make the assumption that the quantum P.B.s,* or at any rate the simpler ones of them, *have the same values as the corresponding classical P.B.s."*

In any case, as a practical matter, one was forced to limit the quantization to relatively "small" Lie subalgebras of classical observables which could be handled without ambiguity (e.g., polynomials which are at most quadratic in the p_i and the q^i, or observables which are affine functions of the positions or of the momenta).

Then, in 1946, Groenewold [Gro] showed that the search for an "acceptable" quantization map was futile. His "no-go" theorem states that one cannot consistently quantize the Poisson algebra of all polynomials in the q^i and p_i on $\mathbf{R}^{2n}$ as symmetric operators on some Hilbert space, subject to the requirement that the q^i and p_i be irreducibly represented. Van Hove subsequently refined Groenewold's result [VH1]. Thus it is *in principle* impossible to quantize—by *any* means—every classical observable on $\mathbf{R}^{2n}$, or even every polynomial observable, in a way consistent with Schrödinger quantization (which, according to the Stone-Von Neumann theorem, is the import of the irreducibility requirement on the p_i and q^i). At most, one can consistently quantize certain Lie subalgebras of observables, for instance the ones mentioned in the preceding paragraph.

Of course, Groenewold's remarkable result is valid only for the classical phase space $\mathbf{R}^{2n}$. The immediate problem is to determine whether similar obstructions appear when trying to quantize other symplectic manifolds. Little was known in this regard, and only in the mid 1990s have other examples come to light. A few years ago an obstruction was found for S^2, representing the (internal) phase space of a massive spinning particle [GGH]. It was shown that one cannot consistently quantize the Poisson algebra of spherical harmonics (thought of as polynomials in the components S_i of the spin angular momentum vector $\mathbf{S}$), subject to the requirement that the S_i be irreducibly represented. This is a direct analogue for S^2 of Groenewold's theorem. Moreover, just recently it was shown that the symplectic cylinder T^*S^1, which plays a role in geometric optics, exhibits a similar obstruction [GGru1]. Combined with the observations that S^2 is in a sense at the opposite extreme from $\mathbf{R}^{2n}$ insofar as symplectic manifolds go, and that T^*S^1 lies somewhere in between, these results indicate that no-go theorems can be expected to hold in some generality. But, interestingly enough, they are *not* universal: It is possible to explicitly construct a quantization of the full Poisson algebra of the torus T^2 in which a suitable irreducibility requirement is imposed [Go3]. It is also possible to quantize certain noncompact phase spaces, e.g. $T^*\mathbf{R}_+$ [GGra1]. An important point, therefore, is to understand the mechanisms which are responsible for these divergent outcomes.

Our goal here is to study obstructions to the quantization of the Poisson algebra of a symplectic manifold. We will review the known examples in some detail, and give a careful presentation of prequantization (in an extended sense) and quantization, with a view to conjecturing a generalized Groenewold-Van Hove theorem and in particular delineating the circumstances under which it can be expected to hold. Already some results have been established along these lines, to the effect that under certain circumstances there are obstructions to quantizing both compact and noncompact symplectic manifolds [GGG, GGra1, GGru2, GM]. Despite these recent advances, many interesting and difficult problems remain. Our discussion will be independent of any particular method of quantization; we concentrate on the structural aspects of quantization theory which are common to all Hilbert space-based quantization techniques.

The present paper is a revised and updated version of the review article "Obstruction Results in Quantization Theory," which was published in 1996 in the *Journal of Nonlinear Science* [GGT]. Since a number of new results and examples have been obtained since that article appeared, we thought it useful to provide a more current summary of the field. As well, a number of the concepts and constructions of that paper have evolved over time, and we have amended the paper accordingly. We have also taken this opportunity to correct a number of misprints and minor errors.

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2 Prequantization

Let (M, ω) be a fixed $2n$-dimensional connected symplectic manifold with associated Poisson algebra $\big(C^\infty(M), \{\cdot\,,\cdot\}\big)$, where $\{\cdot\,,\cdot\}$ is the Poisson bracket.

To start the discussion, we state what it means to "prequantize" a Lie algebra of observables.

Definition 1 Let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$. A *prequantization* of $\mathcal{O}$ is a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the linear space $\mathrm{Op}(D)$ of symmetric operators which preserve a fixed dense domain D in some separable Hilbert space $\mathcal{H}$, such that for all $f, g \in \mathcal{O}$,

(Q1) $\mathcal{Q}(\{f, g\}) = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$,

(Q2) if the constant function $1 \in \mathcal{O}$, then $\mathcal{Q}(1) = I$, and

(Q3) if the Hamiltonian vector field X_f of f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D.

If $\mathcal{O} = C^\infty(M)$, the prequantization is said to be *full*. A prequantization $\mathcal{Q}$ is *nontrivial* provided $\operatorname{codim}\ \ker \mathcal{Q} > 1$; otherwise $\mathcal{Q}$ factors through a representation of $\mathcal{O}/\ker \mathcal{Q}$ with $\dim(\mathcal{O}/\ker \mathcal{Q}) \le 1$.

Remarks. 1. By virtue of (Q1) a prequantization $\mathcal{Q}$ of $\mathcal{O}$ is essentially a Lie representation of $\mathcal{O}$ by symmetric operators. (More precisely: If we set $\pi(f) = \frac{i}{\hbar}\,\mathcal{Q}(f)$, then π is a true Lie representation by skew-symmetric operators on D equipped with the commutator bracket. We will blur the distinction between π and $\mathcal{Q}$.) In this context there are several additional requirements we could place upon $\mathcal{Q}$, such as irreducibility and integrability. However, we do not want to be too selective at this point, so we do not insist on these; they will be discussed as the occasion warrants.

2. Condition (Q2) reflects the fact that if an observable f is a constant c, then the probability of measuring $f = c$ is one regardless of which quantum state the system is in. It also serves to eliminate some "trivial" possibilities, such as the regular representation $f \mapsto -i\hbar X_f$ on $L^2(M, \omega^n)$.

3. Regarding (Q3), we remark that in contradistinction with Van Hove [VH1], we do not confine our considerations to only those classical observables whose Hamiltonian vector fields are complete. Rather than taking the point of view that "incomplete" classical observables cannot be quantized, we simply do not demand that the corresponding quantum operators be essentially self-adjoint ("e.s.a."). We do not imply by

this that symmetric operators which are not e.s.a. are acceptable as physical observables; as is well known, this is a controversial point.

4. Notice that no assumptions are made at this stage regarding the multiplicative structure on $C^\infty(M)$ vis-à-vis $\mathcal{Q}$. This is partly for historical reasons: In classical mechanics the Lie algebra structure has played a more dominant role than the associative algebra structure, so it is natural to concentrate on the former. This is also the approach favored by Dirac [Di] and the geometric quantization theorists [So, Wo]. For more algebraic treatments, see [As, Em, VN]. The associative algebra structure is emphasized to a much greater degree in deformation quantization theory [BFFLS, Ri2, Ri3]. We shall make some comments on this as we go along; see especially §§5.1 and 6.5.

Prequantizations in this broad sense (even full ones) are usually easy to construct, cf. [Ch3, Ur, Wo]. Van Hove was the first to construct a full prequantization of $C^\infty(\mathbf{R}^{2n})$ [VH1]. It goes as follows: The Hilbert space $\mathcal{H}$ is $L^2(\mathbf{R}^{2n})$, the domain D is the Schwartz space $\mathcal{S}(\mathbf{R}^{2n},\mathbf{C})$ of rapidly decreasing smooth complex-valued functions (for instance), and for $f \in C^\infty(\mathbf{R}^{2n})$,

$$\mathcal{Q}(f) = -i\hbar \sum_{k=1}^{n} \left[\frac{\partial f}{\partial p_k} \left(\frac{\partial}{\partial q^k} - \frac{i}{\hbar}\, p_k \right) - \frac{\partial f}{\partial q^k} \frac{\partial}{\partial p_k} \right] + f. \tag{1}$$

As luck would have it, however, prequantization representations tend to be flawed. For example, the Van Hove prequantization of $C^\infty(\mathbf{R}^{2n})$, when restricted to the Heisenberg subalgebra $\mathrm{h}(2n) \cong \mathrm{span}\{1,\, p_i,\, q^i \,|\, i = 1, \ldots, n\}$, is not unitarily equivalent to the Schrödinger representation (which it ought to be, in the context of a particle moving in $\mathbf{R}^n$ with no superselection rules) [Bl1, Ch1]. (Recall that the *Schrödinger representation* of $\mathrm{h}(2n)$ is defined to be[1]

$$q^i \mapsto q^i, \quad p_j \mapsto -i\hbar\, \partial/\partial q^j, \quad \text{and} \quad 1 \mapsto I \tag{2}$$

on the domain $\mathcal{S}(\mathbf{R}^n,\mathbf{C}) \subset L^2(\mathbf{R}^n)$. It is irreducible in the sense given in §4.) There are various ways to see this; we give Van Hove's original proof [VH1, §17] as it will be useful later. Take $n = 1$ for simplicity. First, define a unitary operator F on $L^2(\mathbf{R}^2)$ by

$$(F\psi)(p,q) = \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} e^{ipv/\hbar} \psi(v, q - v)\, dv.$$

Then for each fixed $j = 0, 1, \ldots$ take $\mathcal{H}_j$ to be the closure in $L^2(\mathbf{R}^2)$ of the linear span of elements of the form Fh_{jk}, where $h_{jk}(p,q) = h_j(p)h_k(q)$ and

$$h_k(q) = e^{q^2/2} \frac{d^k}{dq^k} e^{-q^2} \tag{3}$$

is the Hermite function of degree k. Now from (1),

$$\mathcal{Q}(q) = i\hbar \frac{\partial}{\partial p} + q, \quad \mathcal{Q}(p) = -i\hbar \frac{\partial}{\partial q}.$$

[1] We denote multiplication operators as functions.

These operators are e.s.a. on $\mathcal{S}(\mathbf{R}^2,\mathbf{C})$, and one may verify that they strongly commute with the orthogonal projectors onto the closed subspaces $\mathcal{H}_j$.[2] Thus the Van Hove prequantization of $C^\infty(\mathbf{R}^2)$ is reducible when restricted to the Heisenberg subalgebra and hence does not produce the Schrödinger representation. Moreover the association $Fh_{jk}(p,q) \mapsto c_j h_k(q)$, where the c_j are normalization constants, provides a unitary equivalence of each subrepresentation of h(2) on $\mathcal{H}_j$ with the Schrödinger representation on $L^2(\mathbf{R})$, from which we see that the multiplicity of the latter is infinite in the Van Hove representation. The Van Hove representation suffers from other defects as well [Zi, §4.5.B].

Likewise, the Kostant-Souriau prequantizations of S^2 do not reproduce the familiar spin representations of the special unitary algebra su(2). We realize S^2 as a coadjoint orbit in su(2)* according to $\mathbf{S}\cdot\mathbf{S} = s^2$, where $\mathbf{S} = (S_1, S_2, S_3)$ is the spin vector and $s > 0$ is the classical spin. It comes equipped with the symplectic form

$$\omega = \frac{1}{2s^2}\sum_{i,j,k=1}^{3} \epsilon_{ijk}\, S_i\, dS_j \wedge dS_k. \tag{4}$$

Now the de Rham class $[\omega/h]$ is integral iff $s = \frac{n}{2}\hbar$, where n is a positive integer, and the corresponding Kostant-Souriau prequantum line bundles can be shown to be $L^{\otimes n}$ where L is the dual of the universal line bundle over S^2 [Ki]. The corresponding prequantum Hilbert spaces $\mathcal{H}_n$ can thus be identified with spaces of square integrable sections ψ of these bundles w.r.t. the inner product

$$\langle \psi, \phi\rangle = \frac{i}{2\pi}\int_{\mathbf{C}} \frac{\overline{\psi(z)}\phi(z)\, dz \wedge d\bar{z}}{(1+z\bar{z})^{n+2}}$$

where $z = (S_1 + iS_2)/(s - S_3)$, cf. [Wo]. But these $\mathcal{H}_n$ are infinite-dimensional, whereas the standard representation spaces for quantum spin $s = \frac{n}{2}\hbar$ have dimension $n+1$.

In both examples the prequantization Hilbert spaces are "too big." The main problem is how to remedy this, in other words, how to modify the notion of a prequantization so as to yield a genuine *quantization*.

It is here that the ideas start to diverge, because there is less agreement in the literature as to what constitutes a quantization map. Some versions define it as a prequantization, not necessarily defined on the whole of $C^\infty(M)$, which is irreducible on a "basic set" $\mathfrak{b} \subset C^\infty(M)$ [Ki]. This is in line with the group theoretical approach to quantization [Is], in which context $\mathfrak{b}$ is realized as the Lie algebra of a symmetry group;[3] quantization should then yield an irreducible representation of this algebra. For example, when $M = \mathbf{R}^{2n}$, one usually takes $\mathfrak{b}$ to be the Heisenberg algebra h$(2n) \cong$ span$\{1, p_i, q^i \mid i = 1, \dots, n\}$ of polynomials of degree at most one. Similarly, when $M = S^2$, one takes for $\mathfrak{b}$ the special unitary algebra su(2) $\cong$ span$\{S_1, S_2, S_3\}$ of spherical harmonics of degree one. We will plumb in detail the rationale behind these choices of $\mathfrak{b}$ in the next section.

[2] Recall that two e.s.a. (or, more generally, normal) operators *strongly commute* iff their spectral resolutions commute, cf. [ReSi, §VIII.5]. Two operators A, B *weakly commute* on a domain D if they commute in the ordinary sense, i.e., $[A, B]$ is defined on D and vanishes.

[3] We typically identify an abstract Lie algebra with its isomorph in $C^\infty(M)$.

A different approach to quantization is to require a prequantization $\mathcal{Q}$ to satisfy some "Von Neumann rule," that is, some given relation between the classical multiplicative structure of $C^\infty(M)$ and operator multiplication on $\mathcal{H}$. (Note that thus far in our discussion the multiplication on $C^\infty(M)$ has been ignored, and it is reasonable to require that quantization preserve at least some of the associative algebra structure of $C^\infty(M)$, given that the Leibniz rule intertwines pointwise multiplication with the Poisson bracket.) There are many different types of such rules [Co, Fo, KLZ, KS, Ku, MC, VN], the simplest being of the form

$$\mathcal{Q}(\varphi \circ f) = \varphi\big(\mathcal{Q}(f)\big) \tag{5}$$

for some distinguished observables $f \in C^\infty(M)$, and certain smooth functions $\varphi \in C^\infty(\mathbf{R})$. (Technically, if φ is not a polynomial, then $\mathcal{Q}(f)$ must be e.s.a. for $\varphi\big(\mathcal{Q}(f)\big)$ to be defined.) In the case of $M = \mathbf{R}^{2n}$, Von Neumann states that the physical interpretation of the quantum theory requires (5) to hold for *all* $f \in C^\infty(M)$ and $\varphi \in C^\infty(\mathbf{R})$ [VN]. However, it is easy to see that this is impossible (simple demonstrations are given in [AB, Fo] as well as §5.1 following); hence the qualifiers in the definition above. In this example, one typically ends up imposing the squaring Von Neumann rule $\varphi(x) = x^2$ on elements of h$(2n)$. The relevant rule for the sphere turns out to be less intuitive; it takes the form $\mathcal{Q}({S_i}^2) = a\mathcal{Q}(S_i)^2 + cI$ for $i = 1,\ 2,\ 3$, where a, c are undetermined (representation-dependent) constants subject only to the constraint that $a^2 + c^2 \neq 0$. Derivations of these rules in these two examples are given in §5 and [GGH].

Another type of quantization is obtained by "polarizing" a prequantization representation [Wo]. Following Blattner [Bl1], we paraphrase it algebraically as follows. Start with a *polarization*, i.e., a maximally commuting Lie subalgebra $\mathcal{A}$ of $C^\infty(M)$. Then require for the quantization map $\mathcal{Q}$ that the image $\mathcal{Q}(\mathcal{A})$ be "maximally commuting" as operators. (If $\mathcal{Q}(\mathcal{A})$ consists of bounded operators, this means that the weak operator closure of the *-algebra generated by $\mathcal{Q}(\mathcal{A})$ $(= \mathcal{Q}(\mathcal{A})'')$ is maximally commuting in $B(\mathcal{H})$. If $\mathcal{Q}(\mathcal{A})$ contains unbounded operators, one should look for a generating set of normal operators in $\mathcal{Q}(\mathcal{A})$, and require that the Von Neumann algebra generated by their spectral projections is maximally commuting.) One can then realize the Hilbert space $\mathcal{H}$ as an L^2-space over the spectrum of this Von Neumann algebra on which this algebra acts as multiplication operators. There will also be a cyclic and separating vector for such an algebra, which provides a suitable candidate for a vacuum vector. Thus another motivation for polarizations is that a maximally commuting set of observables provides a set of compatible measurements, which can determine the state of a system. When $M = \mathbf{R}^{2n}$, one often takes the "vertical" polarization $\mathcal{A} = \big\{f(q^1, \dots, q^n)\big\}$, in which case one recovers the standard position or coordinate representation. However, in some instances, such as S^2, it is useful to broaden the notion of polarization to that of a maximally commuting subalgebra of the complexified Poisson algebra $C^\infty(M,\mathbf{C})$. Then, thinking of S^2 as $\mathbf{C}P^1$, we may take the "antiholomorphic" polarization $\mathcal{A} = \{f(z)\}$, which leads to the usual representations for spin. For treatments of polarizations in the context of deformation quantization, see [Fr, He].

Thus, informally, a "quantization" could be defined as a prequantization which incorporates one (or more) of the three additional requirements above (or possibly even others). Before proceeding, however, there are two points we would like to make.

The first is that it is of course not enough to simply state the requirements that a quantization map should satisfy; one must also devise methods for implementing them in practice. Thus geometric quantization theory, for example, provides a specific technique for polarizing certain (Kostant-Souriau) prequantization representations [Bl1, Ki, So, Wo]. However, as we are interested here in the structural aspects of quantization theory, and not in specific quantization schemes, we do not attempt to find such implementations.

Second, these three approaches to a quantization map are not independent; there exist subtle connections between them which are not well understood. For instance, demanding that a prequantization be irreducible on some basic algebra typically leads to the appearance of Von Neumann rules; this is how the Von Neumann rules for $\mathbf{R}^{2n}$ and S^2 mentioned above arise. We will delineate these connections in specific cases in §5, and more generally in §7.

At the core of each of the approaches above is the imposition—in some guise—of an irreducibility requirement, which is used to "cut down" a prequantization representation. Since this is most apparent in the first approach, we will henceforth concentrate on it. We will tie in the two remaining approaches as we go along.

So let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$, and suppose that $\mathfrak{b} \subset \mathcal{O}$ is a "basic algebra" of observables. Provisionally, we take a *quantization* of the pair $(\mathcal{O}, \mathfrak{b})$ to mean a prequantization $\mathcal{Q}$ of $\mathcal{O}$ which (among other things) irreducibly represents $\mathfrak{b}$. In the next two sections we will make this more precise, as well as examine in detail the criteria that $\mathfrak{b}$ should satisfy.

Natural issues to address for quantizations are existence, uniqueness and classification, and functoriality. For *pre*quantizations these questions already have partial answers from geometric quantization theory. So in particular we know that if (M, ω) satisfies the integrality condition $[\omega/h] \in H^2(M, \mathbf{Z})$, then full prequantizations of the Poisson algebra $C^\infty(M)$ exist, and that certain types of these—the Kostant-Souriau prequantizations—can be classified cohomologically [Ur, Wo]. For some limited types of manifolds the functorial properties of these prequantizations were considered by Blattner [Bl1]. However, as there are prequantizations not of the Kostant-Souriau type [Av3, Ch3], these questions are still open in general (especially for manifolds which violate the integrality condition [We]).

For quantization maps these questions are far more problematic. Our main focus will be on the existence of both *full quantizations*, by which we mean a quantization of $(C^\infty(M), \mathfrak{b})$ for some appropriately chosen basic algebra $\mathfrak{b}$, and *polynomial quantizations*, by which we mean a quantization of $(P(\mathfrak{b}), \mathfrak{b})$, where $P(\mathfrak{b})$ is the Poisson algebra of polynomials generated by $\mathfrak{b}$. As indicated earlier, these are not completely understood in general, although substantial progress has been made in the past several years. In our terminology, the classical result of Groenewold states that there is no quantization of $\big(P(\mathrm{h}(2n)), \mathrm{h}(2n)\big)$ on $\mathbf{R}^{2n}$, while the more recent results of [GGH] and [GGru1] imply the same for $\big(P(\mathrm{su}(2)), \mathrm{su}(2)\big)$ on S^2 and $\big(P(\mathrm{e}(2)), \mathrm{e}(2)\big)$ on T^*S^1, respectively, where e(2) is the Euclidean algebra (cf. §5.3). On the other hand, nontrivial polynomial quantizations do exist: One can construct such a quantization of $T^*\mathbf{R}_+$ with the affine algebra a(1) [GGra1]. In fact, full quantizations exist as well; there is one of T^2 with $\mathfrak{b}$ the Lie algebra of trigonometric polynomials of mean zero [Go3]. However, it does seem that nonexistence results are the rule. In the absence of a full (resp. a polynomial) quantization, then, it is important to determine the largest Lie subalgebras $\mathcal{O}$ of $C^\infty(M)$ (resp. $P(\mathfrak{b})$) for which $(\mathcal{O}, \mathfrak{b})$ can be quantized. This we will investigate for $\mathbf{R}^{2n}$, S^2, and T^*S^1 in §5. At present, questions of uniqueness and classification can only be answered in specific examples.

3 Basic Algebras of Observables

Our first goal here is to make clear what we mean by a basic algebra of observables $\mathfrak{b} \subset C^\infty(M)$. Such algebras, in one way or another, play an important role in many quantization methods, such as geometric quantization [Ki], deformation quantization [BFFLS, Fr] and also the group theoretic approach [Is].

We start with the most straightforward case, that of an "elementary system" in the terminology of Souriau [So, Wo]. This means that M is a homogeneous space for a Hamiltonian action of a finite-dimensional Lie group G. The appeal of an elementary system is that it is a classical version of an irreducible representation: Using the transitive action of G, one can obtain any classical state from any other one, in direct analogy with the fact that every nonzero vector in a Hilbert space $\mathcal{H}$ is cyclic for an irreducible unitary representation ("IUR") of G on $\mathcal{H}$ [BaRa, §5.4]. Now notice that the span $\mathfrak{j}$ of the components of the associated (equivariant) momentum map satisfies:

(J1) $\mathfrak{j}$ is a finite-dimensional Lie subalgebra of $C^\infty(M)$,

(J2) the Hamiltonian vector fields X_f, $f \in \mathfrak{j}$, are complete, and

(J3) $\{X_f \mid f \in \mathfrak{j}\}$ spans TM.

For both $M = \mathbf{R}^{2n}$ and S^2, the basic algebras are precisely of this type: From the elementary systems of the Heisenberg group H$(2n)$ acting on $\mathbf{R}^{2n}$, and the special unitary group SU(2) acting on S^2, we have for $\mathfrak{j}$ the spaces $\text{span}\{1, p_i, q^i \mid i = 1, \dots, n\}$ and $\text{span}\{S_1, S_2, S_3\}$, respectively. The same is true for $M = T^*S^1$ and $T^*\mathbf{R}_+$, as explained in §5.

Property (J3) is just an infinitesimal restatement of transitivity, and so we call a subset of $C^\infty(M)$ *transitive* if it satisfies this condition. Kirillov [Ki] uses the terminology "complete," motivated by the fact that such a set of observables locally separates classical states. (If a set of observables *globally* separates classical states, we call it *separating*.) In this regard, the finite-dimensionality criterion in (J1) plays an important role operationally: It guarantees that a finite number of measurements using this collection of observables will suffice to distinguish any two nearby states.

A Lie subalgebra $\mathfrak{b} \subset C^\infty(M)$ satisfying (J1)–(J3) is a prototypic basic algebra. However, there need not exist basic algebras in this sense for arbitrary M. For instance, if $M = T^2$, the self-action of the torus is not Hamiltonian, so there is no momentum map. Thinking of T^2 as $\mathbf{R}^2/\mathbf{Z}^2$, a natural choice of basic algebra is then the Lie algebra $\mathfrak{t}$ generated by

$$\mathcal{T} = \{\sin 2\pi x, \cos 2\pi x, \sin 2\pi y, \cos 2\pi y\}.$$

This Lie algebra—viz. the trigonometric polynomials of mean zero—is infinite-dimensional. While perhaps unpleasant, this is in fact unavoidable: It follows from Proposition 2 below that there is no finite-dimensional basic algebra on T^2. However, in keeping with the discussion above, note that $\mathfrak{t}$ is finitely generated, and one can use this generating set to separate states.

We will therefore dispense with the finite-dimensionality assumption, and instead merely require that $\mathfrak{b}$ be finitely generated. One then still has a finite number of "basic observables" with which to distinguish states. Thus we make:

Definition 2 A *basic algebra of observables* $\mathfrak{b}$ is a Lie subalgebra of $C^\infty(M)$ such that:[4]

(B1) $\mathfrak{b}$ is finitely generated,

(B2) the Hamiltonian vector fields X_f, $f \in \mathfrak{b}$, are complete,

(B3) $\mathfrak{b}$ is transitive and separating, and

(B4) $\mathfrak{b}$ is a minimal Lie algebra satisfying these requirements.

We spend some time elaborating on this definition. First, the completeness condition (B2) guarantees that a basic observable generates a one-parameter group of canonical transformations. In view of (Q3), it is the classical analogue of the requirement that an operator representing a physically observable quantity should be e.s.a., whence it generates a one-parameter group of unitary transformations.

Next consider the transitivity requirement in (B3). When $\mathfrak{b}$ is finite-dimensional, it together with (B2) enables us to integrate $\mathfrak{b}$ to a transitive group action on M. Indeed, the map $f \mapsto X_f$ can be thought of as an action of $\mathfrak{b}$ on M. By (B2) the vector fields X_f are complete and so by a theorem of Palais [Va, Thm. 2.16.13] this action of $\mathfrak{b}$ can be integrated to an action of the corresponding simply connected Lie group B. Condition (B3) implies that this action is locally transitive and thus globally transitive as M is connected.

As part of (B3) we also require that $\mathfrak{b}$ globally separate classical states. This ensures that $\mathfrak{b}$ accurately reflects the topology of M [Ve]. Without it, e.g., the Lie algebra $\mathfrak{t}$ defined above could equally well live on either $\mathbf{R}^2$ or T^2 (or even "halfway between," on T^*S^1); measurements using $\mathfrak{t}$ could not distinguish amongst these phase spaces.

Although a transitive set of observables is locally separating, it need not be (globally) separating. Conversely, a separating set of observables need not be everywhere transitive. So these two conditions are distinct.

While (B3) is geometrically natural, there are other conditions one might use in place of it. By way of motivation, consider a unitary representation U of a Lie group G on a Hilbert space $\mathcal{H}$. The representation U is irreducible iff the *-algebra $\mathcal{U}$ of bounded operators generated by $\{U(g) \mid g \in G\}$ is irreducible, in which case we have the following equivalent characterizations of irreducibility:

[4] This definition differs from that given in [GGT] in three regards: It is phrased in terms of basic *algebras* as opposed to basic *sets*, we no longer insist that $1 \in \mathfrak{b}$ (this is superfluous), and we have strengthened (B3) by requiring that $\mathfrak{b}$ be separating.

(I1) The commutant $\mathcal{U}' = \mathbf{C}I$, and

(I2) the weak operator closure of $\mathcal{U}$ is the algebra of all bounded operators: $\overline{\mathcal{U}}^w = B(\mathcal{H}) \quad (= \mathcal{U}'')$.

That (I1) is equivalent to irreducibility is the content of Schur's Lemma. Property (I2) means that all bounded operators can be built from those in $\mathcal{U}$ by weak operator limits. It follows from (I1), the Von Neumann density theorem [BrRo, Cor. 2.4.15], and the fact that $\mathcal{U}' = \big(\overline{\mathcal{U}}^w\big)'$. Clearly (I2) implies (I1).

These restatements of irreducibility have the following classical analogues for a set $\mathcal{F}$ of observables:

(C1) $\{f, g\} = 0$ for all $f \in \mathcal{F}$ implies g is constant, and

(C2) the Poisson algebra of polynomials generated by $\mathcal{F}$ forms a dense subspace in $C^\infty(M)$.

For (C2) a topology on $C^\infty(M)$ must be decided on, and we will use the topology of uniform convergence on compacta of a function as well as its derivatives.

Because the algebraic structures of classical and quantum mechanics are different, (C1) and (C2) lead to inequivalent notions of "classical irreducibility." It is not difficult to verify that (C1) $\Leftarrow$ (B3) $\Leftarrow$ (C2) strictly. In principle either of (C1) or (C2) could serve in place of (B3). Indeed, since on $C^\infty(M)$ one has two algebraic operations, it is natural to consider irreducibility in either context: in terms of the multiplicative structure (C2), or the Poisson bracket (C1). However, it turns out that (C1) is too weak for our purposes, while (C2) is too strong.

The nondegeneracy condition (C1) is equivalent to the statement that observables in $\mathfrak{b}$ locally separate states almost everywhere [Ki]. It is also equivalent to the statement that the Hamiltonian vector fields of elements of $\mathfrak{b}$ span the tangent spaces to M almost everywhere. Consequently, it will not do to replace (B3) by (C1) in the definition of basic algebra, for then as shown below the Lie algebra $\mathfrak{t}$ on T^2 would no longer be minimal, which seems both awkward and unreasonable. Furthermore, unlike (B3), (C1) has the defect that the simply connected covering group of $\mathfrak{b}$ need not act transitively on M. This happens for the symplectic algebra sp(2, $\mathbf{R}$) $\cong$ span$\{p^2, pq, q^2\}$ on $\mathbf{R}^2$. Condition (C2) fails for the affine algebra a(1) $\cong$ span$\{pq, q^2\}$ on $T^*\mathbf{R}_+$ since, e.g., $C^\infty(T^*\mathbf{R}_+)$ contains functions which blow up as $q \to 0$ along with all their q-derivatives, and such functions cannot be approximated by polynomials in the elements of a(1). On the other hand, all these examples satisfy (B3), which shows that this is a reasonable condition to impose.

Finally, the minimality condition (B4) is crucial. From a physical or operational point of view, it is not obvious that it is necessary, as long as $\mathfrak{b}$ is finitely generated. But the quantization of a pair $(\mathcal{O}, \mathfrak{b})$ with $\mathfrak{b}$ nonminimal in this sense can lead to physically incorrect results.

Here is an illustration. First observe that the extended symplectic group HSp($2n$,$\mathbf{R}$) (which is the semidirect product of the symplectic group Sp($2n$,$\mathbf{R}$) with the Heisenberg group H($2n$)) acts transitively on $\mathbf{R}^{2n}$. This action has a momentum map whose components consist of all inhomogeneous quadratic polynomials in the q^i and p_i. The corresponding Lie subalgebra $\mathfrak{j} \cong \mathrm{hsp}(2n, \mathbf{R})$ satisfies all the requirements for a basic algebra save minimality, since $\mathrm{h}(2n)$ is a separating transitive subalgebra of $\mathrm{hsp}(2n, \mathbf{R})$. Now consider again the Van Hove prequantization $\mathcal{Q}$ of $C^\infty(\mathbf{R}^{2n})$ for $n = 1$. In [VH1, §17] it is shown that $\mathcal{Q}$ is completely reducible when restricted to $\mathfrak{j}$. In fact, there exist exactly two nontrivial HSp(2,**R**)-invariant closed subspaces $\mathcal{H}_\pm$ in $L^2(\mathbf{R}^2)$, namely

$$\mathcal{H}_+ = \bigoplus_{j \text{ even}} \mathcal{H}_j \quad \text{and} \quad \mathcal{H}_- = \bigoplus_{j \text{ odd}} \mathcal{H}_j,$$

cf. §2. If we denote the corresponding subrepresentations of $\mathfrak{j}$ on $\mathcal{S}(\mathbf{R}^2, \mathbf{C}) \cap \mathcal{H}_\pm$ by $\mathcal{Q}_\pm$, then it follows that $\mathcal{Q}_\pm$ are quantizations of the pair $(\mathfrak{j}, \mathfrak{j})$. But these quantizations are physically unacceptable, since—just like the full prequantization $\mathcal{Q}$—they are reducible when further restricted to h(2). On the one hand, asking for a quantization of $(\mathfrak{j}, \mathfrak{j})$ in this context is clearly the wrong thing to do, since compatibility with Schrödinger quantization devolves upon the irreducibility of an h(2) algebra, not an $\mathrm{hsp}(2, \mathbf{R})$ one. But on the other hand, this example does make our point.

To illustrate the appropriateness of Definition 2, consider again the torus and let $\mathfrak{t}_k$ be the Lie algebras generated by the sets

$$\mathcal{T}_k = \{\sin 2\pi kx,\ \cos 2\pi kx,\ \sin 2\pi ky,\ \cos 2\pi ky\}$$

for $k = 1, 2, \ldots$ Each $\mathfrak{t}_k$ is transitive. But without the separation axiom in (B3), none of the $\mathfrak{t}_k$ would be minimal, since each contains the infinite descending series $\mathfrak{t}_k \supset \mathfrak{t}_{2k} \supset \cdots$. However, only $\mathfrak{t} = \mathfrak{t}_1$ is separating, and in fact it is a minimal *separating* transitive subalgebra.

Other properties that basic algebras might be required to satisfy are discussed in [Is]. For our purposes, (B1)–(B4) will suffice.

It is difficult to characterize basic algebras on general symplectic manifolds. In the compact case, however, we can be quite precise.

Proposition 1 *Let* $\mathfrak{b}$ *be a finite-dimensional basic algebra on a compact symplectic manifold. Then* $\mathfrak{b}$ *is compact and semisimple. In particular, its center must be zero.*

Proof. Define an inner product on $\mathfrak{b}$ according to

$$\langle f, g \rangle = \int_M fg\ \omega^n.$$

Using the identity

$$\{f, g\}\,\omega^n = n\, d(f\, dg \wedge \omega^{n-1}) \tag{6}$$

together with Stokes' Theorem, we immediately verify that

$$\langle \{f, g\}, h \rangle + \langle g, \{f, h\} \rangle = 0$$

whence $\mathfrak{b}$ is compact [On, §1.2.6]. As a consequence, $\mathfrak{b}$ splits as the Lie algebra direct sum $\mathfrak{z} \oplus \mathfrak{s}$, where $\mathfrak{z}$ is the center of $\mathfrak{b}$ and $\mathfrak{s}$ is semisimple [On, Prop. 1.2.8].

Now transitivity implies that any function which Poisson commutes with every element of $\mathfrak{b}$ must be a constant, so that $\mathfrak{z} \subseteq \mathbf{R}$. But if equality holds then $\mathfrak{s}$ would be a separating transitive subalgebra, thereby violating (B4). Thus $\mathfrak{z} = \{0\}$ and $\mathfrak{b}$ is semisimple. □

In particular, the proof shows that any reductive (and consequently any compact) basic algebra must be semisimple.

There is no guarantee that a given symplectic manifold will carry a basic algebra. Indeed, the next proposition shows that those phase spaces which admit basic algebras form a quite restricted class.

Proposition 2 *If a connected symplectic manifold M admits a finite-dimensional basic algebra $\mathfrak{b}$, then M is a coadjoint orbit in $\mathfrak{b}^*$. In particular, when M is compact it must be simply connected.*

Proof. For if $\mathfrak{b}$ is a finite-dimensional basic algebra on M then, by our considerations above, M must be a homogeneous Hamiltonian B-space, where B is the simply connected covering group of $\mathfrak{b}$ and the momentum map J is given by $\langle J(m), b\rangle = b(m)$ for $b \in \mathfrak{b}$ and $m \in M$. The Kirillov-Kostant-Souriau Coadjoint Orbit Covering Theorem [MR, Thm. 14.6.5] then implies that $J : M \to \mathfrak{b}^*$ is a symplectic local diffeomorphism of M onto a coadjoint orbit $O \subset \mathfrak{b}^*$. Since by (B3) $\mathfrak{b}$ is separating, it follows that J is injective (for otherwise elements of $\mathfrak{b}$ cannot separate points in $J^{-1}(\mu)$ for $\mu \in O$.) Thus M is symplectomorphic to O.

If M is compact, then by Proposition 1 $\mathfrak{b}$ is compact and semisimple. We conclude that B is compact [On, p. 29]. But the coadjoint orbits of a compact connected Lie group are simply connected [Fi, Thm. 2.3.7]. □

As M is a homogeneous space for B, the last paragraph of this proof shows that M is compact iff $\mathfrak{b}$ is compact iff B is compact.

Thus the symplectic algebra sp(2,**R**) $\cong$ span$\{p^2, pq, q^2\}$ is not a basic algebra on $\mathbf{R}^2 \setminus \{\mathbf{0}\}$, since the latter is not a coadjoint orbit (but rather a double covering of one). (Note that sp(2,**R**) satisfies all the criteria for a basic algebra save the separation axiom.) Even if $M \subset \mathfrak{g}^*$ is a coadjoint orbit, $\mathfrak{g}$ need *not* form a basic algebra on M. An example is provided by S^2, which is a coadjoint orbit in $\mathrm{su}(2)^* \oplus \{0\} \subset \mathrm{su}(2)^* \oplus \mathrm{su}(2)^* \cong \mathrm{o}(4)^*$ with basic algebra su(2), not o(4). In the compact case we can be more explicit:

Proposition 3 *Let M be a coadjoint orbit in $\mathfrak{g}^*$, where $\mathfrak{g}$ is a compact semisimple Lie algebra. If either M is principal or $\mathfrak{g}$ is simple, then M admits $\mathfrak{g}$ as a basic algebra.*

The proof is given in [GGG]. This result is not true when $\mathfrak{g}$ is noncompact, cf. §6.5.

Despite all this, M may still carry *infinite*-dimensional basic algebras, as happens for T^2. Not much is known regarding these, and we refer the reader to [Is] for further discussion (cf. especially §4.8.4).

We denote by $P(\mathfrak{b})$ the polynomial algebra generated by $\mathfrak{b}$. Since $\mathfrak{b}$ is a Lie algebra, $P(\mathfrak{b})$ is a Poisson algebra. Note that (*i*) $P(\mathfrak{b})$ is not necessarily freely generated by $\mathfrak{b}$ as an associative algebra (cf. the examples in §5), and (*ii*) by definition $\mathbf{R} \subset P(\mathfrak{b})$. When $P(\mathfrak{b})$ is freely generated by $\mathfrak{b}$, it can be identified with the symmetric algebra $S(\mathfrak{b})$ over $\mathfrak{b}$, but otherwise $P(\mathfrak{b})$ is realized as the quotient of $S(\mathfrak{b})$ by the associative ideal generated by elements of the form $C - c$, where C is a "Casimir" and c is some constant (depending upon M).[5,6] Note that $S(\mathfrak{b})$ is itself a unital Poisson algebra, and that the canonical projection is a Poisson algebra homomorphism. In general we will not distinguish between $P(\mathfrak{b})$ and $S(\mathfrak{b})$, and in examples where Casimirs are present we will often work with representatives, i.e., on $S(\mathfrak{b})$, without explicitly stating so. Let $P^k(\mathfrak{b})$ denote the subspace of polynomials of minimal degree at most k. (Since $P(\mathfrak{b})$ is not necessarily freely generated by $\mathfrak{b}$, the notion of "degree" may not be well-defined, but that of "minimal degree" is.) In the cases when degree does make sense, we let $P_k(\mathfrak{b})$ denote the subspace of homogeneous polynomials of degree k, so that $P^k(\mathfrak{b}) = \oplus_{l=0}^k P_l(\mathfrak{b})$ (vector space direct sum). We then also introduce $P_{(k)}(\mathfrak{b}) = +_{l \geq k} P_l(\mathfrak{b})$. Notice that $P^1(\mathfrak{b}) = \mathfrak{b}$ or $\mathbf{R} \oplus \mathfrak{b}$, depending upon whether $1 \in \mathfrak{b}$ or not. When $\mathfrak{b}$ is fixed in context, we simply write $P = P(\mathfrak{b})$, etc.

4 Quantization

We are now ready to discuss what we mean by a "quantization." Let $\mathcal{O}$ be a Lie subalgebra of $C^\infty(M)$, and suppose that $\mathfrak{b} \subset \mathcal{O}$ is a basic algebra of observables. Two eminently reasonable requirements to place upon a quantization are irreducibility and integrability [BaRa, Fl, Is, Ki].

Irreducibility is of course one of the pillars of the quantum theory, and we have already seen the necessity of requiring that quantization represent $\mathfrak{b}$ irreducibly. We must however be careful to give a precise definition since the operators $\mathcal{Q}(f)$ are in general unbounded (although, according to (B2) and (Q3), all elements of $\mathcal{Q}(\mathfrak{b})$ are e.s.a.). So let $\mathcal{X}$ be a set of e.s.a. operators defined on a common invariant dense domain D in a Hilbert space $\mathcal{H}$. Then $\mathcal{X}$ is *irreducible* provided the only bounded self-adjoint operators which strongly commute with all $X \in \mathcal{X}$ are multiples of the identity. While this definition is fairly standard, and well suited to our needs, we note that other notions of irreducibility can be found in the literature [BaRa, MMSV, Tu]; cf. also the discussion at the end of this section.

Given such a set $\mathcal{X}$ of operators, let $\mathcal{U}(\mathcal{X})$ be the *-algebra generated by the unitary operators $\left\{ \exp(it\overline{X}) \mid t \in \mathbf{R},\ X \in \mathcal{X} \right\}$, where $\overline{X}$ is the closure of X. Then by Schur's Lemma $\mathcal{X}$ is irreducible iff the only closed subspaces of $\mathcal{H}$ which are invariant under $\mathcal{U}(\mathcal{X})$ are $\{0\}$ and $\mathcal{H}$.

Turning now to integrability, we first consider the case when basic algebra is finite-dimensional. Then it is natural to demand that the Lie algebra representation $\mathcal{Q}(\mathfrak{b})$ on $D \subset \mathcal{H}$ be *integrable* in the following sense: There exists a unitary representation Π of some Lie group with Lie algebra $\mathfrak{b}$ on $\mathcal{H}$ such that $\mathcal{Q}(f) = -i\hbar\, d\Pi(f) \restriction D$ for all $f \in \mathfrak{b}$, where $d\Pi$ is the derived representation of Π. For this it is *not* sufficient that elements of $\mathfrak{b}$ quantize to e.s.a. operators on D [ReSi, §VIII.5]. But integrability will follow from the following result of Flato et al., cf. [Fl] and [BaRa, Ch. 11].

Proposition 4 *Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra, and let π be a representation of $\mathfrak{g}$ by skew-symmetric operators on a common dense invariant domain D*

[5] A *Casimir* is an element of the Lie center of $S(\mathfrak{b})$ which has no constant term.

[6] Should $1 \in \mathfrak{b}$, we identify it with the units in both $P(\mathfrak{b})$ and $S(\mathfrak{b})$.

in a Hilbert space $\mathcal{H}$. Suppose that $\{\xi_1, \dots, \xi_k\}$ generates $\mathfrak{g}$ by linear combinations and repeated brackets. If D contains a dense set of separately analytic vectors for $\{\pi(\xi_1), \dots, \pi(\xi_k)\}$, then there exists a unique unitary representation Π of the connected simply connected Lie group with Lie algebra $\mathfrak{g}$ on $\mathcal{H}$ such that $d\Pi(\xi) \restriction D = \pi(\xi)$ for all $\xi \in \mathfrak{g}$.

We recall that a vector ψ is *analytic* for an operator X on $\mathcal{H}$ provided the series

$$\sum_{k=0}^{\infty} \frac{\|X^k \psi\|}{k!} t^k$$

is defined and converges for some $t > 0$. If $\{X_1, \dots, X_k\}$ is a set of operators defined on a common invariant dense domain D, a vector $\psi \in D$ is *separately analytic* for $\{X_1, \dots, X_k\}$ if ψ is analytic for each X_j. By a slight abuse of terminology, we will say that a vector is separately analytic for a Lie algebra of operators $\mathcal{X}$ if it is separately analytic for some Lie generating set $\{X_1, \dots, X_k\}$ of $\mathcal{X}$.

If it happens that $\mathfrak{b}$ is infinite-dimensional, then there need not exist a Lie group having $\mathfrak{b}$ as its Lie algebra. Even if such a Lie group existed, integrability is far from automatic, and technical difficulties abound. Thus we will not insist that a quantization be integrable in general. On the other hand, the analyticity requirement in Proposition 4 makes sense under all circumstances,[7] and does guarantee integrability when $\mathfrak{b}$ is finite-dimensional, so we will adopt it in lieu of integrability.

Finally, we will require that a quantization $\mathcal{Q}$ be faithful on $\mathfrak{b}$. While faithfulness is not usually assumed in the definition of a quantization, it seems to us a reasonable requirement in that a classical observable can hardly be regarded as "basic" in a physical sense if it is in the kernel of a quantization map. In this case, it cannot be obtained in any classical limit from a quantum theory.

Therefore we have at last:

Definition 3 A *quantization* of the pair $(\mathcal{O}, \mathfrak{b})$ is a prequantization $\mathcal{Q}$ of $\mathcal{O}$ on $\mathrm{Op}(D)$ satisfying

(Q4) $\mathcal{Q} \restriction \mathfrak{b}$ is irreducible,

(Q5) D contains a dense set of separately analytic vectors for $\mathcal{Q}(\mathfrak{b})$, and

(Q6) $\mathcal{Q} \restriction \mathfrak{b}$ is faithful.

Remarks. 5. There are a number of analyticity assumptions similar to (Q5) that one could make [Fl]; we have chosen the weakest possible one.

6. (Q5) is not a severe restriction: When $\mathfrak{b}$ is finite-dimensional, it is always possible to find representations of it on domains D which satisfy this property [Fl]. On the other hand, nonintegrable representations do exist in general [Fl, p. 247].

7. Proposition 4 requires that a specific generating set for $\mathcal{Q}(\mathfrak{b})$ be singled out. This also is not a serious restriction: In examples, $\mathfrak{b}$ is often specified in this manner.

[7] As long as $\mathfrak{b}$ is finitely generated, which is assured by (B1).

It is possible that (Q5) could be satisfied for one such set but not another, but Remark 6 shows that the domain D can be chosen in such a way that this cannot happen if $\mathfrak{b}$ is finite-dimensional.

8. It is important to realize that irreducibility does not imply integrability. For instance, there is an irreducible representation of h(2) which is not integrable [ReSi, p. 275].

We briefly comment on the domains D appearing in Definition 3. For a representation π of a Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$, there is typically a multitude of common, invariant dense domains that one can use as carriers of the representation. (See [BaRa, §11.2] for a discussion of some of the possibilities.) But what is ultimately important for our purposes are the closures $\overline{\pi(\xi)}$ for $\xi \in \mathfrak{g}$, and not the $\pi(\xi)$ themselves. So we do not want to distinguish between two representations π on $\mathrm{Op}(D)$ and π' on $\mathrm{Op}(D')$ whenever $\overline{\pi(\xi)} = \overline{\pi'(\xi)}$, in which case we say that π and π' are *coextensive*. In particular, it may happen that the given domain D for a representation π does not satisfy (Q5), but there is an extension to a coextensive representation π' on a domain D' that does.[8] In such cases we will suppose that the representation has been so extended.

We end this section with a refinement of the irreducibility condition (Q4). There is another, simpler notion of irreducibility which is very useful for our purposes: We say that $\mathcal{Q}(\mathfrak{b})$ is *algebraically irreducible* provided the only operators in $\mathrm{Op}(D)$ which (weakly) commute with all elements of $\mathcal{Q}(\mathfrak{b})$ are scalar multiples of the identity. It turns out that a quantization is automatically algebraically irreducible.

Proposition 5 *Let $\mathcal{Q}$ be a representation of a finite-dimensional Lie algebra $\mathfrak{b}$ by symmetric operators on an invariant dense domain D in a separable Hilbert space $\mathcal{H}$. If $\mathcal{Q}$ satisfies* (Q4) *and* (Q5)*, then $\mathcal{Q}(\mathfrak{b})$ is algebraically irreducible.*

The proof, which hinges on an unbounded version of Schur's lemma [Ro], is given in [GGra1, Prop. 3]; cf. also [Go4].

5 Examples

In this section we present the gist of the arguments—more or less as they originally appeared—that there are no nontrivial polynomial quantizations of either $\mathbf{R}^{2n}$, S^2, or T^*S^1, with the basic algebras $\mathrm{h}(2n)$, su(2), and e(2), respectively. The complete proofs can be found in [AM, Ch1, Fo, Go1, Go4, Gro, GS, VH1, VH2] for $\mathbf{R}^{2n}$, [GGH] for S^2, and [GGru1] for T^*S^1. The proofs in all three examples require a detailed knowledge of the structure of the Poisson algebras involved and their representations. Finally, we show following [GGra1] that there is a polynomial quantization of $T^*\mathbf{R}_+$ with the basic algebra a(1), and following [Go3] that there is a full quantization of T^2 with the basic algebra $\mathfrak{t}$. We also take this opportunity to repair a defect in the standard presentations of the Groenewold-Van Hove theorem for $\mathbf{R}^{2n}$.

As an aside, we point out that many of the calculations in §§5.2 and 5.3 were done using the *Mathematica* package *NCAlgebra* [HM].

[8] A simple illustration is provided by the Schrödinger representation (2) with $D = C_c^\infty(\mathbf{R}^n, \mathbf{C})$ and $D' = \mathcal{S}(\mathbf{R}^n, \mathbf{C})$.

5.1 $\mathbf{R}^{2n}$

Before proceeding with the no-go theorem for $\mathbf{R}^{2n}$, we remark that already at a purely mathematical level one can observe a suggestive structural mismatch between the classical and the quantum formalisms. Since a prequantization is essentially a Lie algebra homomorphism, it "compares" the Lie algebra structure of $C^\infty(\mathbf{R}^{2n})$ with the Lie algebra of (skew-) symmetric operators (preserving a dense domain D) equipped with the commutator bracket. But if we take $P \subset C^\infty(\mathbf{R}^{2n})$ to be the subalgebra of polynomials, Joseph [Jo] has shown that P has outer derivations, but the enveloping algebra of the Heisenberg algebra $\mathrm{h}(2n)$—and hence that of the Schrödinger representation thereof on $L^2(\mathbf{R}^n)$—has none. In the next section we generalize this line of reasoning, and present another such "algebraic" no-go theorem to the effect that a unital Poisson algebra can never be realized as an associative algebra with the commutator bracket.

In particular, one can see at the outset that it is impossible for a prequantization to satisfy the "product $\to$ anti-commutator" rule. Taking $n = 1$ for simplicity, suppose $\mathcal{Q}$ were a prequantization of the polynomial algebra $P = \mathbf{R}[q, p]$ for which

$$\mathcal{Q}(fg) = \tfrac{1}{2}\big(\mathcal{Q}(f)\mathcal{Q}(g) + \mathcal{Q}(g)\mathcal{Q}(f)\big)$$

for all $f, g \in P$. Take $f(p, q) = p$ and $g(p, q) = q$. Then

$$\begin{aligned} \tfrac{1}{4}\big(\mathcal{Q}(p)\mathcal{Q}(q) + \mathcal{Q}(q)\mathcal{Q}(p)\big)^2 &= \mathcal{Q}(pq)^2 \\ &= \mathcal{Q}(p^2q^2) \;=\; \tfrac{1}{2}\big(\mathcal{Q}(p)^2\mathcal{Q}(q)^2 + \mathcal{Q}(q)^2\mathcal{Q}(p)^2\big). \end{aligned}$$

Now by (Q1) we have $[\mathcal{Q}(p), \mathcal{Q}(q)] = -i\hbar I$, so that the L.H.S. reduces to

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar\mathcal{Q}(q)\mathcal{Q}(p) - \tfrac{1}{4}\hbar^2 I$$

while the R.H.S. becomes

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar\mathcal{Q}(q)\mathcal{Q}(p) - \hbar^2 I.$$

As the product $\to$ anti-commutator rule is equivalent to the squaring Von Neumann rule $\mathcal{Q}(f^2) = \mathcal{Q}(f)^2$, this contradiction also shows that the latter is inconsistent with prequantization. Note that the contradiction is obtained on quartic polynomials; there is no problem if consideration is limited to observables which are at most cubic.

This argument only used axiom (Q1) in the specific instance $[\mathcal{Q}(p), \mathcal{Q}(q)] = -i\hbar I$. Consequently, one still obtains a contradiction if one drops (Q1) and instead insists that $\mathcal{Q}$ be consistent with Schrödinger quantization (in which context this one commutation relation remains valid, cf. (2)). This *manifest* impossibility of satisfying the product $\to$ anti-commutator rule while being consistent with Schrödinger quantization is one reason we have decided to focus on the Lie structure as opposed to the associative structure of $C^\infty(M)$. See [AB] for further results in this direction.

We now turn to the no-go theorem for $\mathbf{R}^{2n}$. We shall state the main results for $\mathbf{R}^{2n}$ but, for convenience, usually prove them only for $n = 1$. The proofs for higher dimensions are immediate generalizations of these. In what follows $P = \mathbf{R}[q^1, \dots, q^n, p_1, \dots, p_n]$; note that $P^1 \cong \mathrm{h}(2n)$, $P_2 \cong \mathrm{sp}(2n, \mathbf{R})$, and $P^2 \cong \mathrm{hsp}(2n, \mathbf{R})$.

Groenewold's celebrated result is:

Theorem 6 *There is no quantization of* (P, P^1).

Proof. Set $n = 1$ and let $\mathcal{Q}$ be a quantization of (P, P^1). We will show that a contradiction arises when cubic polynomials are considered.

We begin by determining $\mathcal{Q}(q^2)$. Set $\Delta = \mathcal{Q}(q^2) - \mathcal{Q}(q)^2$. Using (Q1) we readily verify that $[\Delta, \mathcal{Q}(q)] = 0$ and $[\Delta, \mathcal{Q}(p)] = 0$. But now algebraic irreducibility (cf. Proposition 5) implies that $\Delta = EI$ for some real constant E. Thus $\mathcal{Q}(q^2) = \mathcal{Q}(q)^2 + EI$.

An identical argument yields $\mathcal{Q}(p^2) = \mathcal{Q}(p)^2 + FI$. Quantizing the relation $4pq = \{p^2, q^2\}$ and using these formulæ then gives

$$\mathcal{Q}(pq) = \tfrac{1}{2}\big(\mathcal{Q}(p)\mathcal{Q}(q) + \mathcal{Q}(q)\mathcal{Q}(p)\big).$$

But upon quantizing $2q^2 = \{pq, q^2\}$ we find that $E = 0$. Similarly $F = 0$. Thus we have the Von Neumann rules

$$\mathcal{Q}(q^2) = \mathcal{Q}(q)^2, \quad \mathcal{Q}(p^2) = \mathcal{Q}(p)^2 \tag{7}$$

and

$$\mathcal{Q}(qp) = \tfrac{1}{2}\big(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)\big). \tag{8}$$

These in turn lead to higher degree Von Neumann rules.

Lemma 1 *For all real-valued polynomials* r,

$$\mathcal{Q}\big(r(q)\big) = r\big(\mathcal{Q}(q)\big), \quad \mathcal{Q}\big(r(p)\big) = r\big(\mathcal{Q}(p)\big),$$

$$\mathcal{Q}\big(r(q)p\big) = \tfrac{1}{2}\big[r\big(\mathcal{Q}(q)\big)\mathcal{Q}(p) + \mathcal{Q}(p)r\big(\mathcal{Q}(q)\big)\big],$$

and

$$\mathcal{Q}\big(qr(p)\big) = \tfrac{1}{2}\big[\mathcal{Q}(q)r\big(\mathcal{Q}(p)\big) + r\big(\mathcal{Q}(p)\big)\mathcal{Q}(q)\big].$$

Proof. We illustrate this for $r(q) = q^3$. The other rules follow similarly using induction. Now $\{q^3, q\} = 0$ whence by (Q1) we have $[\mathcal{Q}(q^3), \mathcal{Q}(q)] = 0$. Since also $[\mathcal{Q}(q)^3, \mathcal{Q}(q)] = 0$, we may write $\mathcal{Q}(q^3) = \mathcal{Q}(q)^3 + T$ for some operator T which (weakly) commutes with $\mathcal{Q}(q)$. We likewise have using (7)

$$[\mathcal{Q}(q^3), \mathcal{Q}(p)] = -i\hbar\, \mathcal{Q}(\{q^3, p\}) = 3i\hbar\, \mathcal{Q}(q^2) = 3i\hbar\, \mathcal{Q}(q)^2 = [\mathcal{Q}(q)^3, \mathcal{Q}(p)]$$

from which we see that T commutes with $\mathcal{Q}(p)$ as well. Consequently, T also commutes with $\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)$. But then from (8),

$$\begin{aligned}
\mathcal{Q}(q^3) &= \tfrac{1}{3}\,\mathcal{Q}\big(\{pq, q^3\}\big) = \tfrac{i}{3\hbar}\,[\mathcal{Q}(pq), \mathcal{Q}(q^3)] \\
&= \tfrac{i}{3\hbar}\left[\tfrac{1}{2}\big(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)\big), \mathcal{Q}(q)^3 + T\right] \\
&= \tfrac{i}{6\hbar}\,[\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q), \mathcal{Q}(q)^3] = \mathcal{Q}(q)^3.
\end{aligned}$$

∇

With this lemma in hand, it is now a simple matter to prove the no-go theorem. Consider the classical equality

$$\tfrac{1}{9}\{q^3, p^3\} = \tfrac{1}{3}\{q^2 p, p^2 q\}.$$

Quantizing and then simplifying this, the formulæ in Lemma 1 give

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar\mathcal{Q}(q)\mathcal{Q}(p) - \tfrac{2}{3}\hbar^2 I$$

for the L.H.S., and

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar\mathcal{Q}(q)\mathcal{Q}(p) - \tfrac{1}{3}\hbar^2 I$$

for the R.H.S., which is a contradiction. □

Remarks. 9. Note that this proof of the no-go theorem does not use the Stone-Von Neumann theorem.

10. There is a small gap in Groenewold's original proof of Theorem 6 [Gro] in that the Von Neumann rules (7) and (8) were not rigorously derived. In effect, Groenewold took the weak commutativity of Δ with $\mathcal{Q}(q)$ to mean strong commutativity.[9] Van Hove supplied an extra assumption which remedied the situation, and which in particular implies: If the Hamiltonian vector fields of f, g are complete and $\{f, g\} = 0$, then $\mathcal{Q}(f)$ and $\mathcal{Q}(g)$ *strongly* commute [VH1]. This assumption along with the Stone-Von Neumann theorem is used to derive the Von Neumann rules (7) and (8) in [AM, Ch1]. However, as our argument based on Proposition 5 shows, it is possible obtain the desired result directly from the quantization axioms, *without* introducing additional assumptions. See [Go4] for further discussion.

11. Van Hove [VH1] gave a slightly different analysis using only those observables $f \in C^\infty(\mathbf{R}^{2n})$ with complete Hamiltonian vector fields, and still obtained an obstruction (but now to quantizing all of $C^\infty(\mathbf{R}^{2n})$). Yet another variant of Groenewold's theorem will be presented in §6.5.

Even though all of P cannot be quantized, there does exist a quantization $d\varpi$ of the pair (P^2, P^1), given by the familiar formulæ

$$d\varpi(q^i) = q^i, \quad d\varpi(1) = I, \quad d\varpi(p_j) = -i\hbar\frac{\partial}{\partial q^j},$$

$$d\varpi(q^iq^j) = q^iq^j, \qquad d\varpi(p_ip_j) = -\hbar^2\frac{\partial^2}{\partial q^i\partial q^j}, \tag{9}$$

$$d\varpi(q^ip_j) = -i\hbar\left(q^i\frac{\partial}{\partial q^j} + \frac{1}{2}\delta^i_j\right),$$

on the domain $\mathcal{S}(\mathbf{R}^n,\mathbf{C}) \subset L^2(\mathbf{R}^n)$. Properties (Q1)–(Q3) and (Q6) are readily verified. (Q4) follows automatically since the restriction of $d\varpi$ to P^1 is just the Schrödinger

[9] This is quite understandable: Groenewold presented his proof in 1946, but the distinction between weak and strong commutativity was apparently not fully appreciated until the late 1950s with the work of Nelson, cf. [ReSi, §VIII.5].

representation. For (Q5), we recall the well-known fact that the Hermite functions (3) form a dense set of separately analytic vectors for $d\varpi(P^1)$. We call $d\varpi$ the "extended metaplectic quantization"; detailed discussions of it may be found in [Fo, GS].

In fact, $d\varpi$ is the "only" quantization of (P^2, P^1):

Proposition 7 *Up to restriction of representations, any quantization of* (P^2, P^1) *is unitarily equivalent to the extended metaplectic quantization.*

Proof. Suppose $\mathcal{Q}$ is a quantization of (P^2, P^1) on a domain D in a Hilbert space $\mathcal{H}$. Then by (Q4) and (Q5) $\mathcal{Q}(P^1)$ can be integrated to an irreducible unitary representation τ of $\mathrm{H}(2n)$. The Stone-Von Neumann theorem then states that τ is unitarily equivalent to the Schrödinger representation Π, and hence $\tau = U\Pi U^{-1}$ for some unitary map $U : L^2(\mathbf{R}^n) \to \mathcal{H}$. Consequently, $\mathcal{Q}(f) = U\overline{d\Pi(f)}U^{-1} \restriction D$ for all $f \in P^1$. It now follows from (2), the invariance of the domain D, and Sobolev's lemma that $U^{-1}D \subset \mathcal{S}(\mathbf{R}^n)$, so that $U^{-1}(\mathcal{Q} \restriction P^1)U$ is in fact the *restriction* of $d\Pi$ to $U^{-1}D \subset \mathcal{S}(\mathbf{R}^n)$. Thus without loss of generality we may assume that $\mathcal{Q}(P^1)$ is the Schrödinger representation (2) and that the domain $D \subset \mathcal{S}(\mathbf{R}^n)$.

Then the first part of the proof of Theorem 6 yields the Von Neumann rules (7) and (8), which in view of (2) imply that $\mathcal{Q}(P^2)$ is given by (9). Thus, up to unitary equivalence, $\mathcal{Q}$ must be either $d\varpi$ or a restriction thereof. □

Since P^2 is a maximal Lie subalgebra of P [GS, §16], (Q1) implies that any quantization which extends $d\varpi$ must be defined on all of P. Theorem 6 then implies

Corollary 8 *The extended metaplectic quantization of* (P^2, P^1) *cannot be extended beyond* P^2 *in* P.

We hasten to add that there are subalgebras of P other than P^2 which can be quantized. For example, let

$$C = \left\{ \sum_{i=1}^{n} f^i(q)p_i + g(q) \right\},$$

where f^i and g are polynomials. Then it is straightforward to verify that for each $\eta \in \mathbf{R}$, the map $\mathcal{Q}_\eta : C \to \mathrm{Op}\big(\mathcal{S}(\mathbf{R}^n, \mathbf{C})\big)$ given by

$$\mathcal{Q}_\eta \left(\sum_{i=1}^{n} f^i(q)p_i + g(q) \right) = -i\hbar \sum_{i=1}^{n} \left(f^i(q)\frac{\partial}{\partial q^i} + \left[\frac{1}{2} + i\eta\right] \frac{\partial f^i}{\partial q^i} \right) + g(q)$$

is a quantization of (C, P^1). $\mathcal{Q}_0$ is the familiar "position" or "coordinate representation." The significance of the parameter η is explained in [ADT]. Since C is also a maximal subalgebra of P, Theorem 6 yields

Corollary 9 *The quantizations* $\mathcal{Q}_\eta$ *of* (C, P^1) *cannot be extended beyond* C *in* P.

We furthermore point out that Proposition 7 in [Go4] yields "uniqueness": Any quantization of (C, P^1) must be unitarily equivalent to (a restriction of) $\mathcal{Q}_\eta$ for some $\eta \in \mathbf{R}$.

A similar analysis applies to the "Fourier transform" of the subalgebra C, i.e. the "momentum subalgebra" of all polynomials which are at most affine in the coordinates q^i. In fact [Go4], it turns out for $n = 1$ that P^2 and C exhaust the list of isomorphism classes of maximal Lie subalgebras of P which contain P^1. This is not true in higher dimensions, however: The subalgebra

$$\{f(q^1)p_1 + g(q^1, q^2, p_2)\}$$

on $\mathbf{R}^4$ is maximal, but not isomorphic to either C or P^2. Furthermore, the subalgebra thereof for which g is at most quadratic in q^2, p_2 is maximal quantizable, but also not isomorphic to either C or P^2. It remains an open problem to determine the largest quantizable Lie subalgebras of P for $n > 1$.

5.2 S^2

Now we turn our attention to the sphere. Since S^2 is compact, all classical observables are complete. Moreover, $\mathrm{su}(2) \cong \mathrm{span}\{S_1, S_2, S_3\}$ is a compact simple Lie algebra. Consequently the functional analytic subtleties present in the case of $\mathbf{R}^{2n}$ disappear. But the actual computations, which were fairly routine for $\mathbf{R}^{2n}$, turn out to be much more complicated for S^2.

The Poisson bracket on $C^\infty(S^2)$ corresponding to (4) is

$$\{f, g\} = -\sum_{i,j,k=1}^{3} \epsilon_{ijk}\, S_i \frac{\partial f}{\partial S_j}\frac{\partial g}{\partial S_k}.$$

In particular, we have the relations $\{S_j, S_k\} = -\sum_{l=1}^{3} \epsilon_{jkl}\, S_l$. In this example P is the polynomial algebra in the components of the spin vector, subject to the relation

$$S_1{}^2 + S_2{}^2 + S_3{}^2 = s^2. \tag{10}$$

We may identify P with the space of spherical harmonics. We have $P_1 \cong \mathrm{su}(2)$ and $P^1 \cong \mathrm{u}(2)$.

Let $\mathcal{Q}$ be a quantization of (P, P_1) on a Hilbert space $\mathcal{H}$, whence

$$[\mathcal{Q}(S_j), \mathcal{Q}(S_k)] = i\hbar \sum_{l=1}^{3} \epsilon_{jkl}\, \mathcal{Q}(S_l) \tag{11}$$

and

$$\mathcal{Q}(\mathbf{S}^2) = s^2 I. \tag{12}$$

By (Q5) and Proposition 4, $\mathcal{Q}(\mathrm{su}(2))$ can be exponentiated to a unitary representation of SU(2) which, according to (Q4), is irreducible. Therefore $\mathcal{H}$ must be finite-dimensional, and $\mathcal{Q}(\mathrm{su}(2))$ must be one of the usual spin angular momentum representations, labeled by $j = 0,\ \frac{1}{2},\ 1, \ldots$ For a fixed value of j, $\dim \mathcal{H} = 2j + 1$ and

$$\sum_{i=1}^{3} \mathcal{Q}(S_i)^2 = \hbar^2 j(j+1)I. \tag{13}$$

Our goal is show that no such (nontrivial) quantization exists. Patterning our analysis after that for $\mathbf{R}^{2n}$, we use irreducibility to derive some Von Neumann rules.

Lemma 2 *For $i = 1, 2, 3$ we have*

$$\mathcal{Q}({S_i}^2) = a\mathcal{Q}(S_i)^2 + cI \tag{14}$$

where a, c are representation dependent real constants with $a^2 + c^2 \neq 0$.

The proof is in [GGH]. From this we also derive

$$\mathcal{Q}(S_i S_k) = \frac{a}{2}\big(\mathcal{Q}(S_i)\mathcal{Q}(S_k) + \mathcal{Q}(S_k)\mathcal{Q}(S_i)\big) \tag{15}$$

for $i \neq k$. (As an aside, these formulæ show that a quantization, if it exists, may be badly behaved with respect to the multiplicative structure on $C^\infty(S^2)$; in particular, the product → anti-commutator rule need not hold. Remarkably, this is as it should be: For *if* this rule were valid, then – subject to a few mild assumptions on $\mathcal{Q}$—the classical spectrum of S_3, say, would have to coincide with that of $\mathcal{Q}(S_3)$ which is contrary to experiment [GGH].) With these tools, we can now prove the main result:

Theorem 10 *There is no nontrivial quantization of (P, P_1).*

Proof. Fix $j > 0$, as $j = 0$ produces a trivial quantization. Assuming that $\mathcal{Q}$ is a quantization of (P, P_1), we can use (11)-(15) to quantize the classical relation

$$s^2 S_3 = \big\{{S_1}^2 - {S_2}^2,\ S_1 S_2\big\} - \big\{S_2 S_3,\ S_3 S_1\big\}, \tag{16}$$

thereby obtaining

$$s^2 = a^2\hbar^2\big(j(j+1) - \tfrac{3}{4}\big) \tag{17}$$

which contradicts $s > 0$ for $j = \frac{1}{2}$. Now assume $j > \frac{1}{2}$, and quantize

$$2s^2 S_2 S_3 = \big\{{S_2}^2, \{S_1 S_2, S_1 S_3\}\big\} - \tfrac{3}{4}\big\{{S_1}^2, \{{S_1}^2, S_2 S_3\}\big\}, \tag{18}$$

similarly obtaining

$$s^2 = a^2\hbar^2\big(j(j+1) - \tfrac{9}{4}\big)$$

which contradicts (17). Thus we have derived contradictions for all $j > 0$, and the theorem is proven. □

In view of the impossibility of quantizing (P, P_1), one can ask what the maximal Lie subalgebras in P are to which we can extend an irreducible representation of P_1. The following chain of results, which we quote without proof (cf. [GGH]), provides the answer.

Proposition 11 *P^1 is a maximal Lie subalgebra of $\mathbf{R} \oplus O \subset P$, where O is the Poisson algebra consisting of polynomials containing only terms of odd degree.*

Next we establish a no-go theorem for $(\mathbf{R} \oplus O, P_1)$. However, the Von Neumann rules listed in Lemma 2 involve only even degree polynomials, so these are not applicable in O. Fortunately, we have another set of Von Neumann rules, also implied by the irreducibility of $\mathcal{Q}(P_1)$, involving only terms of odd degree.

Lemma 3 *If $\mathcal{Q}$ is a quantization of $(\mathbf{R} \oplus O, P_1)$, then for $i = 1, 2, 3$,*

$$\mathcal{Q}({S_i}^3) = b\mathcal{Q}(S_i)^3 + e\mathcal{Q}(S_i)$$

where $b, e \in \mathbf{R}$.

From this we prove (with far greater effort):

Theorem 12 *There is no nontrivial quantization of $(\mathbf{R} \oplus O, P_1)$.*

Now $\mathbf{R} \oplus O$ is itself a maximal Lie subalgebra of P, and in fact the only Lie subalgebras of P strictly containing P_1 are $P^1, \mathbf{R} \oplus O$, and P itself. On the other hand, $P^1 = \mathbf{R} \oplus P_1$ is obviously quantizable. Thus Theorem 12 and Proposition 11 combine to yield our sharpest result for the sphere:

Corollary 13 *No nontrivial quantization of (P^1, P_1) can be extended beyond P^1 in P.*

Thus within the algebra of polynomials, $\big(\mathrm{u}(2),\mathrm{su}(2)\big)$ is the most one can quantize.

There is a crucial structural difference between the Groenewold-Van Hove analysis of $\mathbf{R}^2$ and the current analysis of S^2. Within $P = \mathbf{R}[q, p]$ the Heisenberg algebra has as its Lie normalizer the algebra of polynomials of degree at most 2, and there is no obstruction to quantization in this algebra: The obstruction comes from the cubic polynomials. On the other hand, for the sphere, the special unitary algebra has as its normalizer the algebra of polynomials of degree at most one; we obtain an obstruction in the quadratic polynomials, and find that there is no extension possible for a quantization of P^1.

5.3 T^*S^1

Our final example of an obstruction is provided by the symplectic cylinder, which appears in geometric optics [GS, §17]. Endow T^*S^1 with the canonical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \ell}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \ell},$$

where ℓ is the angular momentum conjugate to θ. While the symplectic self-action of T^*S^1 is not Hamiltonian (thinking of T^*S^1 as $S^1 \times \mathbf{R}$), the cylinder can nonetheless be realized as a coadjoint orbit of the special Euclidean group SE(2) [MR, §14.8]. The corresponding momentum map $T^*S^1 \to \mathrm{e}(2)^*$ has components $\{\sin\theta, \cos\theta, \ell\}$; therefore we take as a basic algebra

$$\mathrm{e}(2) \cong \mathrm{span}\{\sin\theta, \cos\theta, \ell\}.$$

The polynomial algebra P generated by this basic algebra consists of sums of multiples of terms of the form $\ell^r \sin^m\theta \cos^n\theta$ of total degree $r+m+n$ with r, m, n nonnegative integers. Then $P_1 \cong \mathrm{e}(2)$.

Our first task is to determine all possible quantizations of the basic algebra e(2). By virtue of (Q3) and (Q4), for this it suffices to compute the derived representations corresponding to the IURs of the universal covering group of SE(2), which is the semidirect product $\mathbf{R} \ltimes \mathbf{R}^2$ with the composition law

$$(t, x, y) \cdot (t', x', y') = (t + t', x' \cos t + y' \sin t + x, y' \cos t - x' \sin t + y).$$

From the theory of induced representations of semidirect products [Ma] (see also [Is, §5.8]), we find that the only nontrivial IURs are infinite-dimensional; up to unitary equivalence they take the form

$$\big(U_\nu^\lambda(t, x, y)\psi\big)(\theta) = e^{i\lambda(x\cos\theta + y\sin\theta)}e^{i\nu t}\psi(\theta + t)$$

on $L^2(S^1)$. Here λ, ν are real parameters satisfying $\lambda > 0$ and $0 \le \nu < 1$. We identify λ with the reciprocal of $\hbar$, cf. [GGru1, Is]. After rescaling appropriately, the corresponding derived representations become

$$dU_\nu(\ell) = -i\hbar\left(\frac{d}{d\theta} + i\nu I\right), \quad dU_\nu(\sin\theta) = \sin\theta, \quad dU_\nu(\cos\theta) = \cos\theta \tag{19}$$

on the domain $C^\infty(S^1,\mathbf{C})$. Each representation dU_ν is a quantization of P_1; in particular, $\{e^{in\theta} \mid n \in \mathbf{Z}\} \subset C^\infty(S^1,\mathbf{C})$ is a dense set of separately analytic vectors for the above basis of $dU_\nu(P_1)$.

Now suppose that $\mathcal{Q}$ is a quantization of (P, P_1) on a domain D. Arguing as in the proof of Proposition 7, we find that up to unitary equivalence $\mathcal{Q} \restriction P_1$ coincides with dU_ν for some ν and that $D \subset C^\infty(S^1,\mathbf{C})$. Up to coextension of representations (cf. the discussion at the end of §4), then, we may as well assume that $D = C^\infty(S^1,\mathbf{C})$.

Just as with our previous examples, we use irreducibility to obtain Von Neumann rules. In [GGru1] we compute

$$\mathcal{Q}(\ell^2) = \mathcal{Q}(\ell)^2 + cI,$$

where $c \in \mathbf{R}$ is arbitrary. From this and (19) we eventually derive

$$\mathcal{Q}(\ell^2 \sin\theta) = \mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 - i\hbar\mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + \frac{\hbar^2}{4}\mathcal{Q}(\sin\theta) \tag{20}$$

and

$$\mathcal{Q}(\ell^2 \cos\theta) = \mathcal{Q}(\cos\theta)\mathcal{Q}(\ell)^2 + i\hbar\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell) + \frac{\hbar^2}{4}\mathcal{Q}(\cos\theta). \tag{21}$$

Theorem 14 *There is no nontrivial quantization of* (P, P_1).

Proof. We merely use (19)–(21) to quantize the bracket relation

$$2\big\{\{\ell^2 \sin\theta, \ell^2 \cos\theta\}, \cos\theta\big\} = 12\ell^2 \sin\theta. \tag{22}$$

After simplifying, the left hand side reduces to

$$12\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 - 12i\hbar\mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + 5\hbar^2\mathcal{Q}(\sin\theta),$$

whereas the right hand side is

$$12\mathcal{Q}(\sin\theta)\mathcal{Q}(\ell)^2 - 12i\hbar\mathcal{Q}(\cos\theta)\mathcal{Q}(\ell) + 3\hbar^2\mathcal{Q}(\sin\theta),$$

and the required contradiction is evident. □

We next determine the maximal Lie subalgebras of P to which we can extend an irreducible representation of P_1. Such subalgebras certainly exist: For instance, there is a two-parameter family of quantizations of the pair (L^1, P_1), where L^1 denotes the Lie subalgebra of polynomials which are at most affine in ℓ. They are the "position representations" on $C^\infty(S^1,\mathbf{C}) \subset L^2(S^1)$ given by

$$\mathcal{Q}_{\nu,\eta}\big(f(\theta)\ell + g(\theta)\big) = -i\hbar\left(f(\theta)\frac{d}{d\theta} + \left[\frac{1}{2} + i\eta\right]f'(\theta) + i\nu f(\theta)\right) + g(\theta), \quad (23)$$

where ν labels the IURs of the universal cover of SE(2) and η is real.

To this end we classify the maximal Lie subalgebras of P containing P_1. For each $\alpha \in \mathbf{R}$ let V_α be the Lie subalgebra generated by

$$\{1,\ \sin\theta,\ \cos\theta,\ \ell,\ \ell(\ell+\alpha)\cos(2N+1)\theta,\ \ell(\ell+\alpha)\sin(2N+1)\theta \mid N \in \mathbf{N}\}.$$

Although far from obvious, it turns out that [GGru1]

Proposition 15 *L^1 and V_α, $\alpha \in \mathbf{R}$, are the only proper maximal Lie subalgebras of P strictly containing P_1.*

In contrast to L^1, it is possible to show that there is *no* nontrivial quantization of any V_α which represents P_1 irreducibly. (While the method of proof is the same as that of the no-go theorem for P presented above, we must make sure that all constructions take place in V_α. The details may be found in [GGru1].) Since L^1 is maximal, Theorem 14 implies that none of the quantizations $\mathcal{Q}_{\nu,\eta}$ can be extended beyond L^1 in P. Furthermore [GGru1], the quantizations (23) of L^1 are the only possible ones:

Theorem 16 *If $\mathcal{Q}$ is a nontrivial quantization of (L^1, P_1), then $\mathcal{Q} = \mathcal{Q}_{\nu,\eta}$ for some $\nu \in [0, 1)$ and $\eta \in \mathbf{R}$.*

Taken together, these results completely characterize the polynomial quantizations for the basic algebra e(2).

Since T^*S^1 is covered by $\mathbf{R}^2$, and as e(2) is the natural analogue for the cylinder of h(2) for the plane, the quantization of the former might be expected to share some of the features of that of the latter, and we see from the above that in most respects this is so. In both examples there is an obstruction, and a maximal Lie subalgebra of polynomial observables that can be consistently quantized consists of those polynomials which are affine in the momentum.

There are some differences, however, which reflect the non-simple connectivity of T^*S^1. For instance, on $\mathbf{R}^2$, there are exactly two isomorphism classes of maximal polynomial Lie subalgebras containing the basic algebra span$\{1, q, p\}$ which can be consistently quantized, whereas according to Proposition 15 there is only one such containing span$\{\sin\theta, \cos\theta, \ell\}$ for the cylinder. (Since P^2 is not a Lie subalgebra of

P, there is no analogue of the metaplectic representation for T^*S^1 and, since θ is an angular variable, there is also no cylindrical counterpart of the momentum representation.) Thus the possible polynomial quantizations of T^*S^1 are more limited than those of $\mathbf{R}^2$.

One topic for future exploration would be to consider the higher-dimensional analogues of the cylinder, viz. T^*S^n with basic algebra e(n).

5.4 $T^*\mathbf{R}_+$

We have encountered obstructions to quantization in the three examples presented so far, despite the fact that $\mathbf{R}^2$, T^*S^1, and S^2 are quite different structurally. Topologically these phase spaces range from contractible to compact, and algebraically the basic algebras h(2), e(2), and su(2) are nilpotent, solvable, and simple, respectively. Moreover, the representations of these algebras were in some instances unique and in others not, and they were finite- as well as infinite-dimensional. This wide array of behaviors strongly suggests that such obstructions should be ubiquitous. Therefore it comes as a surprise that this is *not* so [GGra1]: there is no obstruction to polynomially quantizing $T^*\mathbf{R}_+ = \{(q, p) \in \mathbf{R}^2 \mid q > 0\}$ with the "affine" basic algebra

$$\mathrm{a}(1) \cong \mathrm{span}\{pq, q^2\}.$$

The simply connected covering group of a(1) is isomorphic to the group $\mathrm{A}_+(1) = \mathbf{R} \rtimes \mathbf{R}_+$ of orientation-preserving affine transformations of the line (hence the terminology). It is straightforward to check that $T^*\mathbf{R}_+$ with the canonical Poisson bracket can be realized as a coadjoint orbit in a(1)* [MR, §14.1(b)].

Upon writing

$$x = pq, \quad y = q^2$$

the bracket relation becomes $\{x, y\} = 2y$. The corresponding polynomial algebra $P = \mathbf{R}[x, y]$ is freely generated, and has the crucial feature that for each $k \geq 0$, the subspaces P_k are *ad*-invariant, i.e.,

$$\{P_1, P_k\} \subset P_k. \tag{24}$$

(Note that $P_1 \cong \mathrm{a}(1)$). Because of this $\{P_k, P_l\} \subset P_{k+l}$, whence each $P_{(k)}$ is a Lie ideal. We thus have the semidirect sum decomposition

$$P = P^1 \ltimes P_{(2)}. \tag{25}$$

Now on to quantization. Since $P_{(2)}$ is a Lie ideal, we can obtain a quantization $\mathcal{Q}$ of *all* of P simply by finding an appropriate representation of $P^1 = \mathbf{R} \oplus P_1$ and setting $\mathcal{Q}(P_{(2)}) = \{0\}$!

Since $\mathrm{A}_+(1)$ is a semidirect product we can generate the required representation of P_1 by induction. Following the recipe in [BaRa, §17.1] we obtain the two one-parameter families of unitary representations $U^\mu_\pm$ of $\mathrm{A}_+(1)$ on $L^2(\mathbf{R}_+, dq/q)$ given by

$$\big(U^\mu_\pm(\nu, \lambda)\psi\big)(q) = e^{\pm i\mu\nu q^2}\psi(\lambda q)$$

with $\mu > 0$. As in the previous subsection, we identify the parameter μ with $\hbar^{-1}$. According to Theorems 4 and 5 in [BaRa, §17.1] the remaining two representations (one for each choice of sign) are irreducible and inequivalent; moreover, these are the *only* irreducible infinite-dimensional unitary ones.

Let $D \subset L^2(\mathbf{R}_+, dq/q)$ be the linear span of the functions $\sqrt{q}\, h_k(q)$, where the h_k are the Hermite functions. Writing $\varrho_\pm = -i\hbar\, dU_\pm$ we get the representations of a(1) on $L^2(\mathbf{R}_+, dq/q)$:

$$\varrho_\pm(pq) = -i\hbar q\frac{d}{dq}, \quad \varrho_\pm(q^2) = \pm q^2.$$

Extend these to P^1 by demanding that $\varrho_\pm(1) = I$ and set $\mathcal{Q}_\pm = \varrho_\pm \oplus 0$, cf. (25). This is clearly a prequantization of P, by construction (Q4) is satisfied, and $\mathcal{Q}_\pm \restriction \mathrm{a}(1) = \varrho_\pm$ is faithful. Finally, it is straightforward to verify that D consists of analytic vectors for both $\varrho_\pm(pq)$ and $\varrho_\pm(q^2)$. Thus $\mathcal{Q}_\pm$ are the required quantization(s) of (P, P_1).

Remarks. 12. The $+$ quantization of a(1) is exactly what one obtains by geometrically quantizing M in the vertical polarization. Carrying this out, one gets $H = L^2(\mathbf{R}_+, dq)$ and

$$pq \mapsto -i\hbar\left(q\frac{d}{dq} + \frac{1}{2}\right), \quad q^2 \mapsto q^2.$$

The $+$ quantization is equivalent to this via the unitary transformation $L^2(\mathbf{R}_+, dq/q) \to L^2(\mathbf{R}_+, dq)$ which takes $f(q) \mapsto f(q)/\sqrt{q}$.

13. Note that $\mathrm{a}(1) \subset \mathrm{sp}(2, \mathbf{R})$. In fact, the $+$ quantization is equivalent to the restrictions to a(1) of the metaplectic representations of $\mathrm{sp}(2, \mathbf{R})$ on both $L^2_{\mathrm{even}}(\mathbf{R}, dq)$ and $L^2_{\mathrm{odd}}(\mathbf{R}, dq)$ (cf. §5.1 and Remark 12.

14. Since $\mathcal{Q}(P_{(2)}) = 0$, the quantization is somewhat 'trivial.' However, there are quantizations which are nonzero on $P_{(2)}$: for instance, set $\mathcal{Q}(x^k) = k\mathcal{Q}(x)$ for $k > 0$, $\mathcal{Q}(x^l y) = \mathcal{Q}(y)$, and $\mathcal{Q}(x^l y^m) = 0$ for $m > 1$.

15. This quantization of $T^*\mathbf{R}_+$ should be contrasted with that given in [Is, §4.5]. Also, we observe that this example is symplectomorphic to $\mathbf{R}^2$ with the basic algebra $\mathrm{span}\{p, e^{2q}\}$.

What makes this example work? After comparing it with our other examples, it is clear that this polynomial quantization exists because we can never decrease degree in P by taking Poisson brackets. Due to this we have (24) as opposed to merely

$$\{P_1, P_k\} \subset P^k.$$

We shall pursue this line of investigation in a more general setting in §7.

5.5 T^2

We have just exhibited a polynomial quantization of $T^*\mathbf{R}_+$. But we can do even more: Here we exhibit a quantization of the *full* Poisson algebra of the torus.

Consider the torus T^2 thought of as $\mathbf{R}^2/\mathbf{Z}^2$, with symplectic form

$$\omega = B\,dx \wedge dy.$$

We study the basic algebra $\mathfrak{t}$ generated by the set

$$\mathcal{T} = \{\sin 2\pi x, \cos 2\pi x, \sin 2\pi y, \cos 2\pi y\}.$$

We already know from Proposition 2 that there are no finite-dimensional basic algebras on the torus; thus $\mathfrak{t}$ is the most natural choice.

Now (T^2, ω) is (geometrically) quantizable provided $B = Nh$ for some nonzero integer N. Fix $N = 1$ and let L be the corresponding Kostant-Souriau prequantization line bundle over T^2 [Ki]. Then the space of smooth sections $\Gamma(L)$ can be identified with the space of "quasi-periodic" functions $\varphi \in C^\infty(\mathbf{R}^2,\mathbf{C})$ satisfying

$$\varphi(x+m,\ y+n) = e^{2\pi i m y}\varphi(x,\ y)\,, \quad n,\ m \in \mathbf{Z},$$

and the prequantization Hilbert space $\mathcal{H}$ with the (completion of) the set of those quasi-periodic φ which are L^2 on $[0,\ 1) \times [0,\ 1)$. The associated prequantization map $\mathcal{Q} : C^\infty(M) \to \mathrm{Op}\big(\Gamma(L)\big)$ (for a specific choice of connection on L) is defined by

$$\mathcal{Q}(f) = -i\hbar\left[\frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y} - \frac{i}{\hbar}x\right) - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right] + f. \tag{26}$$

As the torus is compact, these operators are essentially self-adjoint on $\Gamma(L) \subset \mathcal{H}$.

Theorem 17 *$\mathcal{Q}$ is a quantization of* $\big(C^\infty(T^2), \mathfrak{t}\big)$.

Proof. Since $\mathcal{Q}$ is a prequantization, we need only verify (Q4) and (Q5), (Q6) being obvious from (26). To this end it is convenient to use complex notation and view

$$\mathcal{T}_{\mathbf{C}} = \left\{e^{\pm 2\pi i x}, e^{\pm 2\pi i y}\right\}.$$

The analysis is simplified by applying the Weil-Brezin-Zak transform Z [Fo, §1.10] to the above data. Define a unitary map $Z : \mathcal{H} \to L^2(\mathbf{R})$ by

$$(Z\phi)(x) = \int_0^1 \phi(x, y)\,dy$$

with inverse

$$(Z^{-1}\psi)(x, y) = \sum_{m\in\mathbf{Z}} \psi(x+m)e^{-2\pi i m y}.$$

Under Z the domain $\Gamma(L)$ maps onto the Schwartz space $\mathcal{S}(\mathbf{R},\mathbf{C})$ [Ki]. Setting $A_\pm := Z\mathcal{Q}(e^{\pm 2\pi i x})Z^{-1}$ and $B_\pm := Z\mathcal{Q}(e^{\pm 2\pi i y})Z^{-1}$ we compute, as operators on $\mathcal{S}(\mathbf{R},\mathbf{C})$,

$$\begin{aligned}(A_\pm\psi)(x) &= e^{\pm 2\pi i x}(1 \mp 2\pi i x)\psi(x)\\ (B_\pm\psi)(x) &= \left(1 \mp 2\pi\hbar\frac{d}{dx}\right)\psi(x \pm 1).\end{aligned}$$

Then $A_{\pm}{}^* = \overline{A_{\mp}}$ on the domain $\{\psi \mid x\psi \in L^2(\mathbf{R})\}$, and likewise $B_{\pm}{}^* = \overline{B_{\mp}}$ on $\{\psi \mid d\psi/dx \in L^2(\mathbf{R})\}$.[10] In fact $\overline{A_{\pm}}$ and $\overline{B_{\pm}}$ are normal operators.

To show that $\mathcal{Q}(t)$ is an irreducible set, let us suppose that T is a bounded s.a. operator on $L^2(\mathbf{R})$ which strongly commutes with $\overline{A_{\pm}}$ and $\overline{B_{\pm}}$. Then T must commute (in the weak sense) with these operators on their respective domains.[11] Consequently T commutes with both

$$\overline{A_-}\,\overline{A_+} = I + 4\pi^2 x^2 \tag{27}$$

on the domain $\{\psi \mid x^2\psi \in L^2(\mathbf{R})\}$, and

$$\overline{B_-}\,\overline{B_+} = I - 4\pi^2\hbar^2 \frac{d^2}{dx^2}$$

on $\{\psi \mid d^2\psi/dx^2 \in L^2(\mathbf{R})\}$. From these equations we see that T commutes, and therefore strongly commutes, with the closures of two of the three generators of the metaplectic representation (9) of sp(2, $\mathbf{R}$) on $\mathcal{S}(\mathbf{R},\mathbf{C})$.

Suppose that μ denotes the metaplectic representation of the metaplectic group Mp(2,$\mathbf{R}$) on $L^2(\mathbf{R})$. We have in effect just established that T commutes with the one-parameter groups $\exp\big(is\,\overline{x^2}\big)$ and $\exp\big(-it\hbar^2\,\overline{d^2/dx^2}\big)$. Now classically the exponentials $\exp(sx^2)$ and $\exp(ty^2)$ generate Sp(2, $\mathbf{R}$) [GS, §4]. As Mp(2, $\mathbf{R}$) $\to$ Sp(2, $\mathbf{R}$) is a double covering, the corresponding exponentials in Mp(2,$\mathbf{R}$) generate a neighborhood of the identity in the metaplectic group. Since $\mu\big[\exp(sx^2)\big] = \exp\big(is\,\overline{x^2}\big)$ and $\mu\big[\exp(ty^2)\big] = \exp\big(-it\hbar^2\,\overline{d^2/dx^2}\big)$, it follows that T commutes with $\mu(\mathcal{M})$ for all $\mathcal{M}$ in a neighborhood of the identity in Mp(2,$\mathbf{R}$) and hence, as this group is connected, for all $\mathcal{M} \in$ Mp(2, $\mathbf{R}$).

Although the metaplectic representation μ is reducible, the subrepresentations μ_e and μ_o on each invariant summand of $L^2(\mathbf{R}) = L^2_e(\mathbf{R}) \oplus L^2_o(\mathbf{R})$ of even and odd functions are irreducible [Fo, §4.4]. Writing $T = P_eT + P_oT$, where P_e and P_o are the even and odd projectors, one has

$$[P_eT, \mu(\mathcal{M})] = 0 \tag{28}$$

for any $\mathcal{M} \in$ Mp(2,$\mathbf{R}$). It then follows from the irreducibility of the subrepresentation μ_e that $P_eT = k_eP_e + RP_o$ for some constant k_e and some operator $R : L^2_o(\mathbf{R}) \to L^2_e(\mathbf{R})$. Substituting this expression into (28) yields $[RP_o, \mu(\mathcal{M})] = 0$, and Schur's Lemma then implies that R is either an isomorphism or is zero. But R cannot be an isomorphism as the representations μ_e and μ_o are inequivalent [Fo, Thm. 4.56]. (Recall that two unitary representations are similar iff they are unitarily equivalent.) Thus $P_eT = k_eP_e$. Similarly $P_oT = k_oP_o$, whence $T = k_eP_e + k_oP_o$.

But now a short calculation shows that T commutes with

$$\overline{A_+} - \overline{A_-} = 2i(\sin 2\pi x - 2\pi x \cos 2\pi x)$$

[10] $d\psi/dx$ is to be understood in the sense of tempered distributions.

[11] Here and in what follows we use the fact that a bounded operator weakly commutes with an (unbounded) normal operator iff they strongly commute.

only if $k_e = k_o$. Thus T is a multiple of the identity, and so $\{A_\pm, B_\pm\}$ is an irreducible set, as was to be shown. Thus in particular (Q4) is satisfied.

For (Q5), we claim that the linear span of the Hermite functions form a dense set of separately analytic vectors for the e.s.a. components of $\{A_\pm, B_\pm\}$. From the expression above for $A_\pm$, it is clear that a vector will be analytic for the e.s.a. components of $A_\pm$ iff it is analytic for multiplication by x. But it is well known that the Hermite functions are analytic for this latter operator. The result for $B_\pm$ is obtained directly from this by means of the Fourier transform. □

Remark. 16. The proof also works for $N = -1$ but breaks down when $|N| \neq 1$ [Go3]. It is not known to what extent this theorem will remain valid in general (but see §7). As a consequence the classical limit is unclear; to compute it, one needs to study how the torus quantization behaves for large values of the quantum number N. But for $N > 1$, the *pre*quantizations with Chern class N may not be actual quantizations. If they are not, then one must construct a series of quantizations $\mathcal{Q}_1, \ldots, \mathcal{Q}_N, \ldots$ with $\mathcal{Q}_1 = \mathcal{Q}$ and see what happens to $\mathcal{Q}_N$ as N grows. Without these "interpolating quantizations," the classical limit of $\mathcal{Q}$ cannot be determined.

This full quantization has several remarkable features. (See [Go3, Ve] for detailed discussions). First, in previous examples the irreducibility requirement typically led to Von Neumann rules. But for T^2 both $\mathcal{Q}(f^2)$ and $\mathcal{Q}(f)^2$ are completely determined for any observable f by the simple fact that $\mathcal{Q}$ is a Kostant-Souriau prequantization; irreducibility is irrelevant. Moreover, one sees from (26) that $\mathcal{Q}(f^2)$ is a first order differential operator whereas $\mathcal{Q}(f)^2$ is of second order, indicating that this quantization will not respect the classical multiplicative structure at all.

This is particularly evident when one considers the classical identity $\cos^2 2\pi x + \sin^2 2\pi x = 1$, as emphasized by [Ve]. In view of (27)

$$[\mathcal{Q}(\cos 2\pi x)]^2 + [\mathcal{Q}(\sin 2\pi x)]^2 = I + 4\pi^2 x^2, \tag{29}$$

which bears scant resemblance to

$$\mathcal{Q}(\cos^2 2\pi x) + \mathcal{Q}(\sin^2 2\pi x) = I.$$

So the torus quantization dramatically violates Souriau's requirement that "the quantum spectrum of commuting observables should be concentrated on their classical range" [Zi]. As reflected by (29), the bounded observables $\cos 2\pi x$ and $\sin 2\pi x$ quantize to unbounded operators. While this may be seen as a flaw of the quantization, it cannot be helped: A theorem of Avez states that when the phase space M is compact, the only possible prequantization of $C^\infty(M)$ by *bounded* operators is $f \mapsto \bar{f} I$, where $\bar{f}$ is the mean value f [Av1]. If the torus is to be fully (and nontrivially) quantized, the representation space must thus be infinite-dimensional, whence a certain "amount" of unboundedness must ensue. So in this regard, the torus is not really behaving badly; there is a trade-off involved here.

Finally, the salient feature of this example is that the basic algebra $\mathfrak{t}$ is infinite-dimensional. This also did not happen in any of our other examples. As a consequence the irreducibility requirement on T^2 is substantially weaker than the corresponding requirements on either $\mathbf{R}^{2n}$, S^2, or T^*S^1, and is likely the underlying reason why $\mathcal{Q}$ provides a full quantization of $\big(C^\infty(T^2), \mathfrak{t}\big)$.

6 No-Go Theorems

Our treatment of the examples in §5 relied heavily on an intimate knowledge of the representations of the relevant basic algebras, and involved detailed calculations. Here we present some general results on the occurrence of obstructions. To accomplish this, we focus on the Lie and Poisson structures of basic algebras and the polynomial algebras they generate; necessarily, the representations of these objects now play a more subdued role. Background on Poisson algebras is given in [At, Gra1, Gra2].

The first key result appeared in 1974 and is due to Avez [Av1, Av2]. Recall that the mean value of $f \in C^\infty(M)$ is

$$\bar{f} = \frac{1}{\mathrm{vol}(M)} \int_M f\,\omega^n.$$

Theorem 18 *The only full prequantization of a compact symplectic manifold by bounded operators is given by $f \mapsto \bar{f} I$.*

Thus there can be no nontrivial finite-dimensional full prequantizations of a compact phase space. In the noncompact case, there is the following complementary result due to Doebner and Melsheimer [DM].

Proposition 19 *A nonzero infinite-dimensional representation of a noncompact finite-dimensional Lie algebra by skew-symmetric operators contains at least one unbounded operator.*

Combining these two results, we see that an infinite-dimensional quantization will necessarily involve unbounded operators. Whereas Theorem 18 uses the Poisson structure on $C^\infty(M)$, Proposition 19 is purely representation theoretic. We shall encounter this dichotomy again in §6.3.

The next advance was made by Ginzburg and Montgomery [GM], who generalized Avez's theorem to noncompact M. Let $C_c^\infty(M)$ denote the Poisson algebra of compactly supported smooth functions on M.

Theorem 20 *There is no nontrivial finite-dimensional Lie representation of $C_c^\infty(M)$.*

We do not give the proof, as it is similar to that of Theorem 22 following. Since a prequantization is simply a special type of Lie representation, Theorems 18 and 20 yield the no-go result:

Corollary 21 *There exists no nontrivial finite-dimensional full prequantization of any symplectic manifold M.*

Inspired by this work, we generalize both Theorem 20 and Corollary 21 to polynomial quantizations. Let $\mathfrak{b}$ be a basic algebra of observables and $P(\mathfrak{b})$ the Poisson algebra of polynomials generated by $\mathfrak{b}$. *Throughout this section we assume that $\mathfrak{b}$ is finite-dimensional*. We break the analysis up into four cases, depending upon whether $\mathfrak{b}$, or equivalently M, is compact and its representations are finite-dimensional. It turns out that we are able to obtain obstructions to quantizing $(P(\mathfrak{b}), \mathfrak{b})$ in three of these cases. And in the remaining case (viz. when $\mathfrak{b}$ is noncompact and the representation space is infinite-dimensional), there is no universal obstruction. In this gross sense, then, we have solved the Groenewold-Van Hove problem for polynomial quantizations.

6.1 M Compact, Finite-dimensional Representations

The main result is:

Theorem 22 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a compact symplectic manifold M. There exists no nontrivial finite-dimensional Lie representation of $P(\mathfrak{b})$.*

We begin with a purely algebraic lemma, whose proof is given in [GGG].

Lemma 4 *If L is a finite-codimensional Lie ideal of an infinite-dimensional Poisson algebra $\mathcal{P}$ with identity, then either L contains the commutator ideal $\{\mathcal{P}, \mathcal{P}\}$ or there is a maximal finite-codimensional associative ideal J of $\mathcal{P}$ such that $\{\mathcal{P}, \mathcal{P}\} \subseteq \mathcal{J}$.*

Proof of Theorem 22. Suppose that $\mathcal{Q}$ were a Lie representation of $P(\mathfrak{b})$ on some finite-dimensional vector space. Then $L = \ker \mathcal{Q}$ is a finite-codimensional Lie ideal of $P(\mathfrak{b})$. We will show that L has codimension at most 1, whence the representation is trivial. We accomplish this in two steps, by showing that:

(a) The derived ideal $\{P(\mathfrak{b}), P(\mathfrak{b})\}$ has codimension 1 in $P(\mathfrak{b})$, and

(b) $L \supseteq \{P(\mathfrak{b}), P(\mathfrak{b})\}$.

Let $A(\mathfrak{b})$ denote the Lie ideal of polynomials of zero mean. The decomposition $f \mapsto \bar{f} + (f - \bar{f})$ gives $P(\mathfrak{b}) = \mathbf{R} \oplus A(\mathfrak{b})$. Thus, if we prove that $\{P(\mathfrak{b}), P(\mathfrak{b})\} = A(\mathfrak{b})$, (a) will follow.

Using (6) along with Stokes' Theorem, we immediately have that $\{P(\mathfrak{b}), P(\mathfrak{b})\} \subseteq A(\mathfrak{b})$. To show the reverse inclusion, let $\{b_1, \ldots, b_N\}$ be a basis for $\mathfrak{b}$, so that

$$\{b_i, b_j\} = \sum_{k=1}^{N} c_{ij}^k b_k$$

for some constants c_{ij}^k. Following Avez [Av2], define the "symplectic Laplacian"

$$\Delta f = -\sum_{i=1}^{N} \{b_i, \{b_i, f\}\}.$$

It is clear from these two expressions and the Leibniz rule that the linear operator Δ maps $P^k(\mathfrak{b})$ into $A^k(\mathfrak{b})$. Furthermore, taking into account the transitivity of $\mathfrak{b}$, we can apply [Av2, Prop. 1(4)] to conclude that $\Delta f = 0$ only if f is constant. Thus for each $k \geq 0$, the decomposition $P^k(\mathfrak{b}) = \mathbf{R} \oplus A^k(\mathfrak{b})$ implies $\Delta(P^k(\mathfrak{b})) = A^k(\mathfrak{b})$. It follows that $A(\mathfrak{b}) \subseteq \{P(\mathfrak{b}), P(\mathfrak{b})\}$.

If (b) does not hold, then by Lemma 4 there must be a proper associative ideal J in $P(\mathfrak{b})$ with $\{P(\mathfrak{b}), P(\mathfrak{b})\} \subseteq J$. Since $\{P(\mathfrak{b}), P(\mathfrak{b})\} = A(\mathfrak{b})$ has codimension 1, $A(\mathfrak{b}) = J$. This is, however, impossible, since f^2 has zero mean only if $f = 0$. □

Corollary 23 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a compact symplectic manifold M. There exists no nontrivial finite-dimensional prequantization of $P(\mathfrak{b})$. In particular, there exists no nontrivial finite-dimensional quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

Although not surprising on mathematical grounds, since $P(\mathfrak{b})$ is "large," these corollaries do have physical import, as one expects the quantization of a compact phase space to yield a *finite*-dimensional Hilbert space.

6.2 M Compact, Infinite-dimensional Representations

We reduce this to the previous case as follows. Suppose that $\mathcal{Q}$ were a quantization of $(P(\mathfrak{b}), \mathfrak{b})$ on a Hilbert space. By conditions (Q3) and (Q5), $\mathcal{Q}(\mathfrak{b})$ can be exponentiated to a unitary representation of the simply connected Lie group B with Lie algebra $\mathfrak{b}$ (recall that $\mathfrak{b}$ is assumed finite-dimensional) which, according to (Q4), is irreducible. Since M is compact, B is compact. The representation space must thus be finite-dimensional, and so Corollary 23 applies. This proves

Theorem 24 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a compact symplectic manifold M. There exists no nontrivial quantization of $(P(\mathfrak{b}), \mathfrak{b})$.*

Thus, there is an obstruction to polynomially quantizing a compact symplectic manifold *regardless* of the dimensionality of the representation.

6.3 M Noncompact, Finite-dimensional Representations

Now suppose that M is noncompact. On physical grounds one expects a quantization of M, if it exists, to be infinite-dimensional. This is what we rigorously prove here, following [GGru2].

Already on the basis of representation theory, one can see that it will be difficult to obtain finite-dimensional quantizations of noncompact basic algebras. For instance, it is known that a Lie algebra admits a nontrivial finite-dimensional irreducible representation by skew-symmetric operators iff its Levi factor contains a nontrivial compact ideal [BaRa, Prop. 8.7.3]. Thus in particular a solvable algebra has no nontrivial finite-dimensional irreducible representations. We now prove that a *basic* algebra cannot admit any faithful finite-dimensional representations at all, irreducible or not.

Theorem 25 *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a noncompact symplectic manifold. Then $\mathfrak{b}$ has no faithful finite-dimensional representations by symmetric operators.*

Proof. We argue by contradiction. Suppose there exists a representation ϱ of $\mathfrak{b}$ on some $\mathbf{C}^k$. As $\varrho(\mathfrak{b})$ consists of hermitian matrices, ϱ is completely reducible. Since by assumption ϱ is faithful, one deduces from [Va, Theorem 3.16.3] that $\mathfrak{b}$ is reductive. By the comment following the the proof of Proposition 1, $\mathfrak{b}$ must then be semisimple.

Since M is noncompact, so is the simply connected covering group B of the semisimple algebra $\mathfrak{b}$. Now consider a unitary representation U of B on $\mathbf{C}^k$. Decompose B into a product $B_1 \times \cdots \times B_K$ of simple groups. Then (at least) one of these, say B_1, must be noncompact. But it is well-known that a connected, simple, noncompact Lie group has no nontrivial finite-dimensional unitary representations [BaRa, Theorem. 8.1.2]. Thus $U(b) = I$ for all $b \in B_1$. Since every finite-dimensional representation ϱ of $\mathfrak{b}$ by symmetric operators is a derived representation of some unitary representation U of B, it follows that $\varrho \restriction \mathfrak{b}_1 = 0$, and so ϱ cannot be faithful. □

Since every quantization of $(\mathcal{O}, \mathfrak{b})$ must be faithful on $\mathfrak{b}$, we conclude that *there is no nontrivial finite-dimensional quantization of* $(\mathcal{O}, \mathfrak{b})$ *on a noncompact symplectic manifold,* where $\mathcal{O}$ is *any* Lie algebra containing $\mathfrak{b}$. Combining this with Corollary 23 we can now assert—roughly speaking—that no symplectic manifold with a (finite-dimensional) basic algebra has a finite-dimensional quantization.

6.4 M Noncompact, Infinite-dimensional Representations

So far we have encountered obstructions in every instance. The present case is the exception: We know from §5.4 that there exists a polynomial quantization of $T^*\mathbf{R}_+$ with the basic algebra a(1).

The behavior exhibited by this example is not characteristic of solvable algebras such as a(1), since e(2) for the cylinder is also solvable yet exhibits an obstruction. Likewise, the Heisenberg algebra is nilpotent and is obstructed as well.

6.5 Discussion and Further Results

Theorem 24 asserts that the polynomial algebra $P(\mathfrak{b})$ generated by any finite-dimensional basic algebra $\mathfrak{b}$ on a compact symplectic manifold cannot be consistently quantized. As the torus illustrates, this need not be true if $\mathfrak{b}$ is allowed to be infinite-dimensional. Similarly Theorem 22 and Corollary 23 can fail when the representation space is allowed to be infinite-dimensional: as is well-known, full prequantizations exist provided ω/h is integral. Thus Corollary 23 and Theorem 24 are the optimal no-go results for compact phase spaces.

Proposition 2 enables us to identify $P(\mathfrak{b})$ with the Poisson algebra of polynomials on $\mathfrak{b}^*$ restricted to the coadjoint orbit M. In particular, we can take $M = S^2 \subset \mathrm{su}(2)^*$, $\mathfrak{b}$ the space of spherical harmonics of degree one ($\mathfrak{b} \cong \mathrm{su}(2)$), and $P(\mathfrak{b})$ the space of all spherical harmonics. Thus Theorem 10 follows immediately from Theorem 24. A similar analysis applies to $\mathbf{C}P^n \subset \mathrm{su}(n+1)^*$.

Our results in the compact case lean heavily on the algebraic structure of $P(\mathfrak{b})$, and in particular on the property that $\{P(\mathfrak{b}), P(\mathfrak{b})\}$ has codimension 1 in $P(\mathfrak{b})$. When M is noncompact, codim $\{P(\mathfrak{b}), P(\mathfrak{b})\}$ is not fixed; it takes on the values 0, 1, and even ∞ in examples. Thus the Poisson theoretic techniques that worked for compact phase spaces will not apply to noncompact ones. This partly explains why Theorem 25 is a representation theoretic result. Furthermore, this theorem hinges on the fact that $\mathfrak{b}$, being noncompact and semisimple, cannot have faithful finite-dimensional representations by Hermitian matrices. But when M is compact, $\mathfrak{b}$ is compact semisimple, and these algebras *do* have such representations. Thus the compact and noncompact cases require entirely different approaches.

It is useful to keep track of which hypotheses the five theorems in this section require. They all use (Q1), and Theorems 20 and 22 require only this. Theorem 18 needs (Q2) as well. Theorem 24 uses also (Q3)–(Q5), and lastly Theorem 25 assumes in addition only (Q6). We do not know if a no-go theorem can be proven in the noncompact, finite-dimensional case without the faithfulness assumption (Q6). Irreducibility was only used in the proof of Theorem 24; in the other cases the finite-dimensionality assumption forced the representation to be "small."

We are thus left with trying to understand the noncompact, infinite-dimensional case, which is naturally the most difficult one. Here one has little control over either the types of basic algebras that can appear (in examples they range from solvable to simple; compare Proposition 1), the structure of the polynomial algebras they generate (cf. the above), or their representations. Thus one should try a different tack. Following the lead of Joseph [Jo] (cf. §5.1), let us try to compare the algebraic structures of Poisson algebras on the one hand with associative algebras of operators with the commutator bracket on the other. Grabowski has adopted this approach, and has produced the following "algebraic" no-go theorem, which is proved in [GGra1].

Theorem 26 *Let $\mathcal{P}$ be a unital Poisson subalgebra of $C^\infty(M,\ \mathbf{C})$. If as a Lie algebra $\mathcal{P}$ is not commutative, it cannot be realized as an associative algebra with the commutator bracket.*

Apply this result to polynomial quantizations. Suppose that $\mathcal{Q} : P(\mathfrak{b}) \to \mathrm{Op}(D)$ were a quantization of $(P(\mathfrak{b}),\ \mathfrak{b})$ on some dense invariant domain D in a Hilbert space. By requiring $\mathcal{Q}$ to be complex linear, we may view it as a quantization of the complexified polynomial algebra $\mathcal{P} = P(\mathfrak{b})_{\mathbf{C}}$. Take $\mathcal{A} \subset \mathrm{Op}(\mathcal{D})$ to be the associative algebra generated over $\mathbf{C}$ by $\{\mathcal{Q}(f) \mid f \in \mathfrak{b}\}$ together with I (if $1 \not\in \mathfrak{b}$). If it can be shown that $\mathcal{Q}$ must be a Lie algebra isomorphism of $\mathcal{P}$ onto $\mathcal{A}$, then the algebraic no-go theorem will yield a contradiction.

To see how this works in practice, let us once again look at the Heisenberg algebra on $\mathbf{R}^2$. We shall prove inductively that

$$\mathcal{Q}(q^k p^l) = X^k Y^l + \sum_{k'+l'<k+l} a^{kl}_{k'l'} X^{k'} Y^{l'} \tag{30}$$

for some constants $a^{kl}_{k'l'}$, where $X = \mathcal{Q}(q),\ Y = \mathcal{Q}(p)$. Indeed,

$$\begin{aligned} [\mathcal{Q}(q^k p^l), Y] &= -i\hbar\mathcal{Q}(\{q^k p^l, p\}) \ = \ i\hbar k\mathcal{Q}(q^{k-1}p^l) \\ &= i\hbar k X^{k-1}Y^l + \textit{ lower degree terms}, \end{aligned}$$

where we have used the inductive assumption. Similarly

$$[\mathcal{Q}(q^k p^l), X] = -i\hbar l X^k Y^{l-1} + \textit{ lower degree terms}$$

and, due to $ad_X \circ ad_Y = ad_Y \circ ad_X$, we can find $F^{kl} = X^k Y^l +$ *lower degree terms*, which has the same commutators with X and Y as $\mathcal{Q}(q^k p^l)$. Since by Proposition 5 $\mathcal{Q}$ is algebraically irreducible in the sense that the the only elements of $\mathcal{A}$ which commute with $\mathcal{Q}(\mathrm{h}(2))$ are multiples of the identity, $\mathcal{Q}(q^k p^l)$ differs from F^{kl} by a constant, and that proves the inductive step. It now follows from (30) that $\mathcal{Q}$ is valued in $\mathcal{A}$, and that it is surjective.

It is easy to see that every nontrivial Lie ideal of $P = \mathbf{R}[q,\ p]$ intersects P^1. In particular if $\ker \mathcal{Q} \neq \{0\}$, then we contradict (Q6). Thus $\mathcal{Q}$ must be injective, and so we have an algebraic obstruction to quantizing $(P,\ P^1)$.

The main difficulty in correlating the algebraic approach with our previous considerations is that there is no *a priori* reason why $\mathcal{Q}(\mathcal{P}) \subset \mathcal{A}$. This requirement is reminiscent of a Von Neumann rule, so one might expect that irreducibility can be used to establish this inclusion as in the example above; this is actually the case for nilpotent basic algebras since then $\mathcal{A}$ must be isomorphic to a Weyl algebra, cf. [GGra1, Prop. 8] for the details on how this works.

In fact, the entire argument for h$(2n)$ can be extended to *any* nilpotent basic algebra [GGra1]. First we observe that if $\mathfrak{b}$ is a finite-dimensional nilpotent basic algebra on M, then M must be symplectomorphic to some $\mathbf{R}^{2n}$. (This follows from Proposition 2 and the well-known fact that a coadjoint orbit of a nilpotent Lie algebra must be symplectomorphic to some $\mathbf{R}^{2n}$ [Wi].) The canonical example of a nilpotent basic algebra on $\mathbf{R}^{2n}$ is of course h$(2n)$. It is not difficult to see that, up to isomorphism, h(2) is the only nilpotent *basic* algebra on $\mathbf{R}^2$. This is not true in higher dimensions, however:

$$\text{span}\{1, q_1, p_2, q_1 p_2 + q_2, p_1\}$$

is a nilpotent basic algebra on $\mathbf{R}^4$ which is not isomorphic to h(4).

Theorem 27 *Let $\mathfrak{b}$ be a finite-dimensional nilpotent basic algebra on a connected symplectic manifold. Then there is no quantization of* $(P(\mathfrak{b}), \mathfrak{b})$.

Theorem 27 does not carry over to more general basic algebras. Indeed, it is not true for a(1) on $T^*\mathbf{R}_+$ as we have already seen; note that a(1) is the simplest example of a solvable algebra which is not nilpotent.

Regardless, it appears that this algebraic approach holds promise; at least it enables us to partially suppress the representational aspects over which we have little control.

We turn now to the other extreme case, viz. when $\mathfrak{b}$ is semisimple. Identifying coadjoint and adjoint orbits by means of the Killing form, we know that M must be an adjoint orbit in $\mathfrak{b}$. Unlike the compact case, however, it is difficult to say which adjoint orbits are "basic," i.e. admit $\mathfrak{b}$ as a basic algebra. For example, in sl$(2, \mathbf{R})$ the nonzero adjoint orbits of which are either open half-cones, hyperboloids of one sheet, or components of hyperboloids of two sheets. One can verify that the open half-cones as well as the hyperboloids of one sheet are basic for sl$(2, \mathbf{R})$, but that the components of the hyperboloids of two sheets are not. (Instead they are basic for the subalgebra of upper triangular matrices.) Thus there is no apparent analogue of Proposition 3 in this context. Nonetheless, we are able to prove [GGra2]

Theorem 28 *Let M be a basic nilpotent adjoint orbit in $\mathfrak{b}$, where $\mathfrak{b}$ is a finite-dimensional semisimple Lie algebra. Then there* is *a nontrivial quantization of* $(P(\mathfrak{b}), \mathfrak{b})$.

This result is actually a corollary of Theorem 30 in §7.

Other than Theorem 28, nothing firm is known regarding obstructions to obtaining infinite-dimensional polynomial quantizations of noncompact semisimple basic algebras (but see Conjecture 1 in §7).

7 Speculations

In view of the theorems in the previous section, obstructions to quantization are guaranteed to exist except when the phase space is noncompact and the representations under consideration are infinite-dimensional. Three of our examples fall into this category: $\mathbf{R}^{2n}$, T^*S^1, and $T^*\mathbf{R}_+$. The first two exhibit obstructions, while the last does not. Comparing the behavior of these examples, as well as that of S^2, which is also obstructed, we attempt to extract the key features which govern the appearance of obstructions to a polynomial quantization.

Of course, any conclusions that we can draw at this point are necessarily tentative, due to the paucity of examples against which to test them. There are also various aspects of these examples that still are not completely understood. Nonetheless, some interesting observations can be made, which may prove helpful in subsequent investigations.

A detailed look at the derivations of the Von Neumann rules for $\mathbf{R}^{2n}$, T^*S^1, and S^2, and how they engender obstructions, shows that the controlling factor is apparently that one can decrease degree *in* $P(\mathfrak{b})$ by taking Poisson brackets. This is particularly evident in the classical Poisson bracket relations (16) and (18), and (22), which led to the contradictions for S^2 and T^*S^1, respectively. The situation for $\mathbf{R}^2$ is subtler, but one can spot this phenomenon in the proof of Lemma 1. The analysis in §5.4 shows that it is *not* possible to decrease degree in $P(\mathfrak{b})$ by taking Poisson brackets on $T^*\mathbf{R}_+$.

There are two—and only two—circumstances under which taking Poisson brackets in $P(\mathfrak{b})$ can decrease degree:[12]

(D1) $1 \in \{P(\mathfrak{b}), P(\mathfrak{b})\}$, and

(D2) There exist *nonzero* Casimirs in the symmetric algebra $S(\mathfrak{b})$ of $\mathfrak{b}$.[13]

According to the discussion at the end of §3, (D2) implies that $P(\mathfrak{b})$ is not freely generated by $\mathfrak{b}$ as an associative algebra. Specifically, $S(\mathfrak{b})$ will have nonzero Casimirs whenever $\mathfrak{b}$ is semisimple and has a nonzero compact ideal, and in particular when it is compact (cf. the comment following the proof of Proposition 1). At the other extreme, when $\mathfrak{b}$ is nilpotent, (D1) holds. Indeed, a nilpotent algebra has a center, and (B3) implies that this center consists of constants. An examination of the descending central series for $\mathfrak{b}$ then shows that $1 \in \{\mathfrak{b}, \mathfrak{b}\}$. In the examples, $\mathbf{R}^{2n}$ satisfies (D1) but not (D2), S^2 satisfies (D2) by virtue of (10) but not (D1), and T^*S^1 satisfies both because of

$$1 = \cos^2\theta + \sin^2\theta = \tfrac{1}{2}\big\{\{\ell^2, \sin\theta\}, \sin\theta\big\} + \tfrac{1}{2}\big\{\{\ell^2, \cos\theta\}, \cos\theta\big\}.$$

On the other hand, $T^*\mathbf{R}_+$ satisfies neither condition.

On the basis of this "anecdotal" evidence, we propose that a general Groenewold-Van Hove theorem takes the form:

Conjecture 1 *Let M be a symplectic manifold with a finite-dimensional basic algebra $\mathfrak{b}$. Suppose that the polynomial algebra $P(\mathfrak{b})$ satisfies either* (D1) *or* (D2)*. Then there is no nontrivial quantization of* $(P(\mathfrak{b}), \mathfrak{b})$.

Indeed, is possible to directly verify this conjecture under certain circumstances.

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[12] *A priori,* a third circumstance would be if $1 \in \mathfrak{b}$. Using the minimality condition (B4), it is not difficult to prove that then $1 \in \{\mathfrak{b}, \mathfrak{b}\}$, so this is actually a subcase of (D1).

[13] By this we mean that if C is a Casimir, then its projection to $P(\mathfrak{b})$ is nonvanishing; in other words, when viewed as a function on M, C is nonzero.

Theorem 29 *Conjecture* 1 *is valid when either* M *is compact or the representation space is finite-dimensional.*

Proof. According to Proposition 1, when M is compact $\mathfrak{b}$ is compact. Just as in §6.2 we may then use (Q3)–(Q5) to reduce the case of infinite-dimensional representations to that of finite-dimensional ones. Thus it suffices to prove the theorem for the case when $\mathcal{Q}$ is a quantization of $P(\mathfrak{b})$ on a finite-dimensional Hilbert space, whence $L = \ker \mathcal{Q}$ has finite codimension in $P(\mathfrak{b})$.

Arguing as in the proof of Theorem 25, we have from (Q6) that $\mathfrak{b}$ is semisimple.

We apply Lemma 4 to L. First suppose that $\{P(\mathfrak{b}), P(\mathfrak{b})\} \subseteq L$. Then semisimplicity gives $\mathfrak{b} = \{\mathfrak{b}, \mathfrak{b}\} \subset L$, and so $\mathcal{Q} \restriction \mathfrak{b} = 0$, which contradicts (Q6).

Thus there must exist a maximal finite-codimensional associative ideal J in $P(\mathfrak{b})$ with $\{P(\mathfrak{b}), P(\mathfrak{b})\} \subseteq J$. If (D1) holds, then $1 \in J$, which cannot be as J is proper. Now suppose (D2) holds, so that there is a nonzero Casimir $C \in S(\mathfrak{b})$. If ρ is the projection $S(\mathfrak{b}) \to P(\mathfrak{b})$, then $K = \rho^{-1}(J)$ is a maximal finite-codimensional associative ideal in $S(\mathfrak{b})$ with $\{S(\mathfrak{b}), S(\mathfrak{b})\} \subseteq K$. Since $\mathfrak{b} = \{\mathfrak{b}, \mathfrak{b}\} \subset \{S(\mathfrak{b}), S(\mathfrak{b})\} \subseteq K$, and since $1 \notin K$ (as K is proper), it follows that K is the associative ideal generated by $\mathfrak{b}$. (Actually, this shows that $S(\mathfrak{b}) = \mathbf{R} \oplus K$.)

Since C is nonzero, transitivity implies that $\rho(C) = c$ for some constant $c \neq 0$. By the definition of a Casimir and the above remarks $C \in K$. But then $C - c \notin K$, which is a contradiction since $C - c \in \ker \rho \subset K$. □

While similar to the proof of Theorem 22, this proof has a key advantage: It does not require us to know the detailed structure of the commutator ideal (which we do not, when $\mathfrak{b}$ is noncompact).

Thus Conjecture 1 is consistent with the results of §6. Furthermore, the hypotheses of Conjecture 1 are certainly *necessary*.

Theorem 30 *Suppose that the polynomial algebra* $P(\mathfrak{b})$ *satisfies neither condition* (D1) *nor* (D2). *Then any nontrivial quantization of* $\mathfrak{b}$ *extends to a quantization of* $(P(\mathfrak{b}), \mathfrak{b})$.

Proof. For if $P(\mathfrak{b})$ satisfies neither of these conditions, then the notion of homogeneous polynomial is well-defined and it is not possible to lower degree in $P(\mathfrak{b})$ by taking Poisson brackets. Just as in §5.4, $P_{(2)}(\mathfrak{b})$ is then an ideal in $P(\mathfrak{b})$, and $P(\mathfrak{b}) = P^1(\mathfrak{b}) \ltimes P_{(2)}(\mathfrak{b})$. Let ϱ be the assumed representation of $\mathfrak{b}$; this extends to a representation of $P^1(\mathfrak{b})$. Then $\mathcal{Q} = \varrho \oplus 0$ is the required quantization of $(P(\mathfrak{b}), \mathfrak{b})$. □

As a specific application of this result [GGra2], suppose that $\mathfrak{b}$ is semisimple and M is a basic nilpotent adjoint orbit in $\mathfrak{b}$. (For instance, when $\mathfrak{b} = \mathrm{sl}(2, \mathbf{R})$ one may take M to be either of the open half-cones.) Now a nilpotent orbit is invariant under the scaling action of $\mathbf{R}_+$ on $\mathfrak{b}$. But by Chevalley's theorem, the ideal of Casimirs of $S(\mathfrak{b})$ is generated by a finite collection of homogeneous polynomials of degree two or higher. Since Casimirs are constant on adjoint orbits, they must therefore vanish on conical ones. Thus (D2) cannot be satisfied. Furthermore, (D1) cannot hold either: If $1 \in \{P(\mathfrak{b}), P(\mathfrak{b})\}$, then $1 = \sum_{i=1}^k \{f_i, g_i\}$ for some polynomials f_i, g_i, whence $\sum_{i=1}^k \{f_i, g_i\}$ is a nonzero Casimir. So such orbits are polynomially quantizable.

Lastly, we observe that the finite-dimensionality assumption on $\mathfrak{b}$ in Conjecture 1 is necessary as well: The symmetric algebra $S(\mathfrak{t})$ on T^2 certainly contains Casimirs, but violates the conjecture.

Of our five examples, the torus is clearly much different than the others. It is not a Hamiltonian homogeneous space, and the basic algebra $\mathfrak{t}$ is infinite-dimensional. Because of this, the irreducibility requirement (Q4) loses much of its force – so much so that it precludes the existence of an obstruction. So it seems equally reasonable to propose

Conjecture 2 *Let M be a symplectic manifold and $\mathfrak{b}$ a basic algebra with $P^1(\mathfrak{b})$ dense in $C^\infty(M)$.*[14] *Then there exists a nontrivial quantization of $(C^\infty(M), \mathfrak{b})$.*

A necessary condition for $\mathcal{Q}$ to be a full quantization of $(C^\infty(M), \mathfrak{b})$ is that $\mathcal{Q}$ represent $C^\infty(M)$ itself irreducibly. It turns out [Ch2, Tu] that this is so for all Kostant-Souriau prequantizations[15]; thus it is natural to consider the case when M is prequantizable in this sense. In fact, in this context [Tu] gives even more:

Proposition 31 *Let M be an integral symplectic manifold, L a Kostant-Souriau prequantization line bundle over M and $\mathcal{Q}_L$ the corresponding prequantization map. Let $\mathfrak{b}$ be a basic algebra with $P^1(\mathfrak{b})$ dense in $C^\infty(M)$. Then $\mathcal{Q}_L$ represents $\mathfrak{b}$ irreducibly on the domain which consists of compactly supported sections of L.*

Set $D_c = \Gamma(L)_c$, the compactly supported sections of L. By construction $\mathcal{Q}_L : C^\infty(M) \to \mathrm{Op}(D_c)$ satisfies (Q1)–(Q3) and (Q6). This proposition states that $\mathcal{Q}_L$ satisfies (Q4) as well. Thus to obtain a full quantization it remains to verify (Q5)—perhaps on some appropriately chosen coextensive domain D; unfortunately, it does not seem possible to do this except in specific instances. A first test would be to understand what happens for $\big(C^\infty(T^2), \mathfrak{t}\big)$ with $|N| \neq 1$. In any event, Proposition 31 does provide a certain amount of support for Conjecture 2.

The "gray area" between these two conjectures consists of symplectic manifolds with basic algebras $\mathfrak{b}$ for which $P^1(\mathfrak{b})$ is infinite-dimensional, yet not dense in $C^\infty(M)$. Maybe the infinite-dimensionality of $\mathfrak{b}$ alone is enough to guarantee the existence of a full quantization?

Completing the proof of Conjecture 1—that is, when M is noncompact and the quantizations are infinite-dimensional—seems to be a difficult problem. Perhaps the "algebraic approach" sketched at the end of §6 will continue to prove useful. It will likely be necessary to work through a few more examples of Groenewold-Van Hove obstructions before one is able to gain sufficient insight into this problem. One example worth studying are the various coadjoint orbits for $\mathrm{sp}(2n, \mathbf{R})$. As well, it would be useful to consider basic algebras of a more general type than the ones we have encountered thus far (which were all either solvable or semisimple). We have also restricted

[14] We use $P^1(\mathfrak{b})$ here to ensure that 1 is present: On the torus, $\mathfrak{b}$ consists only of trigonometric polynomials of mean zero, whereas $P^1(\mathfrak{b})$ comprises all trigonometric polynomials.

[15] However, there are other prequantizations which do not represent $C^\infty(M)$ irreducibly; for instance, the prequantization of Avez [Av3, Ch3].

consideration to polynomial subalgebras to a large extent, but there are other subalgebras $\mathcal{O}$ which are of interest (e.g., on $\mathbf{R}^{2n}$, those functions which are constant outside some compact set [Ch3]).

A negative answer to the conjecture might indicate that one should strengthen the conditions defining a basic algebra by, e.g., replacing (B3) by (C2) as discussed in §3 (although this specific change would eliminate a(1) on $T^*\mathbf{R}_+$ from the ranks of basic algebras.) One could also modify the axioms for a quantization, for instance by adopting Souriau's requirement that classical observables with bounded spectra should quantize to operators with bounded spectra. Or, if the conjecture still seems undecidable, perhaps one should abandon the definition of a quantization map solely in terms of basic algebras and consider an alternative. However, the two other ways to define a quantization map listed previously suffer from serious flaws. If one imposes Von Neumann rules at the outset, then one tends to run into difficulties rather quickly—especially if one tries to enforce the rules on all of $C^\infty(M)$ and not some basic algebra thereof—as was shown in §5.1. Furthermore, it is unclear what form Von Neumann rules should take in general, as is illustrated by the unintuitive rules (14) for the sphere. For instance, mimicking the situation for $\mathbf{R}^{2n}$, one might simply postulate that $\mathcal{Q}(f^2) = \mathcal{Q}(f)^2$ for $f \in \mathrm{su}(2)$. While the squaring rule for angular momentum is compatible with (14), one would still "miss" various possibilities (corresponding to the freedom in the choice of parameters $a,\ c$), which do occur in specific representations.[16] And in the case of the torus, Von Neumann rules are effectively moot, since the explicit prequantization map $\mathcal{Q}$ itself determines the quantization of every observable. Von Neumann rules are also irrelevant in the $T^*\mathbf{R}_+$ example, because of the peculiar structure (25) of $P(a(1))$. All in all, it appears as if the Von Neumann rules play a secondary role; the basic algebra $\mathfrak{b}$ is the primary object. It is also more compelling physically and pleasing æsthetically to require $\mathcal{Q}$ to satisfy an irreducibility requirement than a Von Neumann rule. Still, one can argue that such rules serve an important purpose [As, Ve].

There are problems with the polarization approach as well. For one thing, symplectic manifolds need not be polarizable [Go2]. This rare occurrence notwithstanding, there are quantizations which cannot be obtained by polarizing a prequantization: A well-known example is the extended metaplectic quantization of $\big(\mathrm{hsp}(2n,\mathbf{R}),\mathrm{h}(2n)\big)$ [Bl2]. As we shall see presently, the specific predictions of geometric quantization theory are also off the mark in a number of instances.

Finally, it should be emphasized that these three approaches to quantization typically lead to obstructions in one way or another. We have already seen in §5 that Von Neumann rules play a crucial role in deriving the Groenewold-Van Hove obstructions for $\mathbf{R}^{2n}$, S^2 and T^*S^1. In the context of polarizations, the only observables which are consistently quantizable *ab initio* are those whose Hamiltonian vector fields preserve a given polarization [Bl1, Wo]. While this does not preclude the possibility of quantizing more general observables, attempts to quantize observables outside this class in specific examples usually result in inconsistencies. In *all* instances, the set of *a priori* quantizable observables relative to a given polarization forms a proper Lie subalgebra of the Poisson algebra of the given symplectic manifold. This observation provides further corroboration that Groenewold-Van Hove obstructions to quantization

[16] Because of this, [KLZ] would refer to (14) as "*non*-Neumann rules"!

should be the rule rather than the exception.

Setting aside the question of the existence of obstructions, let us now suppose that there is an obstruction to, say, a polynomial quantization, so that it is impossible to consistently quantize all of $P(\mathfrak{b})$. The question is: What are the largest Lie subalgebras $\mathcal{O} \subset P(\mathfrak{b})$ containing the given basic algebra $\mathfrak{b}$ such that $(\mathcal{O}, \mathfrak{b})$ can be quantized? Modulo technical issues, given a representation $\mathcal{Q}$ of $\mathfrak{b}$ on a Hilbert space $\mathcal{H}$, one ought to be able to induce a representation of its Lie normalizer $\mathfrak{n}(\mathfrak{b})$ in $P(\mathfrak{b})$ on $\mathcal{H}$. (Indeed, the structure $(\mathfrak{n}(\mathfrak{b}), \mathfrak{b})$ brings to mind an infinitesimal version of a Mackey system of imprimitivity [BaRa].) Thus it seems reasonable to assert:

Conjecture 3 *Let* $\mathfrak{b}$ *be a finite-dimensional basic algebra. Then every quantization of* $\mathfrak{b}$ *can be extended to a quantization of* $(\mathfrak{n}(\mathfrak{b}), \mathfrak{b})$.[17]

This is in exact agreement with the examples. In particular, for $\mathbf{R}^{2n}$ one has $\mathfrak{n}\big(\mathrm{h}(2n)\big) = \mathrm{hsp}(2n, \mathbf{R})$, and for S^2 one computes $\mathfrak{n}\big(\mathrm{su}(2)\big) = \mathrm{u}(2)$. In both cases, we have shown that these normalizers are in fact the maximal polynomial subalgebras that can be consistently quantized. It is therefore tempting to conjecture that:

No nontrivial quantization of $(\mathfrak{n}(\mathfrak{b}), \mathfrak{b})$ *can be extended beyond* $\mathfrak{n}(\mathfrak{b})$.

If true, this would point where to look for a Groenewold-Van Hove contradiction, viz. just outside the normalizer. Alas, this is *false*: For the cylinder $\mathfrak{n}(\mathrm{e}(2)) = \mathbf{R} \oplus \mathrm{e}(2)$. But from §5.3, we know that the representation (19) can be extended, in infinitely many ways, to the quantizations (23) of (L^1, P_1), where L^1 is the Lie subalgebra of observables which are affine in the angular momentum ℓ. It is not clear how one could "discover" this subalgebra given just the basic algebra $\mathrm{e}(2)$ (but see below). The situation for $T^*\mathbf{R}_+$ is of course even worse than for T^*S^1. An outstanding problem is therefore to determine the maximal Lie subalgebras of quantizable observables.

This is reminiscent of the situation in geometric quantization with respect to polarizations. Suppose that $\mathcal{A}$ is a polarization of $C^\infty(M, \mathbf{C})$. Then one knows that one can consistently quantize those observables which preserve $\mathcal{A}$, i.e., which belong to the real part of $\mathfrak{n}(\mathcal{A})$ [Bl1, Wo]. In this way one obtains a "lower bound" on the set of quantizable functions for a given polarization. If one takes the antiholomorphic polarization on S^2, then it turns out that the set of *a priori* quantizable functions obtained in this manner is precisely the $\mathrm{u}(2)$ subalgebra $\mathrm{span}\{1, S_1, S_2, S_3\}$. But it may happen that the real part of $\mathfrak{n}(\mathcal{A})$ is too small, as for $\mathbf{R}^{2n}$ with the antiholomorphic polarization. In this case the real part of $\mathfrak{n}(\mathcal{A})$ is only a proper subalgebra of P^2, and in particular is not maximal. This illustrates the fact, alluded to previously, that the extended metaplectic representation cannot be derived via geometric quantization. Furthermore, in the case of the torus, introducing a polarization will drastically cut down the set of *a priori* quantizable functions, which is at odds with the existence of a full quantization of this space. So geometric quantization is not a reliable guide insofar as computing maximally quantizable Lie subalgebras of observables. On the other hand, the position subalgebra $S = \{f(q)p + g(q)\}$ (resp. L^1) is just the normalizer of the vertical polarization $\mathcal{A} = \{h(q)\}$ on $\mathbf{R}^2$ (resp. $\{h(\theta)\}$ on T^*S^1), so these subalgebras find natural interpretations in the context of polarizations.

[17] In [GGT] quantizations which satisfy this condition are termed "strong."

Clearly, there must be some connection between polarizations and basic algebras that awaits elucidation. It would be interesting to determine if there is a way to recast the Groenewold-Van Hove results in terms of polarizations. It would also be worthwhile, assuming that it is somehow possible to predict the maximal set(s) of quantizable observables *a priori*, to see whether one can use this knowledge to refine geometric quantization theory, or to develop a new quantization procedure, which is adapted to the Groenewold-Van Hove obstruction in that it will automatically be able to quantize this maximal set.

Here we have focused on the quantization of symplectic manifolds. It is natural to wonder to what extent these results will carry over to Poisson manifolds, or even to abstract Poisson algebras.

One of our goals in this paper was to obtain results which are independent of the particular quantization scheme employed, as long as it is Hilbert-space based. Therefore it is interesting that some of the go and no-go results described in this proposal have direct analogues in deformation quantization theory, since this theory was developed, at least in part, to avoid the use of Hilbert spaces altogether [BFFLS]. So for example, the no-go result for S^2 is mirrored by the fact that there are no strict SU(2)-invariant deformation quantizations of $C^\infty(S^2)$ [Ri1], while the go theorem for T^2 has as a counterpart the result that there do exist strict deformation quantizations of the torus [Ri1]. It is generally believed that the existence of Groenewold-Van Hove obstructions necessitates a weakening of the Poisson bracket $\to$ commutator rule (by insisting that it hold only to order $\hbar$), but these observations indicate that this may not suffice to remove the obstructions. There are undoubtedly important things to be learned by getting to the heart of this analogy.

References

[AM] Abraham, R. & Marsden, J.E. [1978] *Foundations of Mechanics*. Second Ed. (Benjamin-Cummings, Reading, MA).

[AA] Aldaya, V. & Azcárraga, J.A. [1982] Quantization as a consequence of the symmetry group: An approach to geometric quantization. *J. Math. Phys.* **23**, 1297–1305.

[ADT] Angermann, B., Doebner, H.-D. & Tolar, J. [1983] Quantum kinematics on smooth manifolds. In: *Nonlinear Partial Differential Operators and Quantization Procedures*. Andersson, S.I. & Doebner, H.-D., Eds. *Lecture Notes in Math.* **1087**, 171–208.

[AB] Arens, R. & Babbit, D. [1965] Algebraic difficulties of preserving dynamical relations when forming quantum-mechanical operators *J. Math. Phys.* **6**, 1071–1075.

[As] Ashtekar, A. [1980] On the relation between classical and quantum variables. *Commun. Math. Phys.* **71**, 59–64.

[At] Atkin, C.J. [1984] A note on the algebra of Poisson brackets. *Math. Proc. Camb. Phil. Soc.* **96**, 45–60.

[Av1] Avez, A. [1974] Représentation de l'algèbre de Lie des symplectomorphismes par des opérateurs bornés. *C.R. Acad. Sc. Paris Sér. A,* **279**, 785–787.

[Av2] Avez, A. [1974–1975] Remarques sur les automorphismes infinitésimaux des variétés symplectiques compactes. *Rend. Sem. Mat. Univers. Politecn. Torino,* **33**, 5–12.

[Av3] Avez, A. [1980] Symplectic group, quantum mechanics and Anosov's systems. In: *Dynamical Systems and Microphysics*. Blaquiere, A. et al., Eds. (Springer, New York) 301–324.

[BaRa] Barut, A.O. & Rączka, R. [1986] *Theory of Group Representations and Applications*. Second Ed. (World Scientific, Singapore).

[BFFLS] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., & Sternheimer, D. [1978] Deformation theory and quantization I, II. *Ann. Phys.* **110**, 61–110, 111–151.

[Bl1] Blattner, R.J. [1983] On geometric quantization. In: *Non-Linear Partial Differential Operators and Quantization Procedures*. Andersson, S.I. & Doebner, H.-D., Eds. *Lecture Notes in Math.* **1087**, 209–241.

[Bl2] Blattner, R.J. [1991] Some remarks on quantization. In: *Symplectic Geometry and Mathematical Physics*. Donato, P. et al., Eds. *Progress in Math.* **99** (Birkhäuser, Boston) 37–47.

[BrRo] Bratteli, O. & Robinson, D.W. [1979] *Operator Algebras and Quantum Statistical Mechanics I.* (Springer, New York).

[Ch1] Chernoff, P.R. [1981] Mathematical obstructions to quantization. *Hadronic J.* **4**, 879–898.

[Ch2] Chernoff, P.R. [1988] Seminar on representations of diffeomorphism groups. Unpublished notes.

[Ch3] Chernoff, P.R. [1995] Irreducible representations of infinite dimensional transformation groups and Lie algebras I. *J. Funct. Anal.* **130**, 255–282.

[Co] Cohen, L. [1966] Generalized phase-space distribution functions. *J. Math. Phys.* **7**, 781–786.

[Di] Dirac, P.A.M. [1967] *The Principles of Quantum Mechanics*. Revised Fourth Ed. (Oxford Univ. Press, Oxford).

[DM] Doebner, H.D. and Melsheimer, O. [1968] Limitable dynamical groups in quantum mechanics I. *J. Math. Phys.* **9**, 1638–1656.

[Em] Emch, G.G. [1972] *Algebraic Methods in Statistical Mechanics and Quantum Field Theory.* (Wiley, New York).

[Fi] Filippini, R.J. [1995] The symplectic geometry of the theorems of Borel-Weil and Peter-Weyl. Thesis, University of California at Berkeley.

[Fl] Flato, M. [1976] Theory of analytic vectors and applications. In: *Mathematical Physics and Physical Mathematics*. Maurin, K. & Rączka, R., Eds. (Reidel, Dordrecht) 231–250.

[Fo] Folland, G.B. [1989] *Harmonic Analysis in Phase Space*. Ann. Math. Ser. **122** (Princeton University Press, Princeton).

[Fr] Fronsdal, C. [1978] Some ideas about quantization. *Rep. Math. Phys.* **15**, 111–145.

[GM] Ginzburg, V.L. & Montgomery, R. [1997] Geometric quantization and no-go theorems. Preprint dg-ga/9703010.

[GJ] Glimm, J. & Jaffe, A. [1981] *Quantum Physics. A Functional Integral Point of View*. (Springer Verlag, New York).

[Go1] Gotay, M.J. [1980] Functorial geometric quantization and Van Hove's theorem. *Int. J. Theor. Phys.* **19**, 139–161.

[Go2] Gotay, M.J. [1987] A class of non-polarizable symplectic manifolds. *Mh. Math.* **103**, 27–30.

[Go3] Gotay, M.J. [1995] On a full quantization of the torus. In: *Quantization, Coherent States and Complex Structures*, Antoine, J.-P. et al., Eds. (Plenum, New York) 55–62.

[Go4] Gotay, M.J. [1999] On the Groenewold-Van Hove problem for $\mathbf{R}^{2n}$. *J. Math. Phys.* **40**, 2107–2116.

[GGra1] Gotay, M.J. & Grabowski, J. [1999] On quantizing nilpotent and solvable basic algebras. Preprint math-ph/9902012.

[GGra2] Gotay, M.J. & Grabowski, J. [2000] On quantizing semisimple basic algebras. In preparation.

[GGG] Gotay, M.J., Grabowski, J., & Grundling, H.B. [2000] An obstruction to quantizing compact symplectic manifolds. *Proc. Amer. Math. Soc.* **128**, 237–243.

[GGru1] Gotay, M.J. & Grundling, H.B. [1997] On quantizing T^*S^1. *Rep. Math. Phys.* **40**, 107–123.

[GGru2] Gotay, M.J. & Grundling, H. [1999] Nonexistence of finite-dimensional quantizations of a noncompact symplectic manifold. In: *Differential Geometry and Applications*, Kolář, I. et al., Eds. (Masaryk Univ., Brno) 593–596.

[GGH] Gotay, M.J., Grundling, H., & Hurst, C.A. [1996] A Groenewold-Van Hove theorem for S^2. *Trans. Amer. Math. Soc.* **348**, 1579–1597.

[GGT] Gotay, M.J., Grundling, H., & Tuynman, G.T. [1996] Obstruction results in quantization theory. *J. Nonlinear Sci.* **6**, 469–498.

[Gra1] Grabowski, J. [1978] Isomorphisms and ideals of the Lie algebras of vector fields. *Invent. Math.* **50**, 13–33.

[Gra2] Grabowski, J. [1985] The Lie structure of C^* and Poisson algebras. *Studia Math.* **81**, 259–270.

[Gro] Groenewold, H.J. [1946] On the principles of elementary quantum mechanics. *Physica* **12**, 405–460.

[GS] Guillemin, V. & Sternberg, S. [1984] *Symplectic Techniques in Physics*. (Cambridge Univ. Press, Cambridge).

[HM] Helton, J.W. & Miller, R.L. [1994] NC Algebra: A Mathematica Package for Doing Non Commuting Algebra. v0.2, ncalg@ucsd.edu. (USCD, La Jolla).

[He] Hennings, M.A. [1986] Fronsdal *-quantization and Fell inducing. *Math. Proc. Camb. Phil. Soc.* **99**, 179–188.

[Is] Isham, C.J. [1984] Topological and global aspects of quantum theory. In: *Relativity, Groups, and Topology II*. DeWitt, B.S. & Stora, R., Eds. (North-Holland, Amsterdam) 1059–1290.

[Jo] Joseph, A. [1970] Derivations of Lie brackets and canonical quantization. *Commun. Math. Phys.* **17**, 210–232.

[KS] Kerner, E.H. & Sutcliffe, W.G. [1970] Unique Hamiltonian operators via Feynman path integrals. *J. Math. Phys.* **11**, 391–393.

[Ki] Kirillov, A.A. [1990] Geometric quantization. In: *Dynamical Systems IV: Symplectic Geometry and Its Applications*. Arnol'd, V.I. and Novikov, S.P., Eds. *Encyclopædia Math. Sci.* **IV**. (Springer, New York) 137-172.

[Ku] Kuryshkin, V.V. [1972] La mécanique quantique avec une fonction non-négative de distribution dans l'espace des phases. *Ann. Inst. H. Poincaré* **17**, 81–95.

[KLZ] Kuryshkin, V.V., Lyabis, I.A., & Zaparovanny, Y.I. [1978] Sur le problème de la regle de correspondence en théorie quantique. *Ann. Fond. L. de Broglie*. **3**, 45–61.

[Ma] Mackey, G.W. [1976] *The Theory of Unitary Group Representations* (University of Chicago Press, Chicago).

[MC] Margenau, H. & Cohen, L. [1967] Probabilities in quantum mechanics. In: *Quantum Theory and Reality*. Bunge, M., Ed. (Springer-Verlag, New York), 71–89.

[MR] Marsden, J.E. & Ratiu, T.S. [1994] *Introduction to Mechanics and Symmetry.* (Springer-Verlag, New York).

[MMSV] Mnatsakanova, M., Morchio, G., Strocchi, F., & Vernov, Yu. [1998] Irreducible representations of the Heisenberg algebra in Krein spaces. *J. Math. Phys.* **39**, 2969–2982.

[On] Onishchik, A.L. [1994] *Topology of Transitive Transformation Groups.* (Johann Ambrosius Barth, Leipzig).

[ReSi] Reed, M. & Simon, B. [1972] *Functional Analysis I.* (Academic Press, New York).

[Ri1] Rieffel, M.A. [1989] Deformation quantization of Heisenberg manifolds. *Commun. Math. Phys.* **122**, 531–562.

[Ri2] Rieffel, M.A. [1990] Deformation quantization and operator algebras. *Proc. Sym. Pure Math.* **45**, 411–423.

[Ri3] Rieffel, M.A. [1993] Quantization and C^*-algebras. In: *C^*-Algebras: 1943-1993, A Fifty Year Celebration.* Doran, R.S., Ed. *Contemp. Math.* **167**, 67–97.

[Ri4] Rieffel, M.A. [1998] Questions on quantization. In: *Operator Algebras and Operator Theory. Contemp. Math.* **228**, 315–326.

[Ro] Robert, A. [1983] *Introduction to the Representation Theory of Compact and Locally Compact Groups.* London Math. Soc. Lect. Note Ser. **80** (Cambridge U. P., Cambridge).

[So] Souriau, J.-M. [1997] *Structure of Dynamical Systems.* (Birkhäuser, Boston).

[Tu] Tuynman, G.M. [1998] Prequantization is irreducible. *Indag. Mathem.* **9**, 607–618.

[Ur] Urwin, R.W. [1983] The prequantization representations of the Poisson Lie algebra. *Adv. Math.* **50**, 126–154.

[VH1] van Hove, L. [1951] Sur certaines représentations unitaires d'un groupe infini de transformations. *Proc. Roy. Acad. Sci. Belgium* **26**, 1–102.

[VH2] van Hove, L. [1951] Sur le problème des relations entre les transformations unitaires de la mécanique quantique et les transformations canoniques de la mécanique classique. *Acad. Roy. Belgique Bull. Cl. Sci.* (5) **37**, 610–620.

[Va] Varadarajan, V.S. [1984] *Lie Groups, Lie Algebras and Their Representations.* (Springer-Verlag, New York).

[Ve] Velhinho, J. [1998] Some remarks on a full quantization of the torus. *Int. J. Mod. Phys.* A**13**, 3905–3914.

[VN] von Neumann, J. [1955] *Mathematical Foundations of Quantum Mechanics.* (Princeton. Univ. Press, Princeton).

[We] Weinstein, A. [1989] Cohomology of symplectomorphism groups and critical values of hamiltonians. *Math. Z.* **201**, 75–82.

[Wi] Wildberger, N. [1983] Quantization and harmonic analysis on Lie groups. Dissertation, Yale University.

[Wo] Woodhouse, N.M.J. [1992] *Geometric quantization*. Second Ed. (Clarendon Press, Oxford).

[Zi] Ziegler, F. [1996] Quantum representations and the orbit method. Thesis, Université de Provence.