TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 4, April 1996

A GROENEWOLD-VAN HOVE THEOREM FOR S^2

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Abstract. We prove that there does not exist a nontrivial quantization of the Poisson algebra of the symplectic manifold S^2 which is irreducible on the su(2) subalgebra generated by the components $\{S_1, S_2, S_3\}$ of the spin vector. In fact there does not exist such a quantization of the Poisson subalgebra $\mathcal{P}$ consisting of polynomials in $\{S_1, S_2, S_3\}$. Furthermore, we show that the maximal Poisson subalgebra of $\mathcal{P}$ containing $\{1, S_1, S_2, S_3\}$ that can be so quantized is just that generated by $\{1, S_1, S_2, S_3\}$.

1. Introduction

In a striking paper, Groenewold [Gr] showed that one cannot "consistently" quantize all polynomials in the classical positions q^i and momenta p_i on $\mathbb{R}^{2n}$. Subsequently Van Hove [VH1, VH2] refined and extended Groenewold's result, in effect showing that there does not exist a quantization functor which is consistent with the Schrödinger quantization of $\mathbb{R}^{2n}$. (For discussions of Groenewold's and Van Hove's work and related results, see [A-M, C, F, Go1, G-S, J] and references contained therein.) However, these theorems rely heavily on certain properties of $\mathbb{R}^{2n}$, and so it is not clear whether they can be generalized. Naturally, one *expects* similar "no-go" theorems to hold in a wide range of situations, but we are not aware of any previous results along these lines.

In this paper we prove a Groenewold-Van Hove theorem for the symplectic manifold S^2. Our proof is similar to Groenewold's for $\mathbb{R}^{2n}$, although it differs from his in several important respects and is technically more complicated. On the other hand, as S^2 is compact there are no problems with the completeness of the flows generated by the classical observables, and so Van Hove's modification of Groenewold's theorem is unnecessary in this instance.

To set the stage, let (M, ω) be a symplectic manifold. We are interested in quantizing the Poisson algebra $C^\infty(M)$ of smooth real-valued functions on M, or at least some subalgebra $\mathcal{C}$ of it, in the following sense.

Definition 1. A *prequantization* of $\mathcal{C}$ is a linear map $\mathcal{Q}$ from $\mathcal{C}$ to an algebra of self-adjoint operators[1] on a Hilbert space such that

(i) $\mathcal{Q}\big(\{f, g\}\big) = -\mathrm{i}\big[\mathcal{Q}(f), \mathcal{Q}(g)\big]$,

where $\{\ ,\ \}$ denotes the Poisson bracket and $[\ ,\ ]$ the commutator. If $\mathcal{C}$ contains the constant function 1, then we also demand

Received by the editors March 23, 1995.
1991 *Mathematics Subject Classification.* Primary 81S99; Secondary 58F06.

[1] Technical difficulties with unbounded operators will be ignored, as they are not important for what follows.

(ii) $\mathcal{Q}(1) = I$.

As is well known, it is necessary to supplement these conditions for a prequantization to be physically meaningful. To this end, one often requires that a certain subalgebra $\mathcal{B}$ of observables be represented irreducibly. Exactly which observables should be taken as "basic" in this regard depends upon the particular example at hand; one typically uses the components of a momentum map associated to a (transitive) Lie symmetry group. For $\mathbb{R}^{2n}$ the relevant group is the Heisenberg group [F, G-S] and $\mathcal{B} = \operatorname{span}\{1, q^i, p_i \,|\, i = 1, \ldots, n\}$. In the case of S^2 the appropriate group is U(2), whence the basic observables are $\operatorname{span}\{1, S_1, S_2, S_3\}$, the S_i being the components of the spin angular momentum.

Alternatively, one could require the *strong von Neumann rule* [vN]

$$\mathcal{Q}\big(k(f)\big) = k\big(\mathcal{Q}(f)\big)$$

to hold for all polynomials k and all $f \in \mathcal{C}$ such that $k(f) \in \mathcal{C}$. Usually it is necessary to weaken this condition [F], insisting only that it hold for a certain subclass $\mathcal{B}$ of observables f and certain polynomials k. We refer to this simply as a "von Neumann rule." In the case of $\mathbb{R}^{2n}$, the von Neumann rule as applied to the q^i and p_i with $k(x) = x^2$ is actually implied by the irreducibility of the $\mathcal{Q}(q^i)$ and $\mathcal{Q}(p_i)$ [C]. But the corresponding statement is not quite true for S^2, as we will see. We refer the reader to [F] for further discussion of von Neumann rules.

In our view, imposing an irreducibility condition on the prequantization map $\mathcal{Q}$ seems more compelling physically and pleasing aesthetically than requiring $\mathcal{Q}$ to satisfy a von Neumann rule. With this as well as the observations above in mind, we make

Definition 2. A *quantization* of the pair $(\mathcal{C}, \mathcal{B})$ is a prequantization of $\mathcal{C}$ which is irreducible on $\mathcal{B}$, where $\mathcal{C} \subset C^\infty(M)$ is a given Poisson algebra, and $\mathcal{B} \subset \mathcal{C}$ a given subalgebra.

The results of Groenewold and Van Hove may then be interpreted as showing that there does not exist a quantization of the pair $\big(C^\infty(\mathbb{R}^{2n}), \operatorname{span}\{1, q^i, p_i \,|\, i = 1, \ldots, n\}\big)$ nor, for that matter, of the subalgebra of all polynomials in the q^i and p_i. We will prove here that there likewise does not exist a quantization of the pair $\big(C^\infty(S^2), \operatorname{span}\{1, S_1, S_2, S_3\}\big)$ nor, for that matter, of the subalgebra of all polynomials in the components S_i of the spin vector.

2. No-Go theorems

Consider a sphere S^2 of radius $s > 0$. We view this sphere as the "internal" phase space of a massive particle with spin s and realize it as the subset of $\mathbb{R}^3$ given by

$$S_1{}^2 + S_2{}^2 + S_3{}^2 = s^2, \tag{2.1}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is the spin angular momentum. The symplectic form is[2]

$$\omega = \frac{1}{2s^2} \sum_{i,j,k=1}^{3} \epsilon_{ijk} S_i \, dS_j \wedge dS_k$$

[2]Note that ω is $1/s$ times the area form $d\boldsymbol{\sigma}$ on S^2. It is the symplectic form on S^2 viewed as a coadjoint orbit of SU(2).

with corresponding Poisson bracket

$$\{f, g\} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} S_i \frac{\partial f}{\partial S_j} \frac{\partial g}{\partial S_k} \tag{2.2}$$

for $f, g \in C^\infty(S^2)$. We have the relations $\{S_i, S_j\} = \sum_{k=1}^{3} \epsilon_{ijk} S_k$.

The group SU(2) acts transitively on S^2 with momentum map $\mathbf{S} = (S_1, S_2, S_3)$, i.e., the pair $\big(S^2, \mathrm{SU}(2)\big)$ is an "elementary system" in the sense of [W]. Thus it is natural to require that quantization provide an irreducible representation of SU(2). In terms of observables, quantization should produce a representation which is irreducible when restricted to the subalgebra generated by $\{S_1, S_2, S_3\}$. However, this subalgebra does not include the constants. To remedy this, we consider instead the central extension U(2) of SU(2) by the circle group with momentum map $(1, S_1, S_2, S_3)$, and take the subalgebra generated by these observables to be the basic set $\mathcal{B}$ in the sense of the Introduction.

Let $\mathcal{P}$ denote the Poisson algebra of polynomials in the components S_1, S_2, S_3 of the spin vector $\mathbf{S}$ modulo the relation (2.1). (This means we are restricting polynomials as functions on $\mathbb{R}^3$ to S^2.) We shall refer to an equivalence class $p \in \mathcal{P}$ as a "polynomial" and take its degree to be the minimum of the degrees of its polynomial representatives. We denote by $\mathcal{P}^k$ the subspace of polynomials of degree at most k. In particular, $\mathcal{P}^1$ is just the Poisson subalgebra generated by $\{1, S_1, S_2, S_3\}$.

When equipped with the L^2 inner product given by integration over S^2, the vector space $\mathcal{P}^k$ becomes a real Hilbert space which admits the orthogonal direct sum decomposition $\mathcal{P}^k = \bigoplus_{l=0}^{k} \mathcal{H}_l$, where $\mathcal{H}_l$ is the vector space of spherical harmonics of degree l (i.e., the restrictions to S^2 of homogeneous harmonic polynomials of degree l on $\mathbb{R}^3$ [A-B-R]). Note that $\mathcal{H}_1$ is the Poisson subalgebra generated by $\{S_1, S_2, S_3\}$. The collection of spherical harmonics $\big\{Y_l^m,\ l = 0, 1, \ldots, k,\ m = -l, -l+1, \ldots, l\big\}$ forms the standard (complex) orthogonal basis for the complexification $\mathcal{P}^k_{\mathbb{C}}$:

$$\int_{S^2} Y_{l_1}^{m_1 *} Y_{l_2}^{m_2}\, d\sigma = s^2 \delta_{l_1 l_2} \delta_{m_1 m_2}.$$

Thus if $p \in \mathcal{P}^k_{\mathbb{C}}$, we have the harmonic decomposition

$$p = p_k + p_{k-1} + \cdots + p_0, \tag{2.3}$$

where $p_l \in (\mathcal{H}_l)_{\mathbb{C}}$ is given by

$$p_l = \frac{1}{s^2} \sum_{m=-l}^{l} \left(\int_{S^2} Y_l^{m*} p \, d\sigma \right) Y_l^m. \tag{2.4}$$

It is well known that O(3) acts orthogonally on $\mathcal{P}$, and that this action is irreducible on each $\mathcal{H}_l$ where it is the standard real (orbital) angular momentum l representation. The corresponding infinitesimal generators on $\mathcal{H}_l$ are $L_i = \{S_i, \cdot\}$. If we identify o(3) and $\mathcal{H}_1$ as Lie algebras, it follows that the "adjoint" action of $\mathcal{H}_1$ on $\mathcal{H}_l$ given by $S_i \mapsto \{S_i, \cdot\}$ is irreducible as well.

Now suppose $\mathcal{Q}$ is a prequantization of $\mathcal{P}$, so that

$$\big[\mathcal{Q}(S_i), \mathcal{Q}(S_j)\big] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} \mathcal{Q}(S_k) \tag{2.5}$$

and

$$\mathcal{Q}(\mathbf{S}^2) = s^2 I. \tag{2.6}$$

If in addition $\mathcal{Q}$ is a quantization of $(\mathcal{P},\, \mathcal{P}^1)$, then $\mathcal{P}^1$ must be irreducibly represented. Since as a Lie algebra $\mathcal{P}^1$ is isomorphic to $\mathrm{u}(2)$, its irreducible representations are all finite-dimensional.[3] These are just the usual (spin angular momentum) representations labeled by $j = 0, \frac{1}{2}, 1, \dots,$ where[4]

$$\sum_{i=1}^{3} \mathcal{Q}(S_i)^2 = j(j+1)I. \tag{2.7}$$

The Hilbert space corresponding to the quantum number j has dimension $2j+1$, with the standard orthonormal basis $\big\{|\,j,m\rangle,\ m = -j, -j+1, \dots, j\big\}$ consisting of eigenvectors of $\mathcal{Q}(S_3)$. We regard the representation defined by $j = 0$ as trivial, in that it corresponds to quantum spin 0.

The following result shows that quantizability implies a weak type of von Neumann rule on $\mathcal{P}^1$.

Proposition 1. *If $\mathcal{Q}$ is a quantization of $(\mathcal{P},\, \mathcal{P}^1)$, then*

$$\mathcal{Q}\big({S_i}^2\big) = a\mathcal{Q}(S_i)^2 + cI \tag{2.8}$$

for $i = 1, 2, 3$, where a and c are real constants with $a^2 + c^2 \neq 0$. Furthermore, for $i \neq \ell$,

$$\mathcal{Q}(S_i S_\ell) = \frac{a}{2}\big(\mathcal{Q}(S_i)\mathcal{Q}(S_\ell) + \mathcal{Q}(S_\ell)\mathcal{Q}(S_i)\big). \tag{2.9}$$

We have placed the proof, which is rather long and technical, in Appendix A so as not to interrupt the exposition.

Observe that on summing (2.8) over i, we get $s^2 = aj(j+1) + 3c$, which fixes the constant c in terms of s, j and a.

From these relations the main result now follows.

Theorem 2 (No-Go Theorem)**.** *There does not exist a nontrivial quantization of $(\mathcal{P}, \mathcal{P}^1)$.*

Proof. Suppose there did exist a quantization $\mathcal{Q}$ of $(\mathcal{P}, \mathcal{P}^1)$; we shall show that for $j > 0$ this leads to a contradiction.

First observe that we have the classical equality

$$s^2 S_3 = \{{S_1}^2 - {S_2}^2, S_1 S_2\} - \{S_2 S_3, S_3 S_1\}.$$

Quantizing this, a calculation using (2.8), (2.9), (2.5) and (2.7) gives

$$s^2 \mathcal{Q}(S_3) = a^2\left(j(j+1) - \frac{3}{4}\right)\mathcal{Q}(S_3).$$

Thus either $\mathcal{Q}(S_3) = 0$, whence $j = 0$, or $j > 0$, in which case

$$s^2 = a^2\left(j(j+1) - \frac{3}{4}\right). \tag{2.10}$$

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[3] Every irreducible representation of $\mathrm{u}(2) = \mathrm{su}(2) \times \mathbb{R}$ by (essentially) self-adjoint operators on an invariant dense domain in a Hilbert space can be integrated to a continuous irreducible representation of $\mathrm{SU}(2) \times \mathbb{R}$ [B-R, §11.10.7.3]. But it is well known that every such representation of this group is finite-dimensional.

[4] In what follows we use standard quantum mechanical notation, cf. [M].

Observe that when $j = \frac{1}{2}$, this implies that $s = 0$, which is impossible. Henceforth take $j > \frac{1}{2}$.

Next we quantize the relation

$$2s^2 S_2 S_3 = \{S_2{}^2, \{S_1 S_2, S_1 S_3\}\} - \frac{3}{4}\{S_1{}^2, \{S_1{}^2, S_2 S_3\}\}.$$

Using (2.9), (2.8), (2.5) and (2.7), the l.h.s. becomes

$$as^2\big(\mathcal{Q}(S_2)\mathcal{Q}(S_3) + \mathcal{Q}(S_3)\mathcal{Q}(S_2)\big) = as^2\big(2\mathcal{Q}(S_2)\mathcal{Q}(S_3) - \mathrm{i}\mathcal{Q}(S_1)\big)$$

while the r.h.s. reduces to

$$a^3\left(j(j+1) - \frac{9}{4}\right)\big(2\mathcal{Q}(S_2)\mathcal{Q}(S_3) - \mathrm{i}\mathcal{Q}(S_1)\big).$$

Since for $j > \frac{1}{2}$ the matrix element

$$\big\langle j, j \,|\, 2\mathcal{Q}(S_2)\mathcal{Q}(S_3) - \mathrm{i}\mathcal{Q}(S_1) |\, j, j-1\big\rangle = \mathrm{i}\left(\frac{1}{2} - j\right)\sqrt{2j}$$

is nonzero, it follows that

$$as^2 = a^3\left(j(j+1) - \frac{9}{4}\right).$$

If $a = 0$ (2.10) yields $s = 0$, whereas if $a \neq 0$ this conflicts with (2.10). Thus we have derived contradictions provided $j > 0$. Since $j = 0$ is the trivial representation, the theorem is proven. $\square$

This contradiction shows that the quantization of $(\mathcal{P}, \mathcal{P}^1)$ goes awry on the level of quadratic polynomials. On the other hand, there are many quantizations of the pair $(\mathcal{P}^1, \mathcal{P}^1)$ of all polynomials of degree at most one, viz. the irreducible representations $\mathcal{Q}$ of $\mathrm{u}(2)$ with $\mathcal{Q}(1) = I$. Thus it is of interest to determine the largest subalgebra of $\mathcal{P}$ containing $\{1, S_1, S_2, S_3\}$ that can be quantized. We will now show that this largest subalgebra is just $\mathcal{P}^1$ itself. Unfortunately, this is not entirely straightforward, since $\mathcal{P}^1$ is not a maximal Poisson subalgebra of $\mathcal{P}$; indeed, if $\mathcal{O}$ denotes the Poisson subalgebra of odd polynomials (i.e., polynomials all of whose terms are of odd degree), then $\mathcal{P}^1$ is contained in $\tilde{\mathcal{O}} = \mathcal{O} \oplus \mathbb{R}$.

To prove the result, we proceed in two stages. First we show that $\mathcal{P}^1$ is maximal in $\tilde{\mathcal{O}}$, and then we prove a no-go theorem for $\tilde{\mathcal{O}}$.

Proposition 3. *$\mathcal{P}^1$ is maximal in $\tilde{\mathcal{O}}$.*

Proof. Actually, the constants are unimportant, and it will suffice to prove that the Poisson subalgebra $\mathcal{H}_1$ generated by $\{S_1, S_2, S_3\}$ is maximal in $\mathcal{O}$.

Set $\mathcal{O}^l = \mathcal{O} \cap \mathcal{P}^l$, where henceforth l is odd. For k odd it is clear from (2.2) and (2.1) that $\{\mathcal{H}_k, \mathcal{H}_l\} \subset \mathcal{O}^{k+l-1}$.

Let $\mathcal{R}$ be the Poisson algebra generated by a single polynomial $r \in \mathcal{O}^l$ of degree $l > 1$ together with $\mathcal{H}_1$. Evidently $\mathcal{R} \subset \mathcal{O}$; we must show that $\mathcal{O} \subset \mathcal{R}$. We will accomplish this in a series of lemmas.

Lemma 1. *If in its harmonic decomposition an element of $\mathcal{R}$ has a nonzero component in $\mathcal{H}_k$, then $\mathcal{H}_k \subset \mathcal{R}$.*

Proof. Let $\mathcal{R}' \subset \mathcal{R}$ be the span of all elements of the form

$$\{h_n, \ldots \{h_2, \{h_1, r\}\} \ldots\}$$

for $h_i \in \mathcal{H}_1$ and $n \in \mathbb{N}$. Then $\mathcal{R}'$ is an o(3)-invariant subspace of $\mathcal{O}^l \subset \mathcal{P}$. Since the representation of o(3) on $\mathcal{P}$ is completely reducible, $\mathcal{R}'$ must be the direct sum of certain $\mathcal{H}_k$ with $k \leq l$. Consequently, if when harmonically decomposed an element of $\mathcal{R}'$ has a nonzero component in some $\mathcal{H}_k$, then $\mathcal{H}_k \subset \mathcal{R}' \subset \mathcal{R}$. $\triangledown$

Now by assumption $r_l \neq 0$ in $\mathcal{H}_l$ and hence $\mathcal{H}_l \subset \mathcal{R}$. Then $\{\mathcal{H}_l, \mathcal{H}_l\} \subset \mathcal{O}^{2l-1} \cap \mathcal{R}$. We will use this fact to show that $\mathcal{O}^{2l-1} \subset \mathcal{R}$. The proof devolves upon an explicit computation of the harmonic decomposition of $\{Y_l^m, Y_l^n\}$.

Lemma 2. *For each j in the range $0 < j \leq 2l$, we have*

$$\{Y_l^{l-j}, Y_l^l\} = \sum_{k=1}^{l} y_{2k-1}(l-j, l) Y_{2k-1}^{2l-j}.$$

In particular, when $j = 1$ the top coefficients $y_{2l-1}(l-1, l)$ are nonzero. Furthermore, provided $l \geq 5$, $k \geq \frac{l-1}{2}$ and $k > l - \frac{j+1}{2}$, the coefficients $y_{2k-1}(l-j, l)$ are nonzero.

Since the proof requires an extended calculation, we defer it until Appendix B.

Lemma 3. $\mathcal{O}^{2l-1} \subset \mathcal{R}$.

Proof. Decompose $Y_l^m = R_l^m + \mathrm{i} I_l^m$ into real and imaginary parts, with $R_l^m,\ I_l^m \in \mathcal{H}_l$. So in $\mathcal{P}_{\mathbb{C}}$ we observe that both $\Re\{Y_l^m,\, Y_l^n\} = \{R_l^m, R_l^n\} - \{I_l^m, I_l^n\}$ and $\Im\{Y_l^m, Y_l^n\} = \{R_l^m, I_l^n\} + \{I_l^m, R_l^n\}$ belong to $\{\mathcal{H}_l,\, \mathcal{H}_l\}$. Thus if the harmonic decomposition of $\{Y_l^m, Y_l^n\}$ has a nonzero k^{th} component, then either its real or imaginary part must be nonzero, which allows us to conclude that $\{\mathcal{H}_l, \mathcal{H}_l\}$ contains an element with nonzero component in $\mathcal{H}_k$.

If $l = 3$, then $\mathcal{H}_3 \subset \mathcal{R}$. Now consider the bracket $\{Y_3^2,\, Y_3^3\}$. By Lemma 2 with $j = 1$ it has a nonzero 5^{th} component, so by the preceding and Lemma 1 it follows that $\mathcal{H}_5 \subset \mathcal{R}$. Since by definition $\mathcal{H}_1 \subset \mathcal{R}$, we then have $\mathcal{O}^5 = \mathcal{H}_1 \oplus \mathcal{H}_3 \oplus \mathcal{H}_5 \subset \mathcal{R}$.

If $l \geq 5$, we consider $\{Y_l^{-2},\, Y_l^l\} \in \mathcal{R}_{\mathbb{C}}$. By Lemma 2 with $j = l+2$, the preceding and Lemma 1 we conclude that

$$\mathcal{H}_{l-2} \oplus \mathcal{H}_l \oplus \cdots \oplus \mathcal{H}_{2l-1} \subset \mathcal{R}.$$

Hence $\mathcal{H}_{l-2} \subset \mathcal{R}$, so by the same argument applied to $\{Y_{l-2}^{-2}, Y_{l-2}^{l-2}\}$ we get that $\mathcal{H}_{l-4} \subset \mathcal{R}$. Continuing in this way we obtain $\mathcal{H}_{l-2n} \subset \mathcal{R}$ for all n with $l - 2n \geq 3$. In particular, taking $n = \frac{l-3}{2}$, we get $\mathcal{H}_3 \subset \mathcal{R}$. But we have already remarked that $\mathcal{H}_1 \subset \mathcal{R}$, so the lemma is proven. $\triangledown$

Thus $\mathcal{R}$ must contain all odd polynomials of degree at most $2l - 1$. To obtain higher degree polynomials, we need only bracket $\mathcal{H}_{2l-1} \subset \mathcal{R}$ with itself and apply the argument above to conclude that $\mathcal{O}^{4l-3} \subset \mathcal{R}$. Continuing in this manner, we have finally that $\mathcal{O} \subset \mathcal{R}$, and this proves the proposition. $\square$

Our strategy in proving the no-go theorem for $\tilde{\mathcal{O}}$ is the same as for $\mathcal{P}$. To begin, we derive a weak version of a cubic von Neumann rule.

Proposition 4. *If $\mathcal{Q}$ is a quantization of $(\tilde{\mathcal{O}}, \mathcal{P}^1)$, then*

$$\mathcal{Q}(S_i{}^3) = a\mathcal{Q}(S_i)^3 + c\mathcal{Q}(S_i) \tag{2.11}$$

for $i = 1, 2, 3,$ *where* a *and* c *are real constants. Furthermore, when* $i \neq \ell$,

$$\mathcal{Q}(S_i S_\ell S_i) = a\mathcal{Q}(S_i)\mathcal{Q}(S_\ell)\mathcal{Q}(S_i) + \frac{1}{3}(a+c)\mathcal{Q}(S_\ell). \tag{2.12}$$

Finally,

$$\mathcal{Q}(S_1S_2S_3) = a\mathcal{Q}(S_1)\mathcal{Q}(S_2)\mathcal{Q}(S_3) + \frac{a}{2\mathrm{i}}\big(\mathcal{Q}(S_1)^2 - \mathcal{Q}(S_2)^2 + \mathcal{Q}(S_3)^2\big). \tag{2.13}$$

Again the proof is placed in Appendix A.

We derive some consequences of these results. Multiplying (2.1) through by S_ℓ and quantizing gives

$$\sum_{i=1}^{3} \mathcal{Q}(S_i S_\ell S_i) = s^2 \mathcal{Q}(S_\ell).$$

Applying (2.12) and (2.11), this in turn becomes

$$a\sum_{i=1}^{3} \mathcal{Q}(S_i)\mathcal{Q}(S_\ell)\mathcal{Q}(S_i) = \left(s^2 - \frac{2a}{3} - \frac{5c}{3}\right)\mathcal{Q}(S_\ell). \tag{2.14}$$

We also find, by rearranging the factors in $\sum_{i=1}^{3} \mathcal{Q}(S_i)\mathcal{Q}(S_\ell)\mathcal{Q}(S_i)$, that

$$\sum_{i=1}^{3} \mathcal{Q}(S_i)\mathcal{Q}(S_\ell)\mathcal{Q}(S_i) = \big(j(j+1) - 1\big)\mathcal{Q}(S_\ell). \tag{2.15}$$

A comparison of (2.14) and (2.15) yields

$$s^2 = a\left(j(j+1) - \frac{1}{3}\right) + \frac{5c}{3} \tag{2.16}$$

provided $j > 0$.

Theorem 5. *There does not exist a nontrivial quantization of* $(\tilde{\mathcal{O}}, \mathcal{P}^1)$.

Proof. Suppose $\mathcal{Q}$ is a quantization of $(\tilde{\mathcal{O}}, \mathcal{P}^1)$; we will show that this entails $j = 0$.

Consider the Poisson bracket relation

$$\begin{aligned} 3s^4 S_3 \;&=\; 4\{{S_1}^3, S_2{S_3}^2\} - 4\{{S_2}^3, {S_3}^2 S_1\} + \{{S_2}^2 S_1, {S_2}^3\} \\ &\quad - \{S_2{S_1}^2, {S_1}^3\} - 6\{{S_2}^3, {S_1}^3\} - 3\{S_2{S_3}^2, {S_3}^2 S_1\}. \end{aligned}$$

Upon quantizing, an enormous calculation using (2.11), (2.12), (2.5) and (2.15) gives[5]

$$\begin{aligned} 3s^4\mathcal{Q}(S_3) \;&=\; \bigg(3a^2j^4 + 6a^2j^3 + 14acj^2 + 8a^2j^2 \\ &\qquad + 14acj + 5a^2j + \frac{29c^2}{3} - 14\frac{ac}{3} - \frac{7a^2}{3}\bigg)\mathcal{Q}(S_3) \\ &\qquad - (10a^2 + 4ac)\mathcal{Q}(S_3)^3 \end{aligned}$$

which in view of (2.16) simplifies to

$$\left[\frac{4}{3}(2a-c)(a+c) - a(7a+4c)j(j+1)\right]\mathcal{Q}(S_3) + a(10a+4c)\mathcal{Q}(S_3)^3 = 0. \tag{2.17}$$

[5]This calculation was done using the *Mathematica* package *NCAlgebra* [H-M].

Now suppose $j = \frac{1}{2}$, so that $\mathcal{Q}(S_3)^2 = \frac{1}{4}I$. Then (2.17) implies that $a = -4c$, which when substituted into (2.16) yields $s = 0$. Similarly, when $j = 1$, $\mathcal{Q}(S_3)^3 = \mathcal{Q}(S_3)$. In this case (2.17) implies that $a = -c$, and again (2.16) requires $s = 0$. Thus we have derived contradictions for these two values of j. Henceforth take $j > 1$.

Next we quantize

$$6s^2 S_1S_2S_3 = \{{S_1}^3, {S_2}^2S_1\} + \{{S_2}^3, {S_3}^2S_2\} + \{{S_3}^3, {S_1}^2S_3\}.$$

Another computer calculation using (2.11), (2.12), (2.13), (2.5) and (2.15) yields

$$\begin{aligned} 6s^2\Big[a\mathcal{Q}(S_1)\mathcal{Q}(S_2)\mathcal{Q}(S_3) + \frac{a}{2\mathrm{i}}\big(\mathcal{Q}(S_1)^2 - \mathcal{Q}(S_2)^2 + \mathcal{Q}(S_3)^2\big)\Big] \\ = -3a\mathrm{i}\big(c - 2a + aj(j+1)\big)\big[\mathcal{Q}(S_1)^2 - \mathcal{Q}(S_2)^2 + \mathcal{Q}(S_3)^2 + 2\mathrm{i}\mathcal{Q}(S_1)\mathcal{Q}(S_2)\mathcal{Q}(S_3)\big]. \end{aligned}$$

Since for $j > 1$ the matrix element

$$\begin{aligned} \Big\langle j, j-2 \Big| \mathcal{Q}(S_1)\mathcal{Q}(S_2)\mathcal{Q}(S_3) + \frac{1}{2\mathrm{i}}\big(\mathcal{Q}(S_1)^2 - \mathcal{Q}(S_2)^2 + \mathcal{Q}(S_3)^2\big)\Big| j, j\Big\rangle \\ = \frac{1}{2\mathrm{i}}(1-j)\sqrt{j(2j-1)} \end{aligned}$$

is nonzero, we conclude that either $a = 0$ or

$$s^2 = c - 2a + aj(j+1). \tag{2.18}$$

If $a = 0$, (2.17) implies that $c = 0$, and then (2.16) leads to a contradiction, so (2.18) must hold. Subtracting (2.18) from (2.16) gives $c = -5a/2$; substituting this into (2.17) produces

$$3a^2(j^2 + j - 3)\mathcal{Q}(S_3) = 0.$$

But then $j = (-1 \pm \sqrt{13})/2$, neither of which is permissible.

Thus we have derived contradictions for all $j > 0$, and the theorem follows. □

We remark that Theorem 5 is actually sharper than Theorem 2; we have included the latter because it is simpler.

As Proposition 3 shows, by augmenting $\mathcal{P}^1$ with a single odd polynomial, we generate $\tilde{\mathcal{O}}$. Similarly we can generate all of $\mathcal{P}$ from $\mathcal{P}^1$ and a single polynomial not in $\tilde{\mathcal{O}}$, which implies that the only Poisson subalgebras of $\mathcal{P}$ strictly containing $\mathcal{P}^1$ are $\tilde{\mathcal{O}}$ and $\mathcal{P}$ itself. This can be proven in the same manner as Proposition 3, but in the interests of economy, we make do with:

Lemma 4. *Any Poisson subalgebra of $\mathcal{P}$ which strictly contains $\mathcal{P}^1$ also contains $\tilde{\mathcal{O}}$.*

Proof. By Proposition 3 it suffices to consider the Poisson algebra $\mathcal{T}$ generated by $\mathcal{P}^1$ and a polynomial p not in $\tilde{\mathcal{O}}$. Then p has a component in some $\mathcal{H}_{2k}$ for $k > 0$, so by Lemma 1 it follows that $\mathcal{H}_{2k} \subset \mathcal{T}$. Now consider the bracket $\{Y_{2k}^{2k-1}, Y_{2k}^{2k}\}$. According to Lemma 2, either its real or imaginary part has a nonzero component in $\mathcal{H}_{4k-1}$. Hence $\mathcal{H}_{4k-1} \subset \mathcal{T}$, and so by Proposition 3 we have $\tilde{\mathcal{O}} \subset \mathcal{T}$. □

Now, given any Poisson subalgebra of $\mathcal{P}$ strictly containing $\mathcal{P}^1$, we only have to apply Theorem 5 to the subalgebra $\tilde{\mathcal{O}}$ inside it to obtain a contradiction. Hence:

Theorem 6. *No nontrivial quantization of $\mathcal{P}^1$ can be extended beyond $\mathcal{P}^1$.*

This result stands in marked contrast to the analogous one for $\mathbb{R}^{2n}$. There one runs into difficulties with *cubic* polynomials in $\{1, q^i, p_i\}$, so that $\mathcal{P}^2$ (the Poisson algebra of polynomials of degree at most two) is a maximal polynomial subalgebra containing $\mathcal{P}^1$ that can be quantized [G-S]. This dichotomy seems to be connected with the fact that for $\mathbb{R}^{2n}$, $\mathcal{P}^2$ is the Poisson normalizer of $\mathcal{P}^1$, whereas for S^2 the normalizer of $\mathcal{P}^1$ is itself. On the other hand, it should be noted that there are other maximal polynomial subalgebras of $C^\infty(\mathbb{R}^{2n})$ containing $\mathcal{P}^1$ that can be quantized, for instance the "coordinate" (or "position") subalgebra

$$\left\{\sum_{i=1}^{n} h^i(q^1, \dots, q^n)p_i + k(q^1, \dots, q^n)\right\},$$

where the h^i and k are polynomials. Here one encounters problems when one tries to extend to terms which are quadratic in the momenta.

Finally, a word is in order regarding the case $j = 0$—the one instance in which we did not derive a contradiction. It happens that the spin 0 representation of $\mathcal{P}^1$ *can* be extended, in a unique way, to a quantization $\mathcal{Q}$ of *all* of $\mathcal{P}$. Indeed, given $p \in \mathcal{P}$, let p_0 denote the constant term in the harmonic decomposition (2.3) of p. Then $\mathcal{Q} : \mathcal{P} \to \mathbb{C}$ defined by $\mathcal{Q}(p) = p_0$ is, technically, a quantization of $(\mathcal{P}, \mathcal{P}^1)$. To prove this, it is only necessary to show that $\mathcal{Q}$ so defined is a Lie algebra homomorphism, which in this context means $\{p, p'\}_0 = 0$. But from (2.4) and (2.2), in vector notation,

$$4\pi s^2\{p, p'\}_0 = \int_{S^2} \{p, p'\}\, d\sigma = \int_{S^2} \mathbf{S} \cdot (\nabla p \times \nabla p')\, d\sigma = s \int_{S^2} (\nabla p \times \nabla p') \cdot d\boldsymbol{\sigma},$$

which vanishes by the divergence theorem.[6]

To show that this quantization is unique, we need

Lemma 5. *For $l > 0$, $\mathcal{H}_l = \{\mathcal{H}_1, \mathcal{H}_l\}$.*

Proof. For $l > 0$, $\{\mathcal{H}_1, \mathcal{H}_l\}$ is a nontrivial invariant subspace of $\mathcal{H}_l$, and hence, by the irreducibility of the $\mathcal{H}_1$-action on $\mathcal{H}_l$, $\{\mathcal{H}_1, \mathcal{H}_l\} = \mathcal{H}_l$. □

Now suppose χ is a quantization of $(\mathcal{P}, \mathcal{P}^1)$ with $j = 0$ so that by the above, χ is a linear map $\mathcal{P} \to \mathbb{C}$ which must annihilate Poisson brackets. But then $\chi(p) = \chi(p_0)$, since by Lemma 5 each term $p_l \in \mathcal{H}_l$ for $l > 0$ in the harmonic decomposition of p must be a sum of terms of the form $\{h_l, r_l\}$ for some $h_l \in \mathcal{H}_1$ and $r_l \in \mathcal{H}_l$. It follows that χ is uniquely determined by its value on the constants, and as $\chi(1) = I = \mathcal{Q}(1)$ we must have $\chi = \mathcal{Q}$.

Although the corresponding representation of $\mathcal{P}^1$ is trivial in that $\mathcal{Q}(S_i) = 0$ for all i, it is worth emphasizing that $\mathcal{Q}$ is *not* zero on the remainder of $\mathcal{P}$. For example, $\mathcal{Q}({S_i}^2) = \frac{s^2}{3}I$ for all i, consistent with Proposition 1 and (2.6). The existence of this trivial yet "not completely trivial" representation of $\mathcal{P}$—for any nonzero value of the classical spin s—is curious. We wonder if it is related to a well-known "anomaly" in the geometric quantization of spin, cf. [T, §7].

[6]This actually is a consequence of a general fact about momentum maps on compact symplectic manifolds, cf. [G-S, p. 187].

3. Discussion

Theorems 2 and 6 might have been "predicted" on the basis of geometric quantization theory. Here one knows that one can quantize those classical observables $f \in C^\infty(M)$ whose flows preserve a given polarization P. In our case we take P to be the antiholomorphic polarization on S^2 (thought of as $\mathbb{C}P^1$); then $\mathcal{P}^1$ is exactly the set of polarization-preserving observables. However, in geometric quantization theory one does not expect to be able to consistently quantize observables outside this class [W].

Further corroboration for our results is provided by Rieffel [R], who showed that there are no strict SU(2)-invariant deformation quantizations of $C^\infty(S^2)$. In fact, it seems that only the polynomial algebra $\mathcal{P}^1 \subset C^\infty(S^2)$ can be rigorously deformation quantized in an SU(2)-invariant way [K].

There are several points we would like to make concerning the von Neumann rules for S^2, especially in comparison with those for $\mathbb{R}^{2n}$. Our no-go theorems may be interpreted as stating that the "Poisson bracket $\to$ commutator" rule is *totally* incompatible with even the relatively weak von Neumann rules given in Propositions 1 and 4. On $\mathbb{R}^{2n}$, on the other hand, the von Neumann rule $k(x) = x^n$ with $n = 2$ does hold for $x \in \mathcal{P}^1 \subset \mathcal{P}^2$ in the metaplectic representation [C, G-S], and with any $n \geq 0$ for $x = q^i$ in the coordinate (or position) representation. Moreover it is curious that, according to Propositions 1 and 4, requiring that the S_i be irreducibly represented does not yield strict von Neumann rules as happens for $\{q^i, p_i\}$ on $\mathbb{R}^{2n}$, cf. [Gr, C]. It is not clear why S^2 and $\mathbb{R}^{2n}$ behave differently in these regards.

We remark that it is substantially easier to prove the no-go Theorems 2 and 5 if one assumed from the start that the strict von Neumann rules $\mathcal{Q}({S_i}^2) = \mathcal{Q}(S_i)^2$ and $\mathcal{Q}({S_i}^3) = \mathcal{Q}(S_i)^3$ hold. We can gain some insight into this as follows. Suppose the strong von Neumann rule applied to one of the S_i, say S_3, so that $\mathcal{Q}({S_3}^n) = \mathcal{Q}(S_3)^n$ for all positive integers n. Suppose furthermore that $\mathcal{Q}$ is injective when restricted to the polynomial algebra $\mathbb{R}[S_3]$ generated by S_3. Provided it is appropriately continuous, $\mathcal{Q}$ will then extend to an isomorphism from the real C^*–algebra $C^*(S_3)$ (consisting of the closure of $\mathbb{R}[S_3]$ in the supremum norm on S^2 with pointwise operations) to the real C^*–algebra generated by $\mathcal{Q}(S_3)$. But this implies that the classical spectrum of S_3 (i.e., the set of all values it takes) is the same as the operator spectrum of $\mathcal{Q}(S_3)$. Since the classical spectrum of S_3 is $[-s, s]$ whereas the quantum spectrum is discrete, it is clear—in retrospect—why no (strong) von Neumann rule can apply to S_3.

Since S^2 is in a sense the opposite extreme from $\mathbb{R}^{2n}$ insofar as symplectic manifolds go, our result lends support to the contention that no-go theorems should hold in some generality. Nonetheless, these two examples are special in that they are symplectic homogeneous spaces $\big(\mathbb{R}^{2n}$ for the translation group $\mathbb{R}^{2n}$, and S^2 for $\mathrm{SU}(2)\big)$. Thus in both cases we are quantizing finite-dimensional Lie algebras $\big($the Heisenberg algebra and u(2), respectively, which are certain central extensions by $\mathbb{R}$ of $\mathbb{R}^{2n}$ and su(2)$\big)$. Will a similar analysis work for other symplectic homogeneous spaces, e.g., $\mathbb{C}P^n$ with group $\mathrm{SU}(n+1)$? How does one proceed in the case of symplectic manifolds which do not have such a high degree of symmetry? What set of observables will play the role of the distinguished subalgebras generated by $\{1, q^i, p_i\}$ and $\{1, S_1, S_2, S_3\}$ (which are the components of the momentum mappings for the Hamiltonian actions of the Heisenberg group and U(2), resp.)? For a cotangent bundle T^*Q, the obvious counterpart would be the infinite-dimensional abgebra of linear momentum observables $P_X + f$, where P_X is the momentum in the direction of the vector field X on Q and f is a function on Q. (These are the components of the momentum mapping for the transitive action of $\mathrm{Diff}(Q) \ltimes C^\infty(Q)$ on T^*Q.) In this regard, it is known that geometric quantization (formally) obeys the von Neumann rules $\mathcal{Q}(P_X{}^2) = \mathcal{Q}(P_X)^2$ and $\mathcal{Q}(f^n) = \mathcal{Q}(f)^n$ [Go2]. We hope to explore some of these issues in future papers.

Acknowledgments

One of us (M.J.G.) would like to thank L. Bos, G. Emch, G. Goldin, M. Karasev and J. Tolar for enlightening conversations, and the University of New South Wales for support while this research was underway. He would also like to express his appreciation to both the organizers of the XIII Workshop on Geometric Methods in Physics and the University of Warsaw, who provided a lively and congenial atmosphere for working on this problem. This research was supported in part by NSF grant DMS-9222241.

Another one of us (H.G.) would like to acknowledge the support of an ARC–grant.

Appendix A. von Neumann rules

Here we provide the proofs of Propositions 1 and 4.

Fix an irreducible representation of $\mathrm{u}(2) = \mathrm{su}(2) \times \mathbb{R}$ labeled by the quantum number j. The proofs depend on the following three facts:

(i) As the representation is irreducible, any endomorphism of the representation space is an element of the enveloping algebra of the generators $\mathcal{Q}(S_i)$ [D, Prop. 2.6.5], and hence can be expressed as a polynomial in the $\mathcal{Q}(S_i)$.

(ii) Under the su(2)-action, a monomial $\mathcal{Q}(S_1)^{n_1}\mathcal{Q}(S_2)^{n_2}\mathcal{Q}(S_3)^{n_3}$ of degree $|n| = n_1 + n_2 + n_3$ transforms as a tensor operator of rank $|n|$.

(iii) Under the induced action of su(2) on $\mathcal{P}$, a monomial $S_1{}^{n_1}S_2{}^{n_2}S_3{}^{n_3}$ transforms as a symmetric tensor of rank $|n|$. Since the quantization $\mathcal{Q}$ is (infinitesimally) equivariant, it follows that $\mathcal{Q}(S_1{}^{n_1}S_2{}^{n_2}S_3{}^{n_3})$ also transforms as a symmetric tensor operator of rank $|n|$.

When $|n| > 1$, the tensor operators $\mathcal{Q}(S_1)^{n_1}\mathcal{Q}(S_2)^{n_2}\mathcal{Q}(S_3)^{n_3}$ and $\mathcal{Q}(S_1{}^{n_1}S_2{}^{n_2}S_3{}^{n_3})$ are reducible.

Equation (2.8) then follows from the observation that as $\mathcal{Q}(S_i{}^2)$ is a reducible symmetric tensor operator of rank 2, its irreducible constituents must be of even rank, and hence it must be equal to an even polynomial in the $\mathcal{Q}(S_i)$ of degree at most 2. Equations (2.9), (2.11), (2.12) and (2.13) follow from similar observations. The rest of the argument is the working out of these observations.

Letting $n = (n_1, n_2, n_3)$ be a multi-index of length $|n| = n_1 + n_2 + n_3$, we denote $\mathcal{Q}(S_1)^{n_1}\mathcal{Q}(S_2)^{n_2}\mathcal{Q}(S_3)^{n_3}$ by $\mathbb{Q}^{|n|}_{\mathbf{n}}$ and $S_1{}^{n_1}S_2{}^{n_2}S_3{}^{n_3}$ by $\mathbb{S}^{|n|}_n$. Since the commutation relations (2.5) are nonlinear, the difference between each $\mathbb{Q}^{|n|}_{\mathbf{n}}$ and its symmetrization $\mathbb{Q}^{|n|}_{(\mathbf{n})}$ is a linear combination of tensor operators $\mathbb{Q}^{|m|}_{\mathbf{m}}$ of lower rank $|m|$. Thus we may use the symmetrized tensor operators $\mathbb{Q}^{|n|}_{(\mathbf{n})}$ as a basis for the enveloping algebra of the operators $\mathcal{Q}(S_i)$. Then by (i) we can expand

$$\mathcal{Q}\big(\mathbb{S}_m^{|m|}\big) = \sum_{|n|=0}^{d} [\, m \,|\, n \,]\, \mathbb{Q}_{(\mathbf{n})}^{|n|} \tag{A.1}$$

as a polynomial of degree d, say.[7] We can further decompose each

$$\mathbb{Q}_{(\mathbf{n})}^{|n|} = \sum_{\lambda=0}^{|n|} \sum_{\mu=-\lambda}^{\lambda} (\, n \,|\, \lambda\, \mu \,)\, T_\mu^\lambda \tag{A.2}$$

into a sum of irreducible spherical tensor operators T_μ^λ of rank λ (with $\mu = -\lambda, \dots, \lambda$),[8] so that (A.1) becomes

$$\mathcal{Q}\big(\mathbb{S}_m^{|m|}\big) = \sum_{|n|=0}^{d} \sum_{\lambda=0}^{|n|} \sum_{\mu=-\lambda}^{\lambda} [\, m \,|\, n \,]\, (\, n \,|\, \lambda\, \mu \,)\, T_\mu^\lambda. \tag{A.3}$$

On the other hand, for $|m| = 2$ we may directly decompose

$$\mathcal{Q}\big(\mathbb{S}_m^{2}\big) = \sum_{\nu=-2}^{2} (\, m \,|\, 2\, \nu \,)\, V_\nu^2 + (\, m \,|\, 0\, 0 \,)\, V_0^0, \tag{A.4}$$

where the irreducible constituents V_ν^η of $\mathcal{Q}\big(\mathbb{S}_m^{2}\big)$ are given by

$$V_\nu^2 = \sum_{|m|=2} (\, 2\, \nu \,|\, m \,)\, \mathcal{Q}(\mathbb{S}_m^2), \tag{A.5}$$

$$V_0^0 = \sum_{|m|=2} (\, 0\, 0 \,|\, m \,)\, \mathcal{Q}(\mathbb{S}_m^2). \tag{A.6}$$

Here we have used the relation $\sum_{\lambda \geq 0} \sum_{\nu=-\lambda}^{\lambda} (\, m \,|\, \lambda\ \nu \,)\, (\, \lambda\ \nu \,|\, m' \,) = \delta_{mm'}$. Note that, as it is symmetric, $\mathcal{Q}\big(\mathbb{S}_m^{2}\big)$ has no irreducible rank 1 constituent. Combining (A.5) and (A.6) with (A.3), we obtain

$$V_\nu^2 = \sum_{|m|=2} \sum_{|n|=0}^{d} \sum_{\lambda=0}^{|n|} \sum_{\mu=-\lambda}^{\lambda} (\, 2\, \nu \,|\, m \,)\, [\, m \,|\, n \,]\, (\, n \,|\, \lambda\, \mu \,)\, T_\mu^\lambda, \tag{A.7}$$

$$V_0^0 = \sum_{|m|=2} \sum_{|n|=0}^{d} \sum_{\lambda=0}^{|n|} \sum_{\mu=-\lambda}^{\lambda} (\, 0\, 0 \,|\, m \,)\, [\, m \,|\, n \,]\, (\, n \,|\, \lambda\, \mu \,)\, T_\mu^\lambda. \tag{A.8}$$

Now apply a rotation $R \in \mathrm{SU}(2)$ to (A.7) to obtain

$$\sum_{\nu'=-2}^{2} D_{\nu\nu'}^2(R) V_{\nu'}^2$$
$$= \sum_{|m|=2} \sum_{|n|=0}^{d} \sum_{\lambda=0}^{|n|} \sum_{\mu=-\lambda}^{\lambda} \sum_{\mu'=-\lambda}^{\lambda} (\, 2\, \nu \,|\, m \,)\, [\, m \,|\, n \,]\, (\, n \,|\, \lambda\, \mu \,)\, D_{\mu\mu'}^\lambda(R)\, T_{\mu'}^\lambda.$$

[7] It can be shown that $d \leq 4j$, but we do not need this fact.

[8] Since they are symmetric the tensor operators $\mathbb{Q}_{(\mathbf{n})}^{|n|}$ are "simply reducible," i.e., in the decomposition (A.2) there is at most one irreducible constituent T_μ^λ for each weight λ. This follows from a consideration of Young tableaux. Likewise, the $\mathcal{Q}\big(\mathbb{S}_m^{|m|}\big)$, being symmetric, are simply reducible, and hence there is no degeneracy in either (A.4) or (A.12) below.

Multiplying both sides of this by $D^2_{\nu\rho'}(R)^*$ and integrating over the group manifold, the orthogonality theorem for products of representations [B-R, §7.1] yields

$$V^2_{\rho'} = \left[\sum_{|m|=2}\sum_{|n|=0}^{d} (\,2\,\nu\,|\,m\,)\,[\,m\,|\,n\,]\,(\,n\,|\,2\,\nu\,)\right] T^2_{\rho'}. \tag{A.9}$$

Applying the same procedure to (A.8) but using $D^0_{00}(R)^*$ instead, we similarly obtain

$$V^0_0 = \left[\sum_{|m|=2}\sum_{|n|=0}^{d} (\,0\,0\,|\,m\,)\,[\,m\,|\,n\,]\,(\,n\,|\,0\,0\,)\right] T^0_0. \tag{A.10}$$

From (A.9) and (A.10), we see that $V^2_{\rho'}$ is proportional to $T^2_{\rho'}$, and V^0_0 to T^0_0; let the corresponding constants of proportionality be a and b. Substituting (A.9) and (A.10) into (A.4) and inverting (A.2) gives

$$\begin{aligned}
\mathcal{Q}\big(\mathbb{S}^2_m\big) &= a\sum_{\nu=-2}^{2}(\,m\,|\,2\,\nu\,)\,T^2_\nu + b(\,m\,|\,0\,0\,)\,T^0_0 \\
&= a\sum_{|n|=2}\sum_{\nu=-2}^{2}(\,m\,|\,2\,\nu\,)\,(\,2\,\nu\,|\,n\,)\,\mathbb{Q}^2_{(n)} + b(\,m\,|\,0\,0\,)\,(\,0\,0\,|\,0\,)\,\mathbb{Q}^0_0 \\
&= a\sum_{|n|=2}P_2(m,n)\,\mathbb{Q}^2_{(n)} + bP_0(m,0)I.
\end{aligned} \tag{A.11}$$

In this last expression, $P_\lambda(m,n) = \sum_{\nu=-\lambda}^{\lambda}(\,m\,|\,\lambda\,\nu\,)\,(\,\lambda\,\nu\,|\,n\,)$ is the projector which picks off the m^{th} component (with respect to $\mathbb{Q}^{|m|}_{(m)}$) of the irreducible rank λ constituent of $\mathbb{Q}^{|n|}_{(n)}$. Since m has length 2, m must be of the form $1_i + 1_\ell$, where 1_i is the multi-index with 1 in the i^{th} slot and zeros elsewhere. Similarly, when $|n| = 2$, $n = 1_p + 1_q$. Then we have

$$\begin{aligned}
P_2(1_i + 1_\ell, 1_p + 1_q) &= \frac{1}{2}\big(\delta_{ip}\delta_{\ell q} + \delta_{iq}\delta_{\ell p}\big) - \frac{1}{3}\delta_{i\ell}\delta_{pq}, \\
P_0(1_i + 1_\ell, 0) &= \frac{1}{3}\delta_{i\ell}.
\end{aligned}$$

Setting $\ell = i$, (A.11) and (2.7) then yield

$$\mathcal{Q}\big({S_i}^2\big) = a\left(\mathcal{Q}(S_i)^2 - \frac{1}{3}j(j+1)I\right) + \frac{1}{3}bI.$$

The constants a and b must both be real $\big($as $\mathcal{Q}\big({S_i}^2\big)$ is self-adjoint$\big)$, and both cannot be simultaneously zero $\big($for this would contradict (2.6)$\big)$. Thus, upon setting $c = \big(b - aj(j+1)\big)/3$, we obtain (2.8).

Taking $\ell \neq i$, we similarly obtain (2.9).

The same arguments can be applied, *mutatis mutandis*, to $\mathcal{Q}(\mathbb{S}^3_m)$. Then (A.11) is replaced by

$$\begin{aligned}\mathcal{Q}(\mathbb{S}^3_m) &= a\sum_{\nu=-3}^{3}(\,m\,|\,3\,\nu\,)\,T^3_\nu + b\sum_{\nu=-1}^{1}(\,m\,|\,1\,\nu\,)T^1_\nu \\ &= a\sum_{|n|=3}P_3(m,n)\,\mathbb{Q}^3_{(n)} + b\sum_{|n|=1}P_1(m,n)\,\mathbb{Q}^1_n.\end{aligned} \tag{A.12}$$

The projectors in this instance are

$$\begin{aligned}&P_3(1_i+1_k+1_\ell, 1_p+1_q+1_r)\\ &\qquad= \frac{1}{6}\sum\delta_{ip}\delta_{kq}\delta_{\ell r} - \frac{1}{30}\sum\delta_{ik}(\delta_{pq}\delta_{\ell r}+\delta_{pr}\delta_{\ell q}+\delta_{qr}\delta_{\ell p}),\\ &P_1(1_i+1_k+1_\ell,1_p) = \frac{1}{15}(\delta_{ik}\delta_{\ell p}+\delta_{k\ell}\delta_{ip}+\delta_{\ell i}\delta_{kp}),\end{aligned}$$

where the sums are over all permutations of $\{i,k,\ell\}$.

Setting $i=k=\ell$, (A.12) reduces to

$$\mathcal{Q}({S_i}^3) = a\left(\mathcal{Q}(S_i)^3 - \frac{3}{5}\Big(j(j+1)-\frac{1}{3}\Big)\mathcal{Q}(S_i)\right) + \frac{1}{5}b\mathcal{Q}(S_i).$$

Again by self-adjointness, we see that a and b must be real. Upon setting $c = \frac{1}{5}\big(b-3\big[j(j+1)-\frac{1}{3}\big]\big)$ and rearranging the above, we obtain (2.11).

Finally, taking $i=k\neq\ell$ and $(i,k,\ell)=(1,2,3)$ in (A.12), we similarly obtain (2.12) and (2.13), respectively.

Although it is not apparent from this derivation, it can be shown that (2.9) is actually a consequence of (2.8) and, likewise, that (2.12) and (2.13) both follow from (2.11).

Appendix B. On the harmonic decomposition of $\{Y_l^{l-j}, Y_l^l\}$

In this appendix we prove Lemma 2, which computes the harmonic decomposition of $\{Y_l^{l-j}, Y_l^l\}$ for $l>0$. Specifically, for each j, $0<j\le 2l$, we have

$$\{Y_l^{l-j},Y_l^l\} = \sum_{k=1}^{l} y_{2k-1}(l-j,l)Y_{2k-1}^{2l-j},$$

where the coefficients $y_{2k-1}(l-j,l)$ have the following properties:

(i) when $j=1$, the top coefficients $y_{2l-1}(l-1,l)\neq 0$, and

(ii) provided $l\ge 5$, $k\ge\frac{l-1}{2}$ and $k>l-\frac{j+1}{2}$, we have $y_{2k-1}(l-j,l)\neq 0$.

The proof will be presented in several steps. We refer the reader to [M, Chapter XIII and Appendix C] for the relevant background and conventions on spherical harmonics.

Step 1. It is convenient to work in spherical coordinates (θ,ϕ) on S^2. The generators L_3 and $L_\pm = L_1\pm \mathrm{i}L_2$ of O(3) then take the form

$$L_3 = \frac{\partial}{\partial\phi} \ \text{ and } \ L_\pm = \pm\mathrm{i}e^{\pm\mathrm{i}\phi}\left(\frac{\partial}{\partial\theta}\pm\mathrm{i}\cot\theta\frac{\partial}{\partial\phi}\right).$$

They satisfy

$$L_3Y_l^m = \mathrm{i}mY_l^m,\quad L_+Y_l^m = \mathrm{i}\beta_{l,m}Y_l^{m+1}\quad\text{and}\quad L_-Y_l^m = \mathrm{i}\beta_{l,m-1}Y_l^{m-1}$$

where $\beta_{l,m} = \sqrt{(l+m+1)(l-m)}$.

The Poisson bracket (2.2) becomes

$$\{f,g\} = \frac{\csc\theta}{s}\left(\frac{\partial f}{\partial\phi}\frac{\partial g}{\partial\theta} - \frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\phi}\right),$$

which can be rewritten in terms of the L_i as

$$\{f,g\} = -\frac{\mathrm{i}}{s}\csc\theta\, e^{-\mathrm{i}\phi}\Big(L_3(f)L_+(g) - L_+(f)L_3(g)\Big).$$

Step 2. As a specific instance of this formula, we compute

$$\{Y_l^m, Y_l^n\}\sin\theta\, e^{\mathrm{i}\phi} = \frac{\mathrm{i}}{s}\Big(m\,\beta_{l,n}Y_l^mY_l^{n+1} - n\,\beta_{l,m}Y_l^{m+1}Y_l^n\Big). \tag{B.1}$$

Now we have the product decomposition

$$\begin{aligned} Y_{l_1}^{m_1}Y_{l_2}^{m_2} = &\sum_{l=|l_1-l_2|}^{l_1+l_2}\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\,\langle\, l_1\, l_2\, 0\, 0\,|\, l\, 0\,\rangle \\ &\times\langle\, l_1\, l_2\, m_1\, m_2\,|\, l\ m_1{+}m_2\,\rangle Y_l^{m_1+m_2}, \end{aligned} \tag{B.2}$$

where the quantities $\langle\, l_1\, l_2\, m_1\, m_2\,|\, L\, M\,\rangle$ are Clebsch-Gordan coefficients. Applying (B.2) to the r.h.s. of (B.1) gives

$$\begin{aligned} \{Y_l^m, Y_l^n\}\sin\theta\, e^{\mathrm{i}\phi} = &\frac{\mathrm{i}}{s}\sum_{j=0}^{2l}\frac{2l+1}{\sqrt{4\pi(2j+1)}}\langle\, l\, l\, 0\, 0\,|\, j\, 0\,\rangle \\ &\times\Big[m\,\beta_{l,n}\langle\, l\, l\, m,\ n{+}1\,|\, j,\ m{+}n{+}1\,\rangle \\ &\quad - n\,\beta_{l,m}\langle\, l\, l\, m{+}1,\ n\,|\, j,\ m{+}n{+}1\,\rangle\Big]Y_j^{m+n+1}. \end{aligned} \tag{B.3}$$

On the other hand, since $\{\mathcal{H}_l, \mathcal{H}_l\} \subset \mathcal{O}^{2l-1}$ we can expand

$$\{Y_l^m, Y_l^n\} = \sum_{k=1}^{l}\sum_{r=-2k+1}^{2k-1} y_{2k-1,r}(m,n)Y_{2k-1}^r.$$

Multiply this equation through by $\sin\theta\, e^{\mathrm{i}\phi}$. Since $\sin\theta\, e^{\mathrm{i}\phi} = -\sqrt{8\pi/3}\,Y_1^1$, we can apply the product decomposition (B.2) to the r.h.s. of the resulting expression to obtain

$$\begin{aligned} \{Y_l^m, Y_l^n\}\sin\theta\, e^{\mathrm{i}\phi} &= -\sqrt{\frac{8\pi}{3}}\left(\sum_{k=1}^{l}\sum_{r=-2k+1}^{2k-1} y_{2k-1,r}(m,n)\big[Y_{2k-1}^rY_1^1\big]\right) \\ &= \sum_{k=1}^{l}\sum_{r=-2k+1}^{2k-1} y_{2k-1,r}(m,n)\Big(e_{2k-1,r}Y_{2k-2}^{r+1} - d_{2k-1,r}Y_{2k}^{r+1}\Big), \end{aligned} \tag{B.4}$$

where

$$d_{t,r} = \sqrt{\frac{(t+r+1)(t+r+2)}{(2t+1)(2t+3)}} \quad\text{and}\quad e_{t,r} = \sqrt{\frac{(t-r)(t-r-1)}{(2t+1)(2t-1)}}.$$

Comparing (B.3) with (B.4), we see that $r = m + n$ and hence

$$\begin{aligned}
&\sum_{k=1}^{l} y_{2k-1,m+n}(m,n)\Big(e_{2k-1,m+n}Y_{2k-2}^{m+n+1} - d_{2k-1,m+n}Y_{2k}^{m+n+1}\Big)\\
&\qquad = \frac{\mathrm{i}}{s}\sum_{j=0}^{2l}\frac{2l+1}{\sqrt{4\pi(2j+1)}}\langle\, l\, l\, 0\, 0\,|\, j\, 0\,\rangle\\
&\qquad\quad\times\Big[m\,\beta_{l,n}\langle\, l\, l\, m,\ n{+}1\,|\, j,\ m{+}n{+}\,1\,\rangle\\
&\qquad\qquad -n\,\beta_{l,m}\langle\, l\, l\, m{+}1,\ n\,|\, j,\ m{+}n{+}\,1\,\rangle\Big]Y_j^{m+n+1}.
\end{aligned}$$

Note that the sum on the l.h.s. contains only even degree harmonics, and hence the sum on the r.h.s. must as well. (This is reflected by the vanishing of the Clebsch-Gordan coefficients $\langle\, l\, l\, 0\, 0\,|\, j\, 0\,\rangle$ for j odd, cf. [M, eq. (C23.a)].) Thus we may reindex $j = 2k$, $k = 0, \dots, l$, on the r.h.s. Upon reindexing $k \mapsto k + 1$ in the first term on the l.h.s. and equating coefficients of Y_{2k}^{m+n+1} on both sides, we obtain the recursion relation

$$\begin{aligned}
y_{2k-1}(m,n) &= \frac{e_{2k+1,m+n}}{d_{2k-1,m+n}}\,y_{2k+1}(m,n)\\
&\quad-\frac{\mathrm{i}}{s}\frac{2l+1}{\sqrt{4\pi(4k+1)}\,d_{2k-1,m+n}}\,\langle\, l\, l\, 0\, 0\,|\, 2k\, 0\,\rangle\\
&\quad\times\Big[m\,\beta_{l,n}\langle\, l\, l\, m\, n{+}1\,|\, 2k,\ m{+}n{+}\,1\,\rangle\\
&\qquad -n\,\beta_{l,m}\langle\, l\, l\, m{+}1,\ n\,|\, 2k,\ m{+}n{+}\,1\,\rangle\Big]
\end{aligned}\tag{B.5}$$

where we have abbreviated $y_{2k-1,m+n}(m,n) =: y_{2k-1}(m,n)$.

Step 3. Now we specialize even further, setting $m = l - j$ and $n = l$ for $j = 1, \cdots, 2l$. Then we have

$$\{Y_l^{l-j}, Y_l^l\} = \sum_{k=1}^{l} y_{2k-1}(l-j,l)Y_{2k-1}^{2l-j},\tag{B.6}$$

as in the statement of the lemma, and (B.5) reduces to

$$\begin{aligned}
y_{2k-1}(l-j,l) &= \frac{e_{2k+1,2l-j}}{d_{2k-1,2l-j}}\,y_{2k+1}(l-j,l)\\
&\quad+\frac{\mathrm{i}}{s}\frac{l(2l+1)}{\sqrt{4\pi(4k+1)}\,d_{2k-1,2l-j}}\,\beta_{l,l-j}\langle\, l\, l\, 0\, 0\,|\, 2k\, 0\,\rangle\\
&\quad\times\langle\, l\, l\, l{-}j{+}1,\ l\,|\, 2k,\ 2l{-}j{+}\,1\,\rangle.
\end{aligned}\tag{B.7}$$

The main reason for this choice of m and n is that it simplifies the last term in (B.5), since $\beta_{l,l} = 0$.

Before proceeding, we must evaluate the Clebsch-Gordan coefficients in (B.7). Using the Racah formula [M, (C.21), (C.22) and (C.23b)], we compute

$$\langle\, l\, l\, 0\, 0\,|\, 2k\, 0\,\rangle = (-1)^{k+l}\sqrt{4k+1}\sqrt{\frac{(2l-2k)!}{(2l+2k+1)!}}\,\frac{(2k)!(l+k)!}{(k!)^2(l-k)!}$$

and

$$\langle\, l\, l\, l{-}j{+}1,\ l \,|\, 2k,\ 2l{-}j{+}1\,\rangle$$

$$= \sqrt{4k+1}\sqrt{\frac{(2l)!(j-1)!(2k+2l-j+1)!}{(2l-j+1)!(2l+2k+1)!(2l-2k)!(2k-2l+j-1)!}}.$$

Substituting these expressions as well as those for $d_{2k-1,2l-j}$, $e_{2k+1,2l-j}$ and $\beta_{l,l-j}$ into (B.7), the recursion relation becomes

$$\begin{aligned} \tilde{y}_{2k-1} \quad &= \quad \sqrt{\frac{(2k-2l+j+1)(2k-2l+j)(4k-1)}{(2k+2l-j)(2k+2l-j+1)(4k+3)}}\,\tilde{y}_{2k+1} \\ &\qquad + (-1)^k\sqrt{\frac{(2k+2l-j-1)!}{(2k-2l+j-1)!}}\,\frac{\sqrt{4k-1}(4k+1)(l+k)!(2k)!}{(2l+2k+1)!(l-k)!(k!)^2}, \end{aligned}$$

where $\tilde{y}_{2k-1}$ is defined according to

$$y_{2k-1}(l-j,l) = \frac{\mathrm{i}}{s}(-1)^l\frac{l(2l+1)}{\sqrt{4\pi}}\sqrt{\frac{j!(2l)!}{(2l-j)!}}\,\tilde{y}_{2k-1}.$$

Since $l>0$, $y_{2k-1}(l-j,l)=0$ iff $\tilde{y}_{2k-1}=0$.

Finally, we rewrite this in the form

$$\tilde{y}_{2k-1} = Z_{2k-1}\big[\tilde{y}_{2k+1} + (-1)^k W_{2k-1}\big] \tag{B.8}$$

where

$$Z_{2k-1} = \sqrt{\frac{(2k-2l+j+1)(2k-2l+j)(4k-1)}{(2k+2l-j)(2k+2l-j+1)(4k+3)}}$$

and

$$W_{2k-1} = \sqrt{\frac{(2k+2l-j+1)!}{(2k-2l+j+1)!}}\,\frac{\sqrt{4k+3}(4k+1)(l+k)!(2k)!}{(2l+2k+1)!(l-k)!(k!)^2}.$$

Notice that for fixed j and l, $y_{2k-1}(l-j,l)=0$ if $2k-1\le 2l-j-2$.

Upon setting $k=l$ and $j=1$, (B.8) gives $\tilde{y}_{2l-1}=(-1)^l Z_{2l-1}W_{2l-1}\ne 0$ and hence $y_{2l-1}(l-1,l)\ne 0$. This proves (i).

Step 4. Fix l and j, in which case the coefficients Z_{2k-1} and W_{2k-1} are nonzero whenever $2k-1>2l-j-2$. We claim that

$$Z_{2k-1}W_{2k-1} < W_{2k-3} \tag{B.9}$$

provided $l\ge 5$ and $k\ge \frac{l-1}{2}$. Indeed, we compute

$$\frac{Z_{2k-1}W_{2k-1}}{W_{2k-3}} = \frac{(4k+1)(l-k+1)(2k-1)}{(4k-3)(2l+2k+1)k}.$$

But now one verifies that the maximum of the r.h.s. of this expression is $18/25$ on the domain in the k,l-plane determined by the above inequalities along with the fact that $k\le l$.

Step 5. We can now prove statement (*ii*) of the lemma. Fix $l \geq 5$ and $j \in \{1, \ldots, 2l\}$. The solution of the recursion relation (B.8) with initial condition $\tilde{y}_{2l+1} = 0$ is

$$\tilde{y}_{2l-2n-1} = (-)^l \sum_{k=0}^{n} (-1)^k U_k$$

where $U_k = W_{2l-2k-1} \prod_{t=0}^{n-k} Z_{2l-2n+2t-1}$, and $n = 0, 1, \ldots, \max\{\frac{l+1}{2}, \frac{j-1}{2}\}$. For the alternating sum $\sum_{k=0}^{n}(-1)^k U_k$ we have by (B.9) that $U_{k+1}/U_k > 1$, so if n is even the sum is

$$U_0 + (U_2 - U_1) + \cdots + (U_n - U_{n-1}) > 0$$

since U_0 and all bracketed terms are positive, and if n is odd

$$(U_0 - U_1) + (U_2 - U_3) + \cdots + (U_{n-1} - U_n) < 0$$

likewise. So the sum is always nonzero in the given ranges of the parameters. Hence $\tilde{y}_{2k-1} \neq 0$, as claimed.

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