3. Invariant factors and module decompositions.

In exercises 1 to 3 below, $\mathcal{Z}$ is some Euclidean domain. Actually, one only needs $\mathcal{Z}$ to be a domain in which every ideal is *principal*, so try to make your arguments work in that more general context. Incidentally, the ring of integers in the field $\mathbf{Q}(\sqrt{-19})$ is principal but not Euclidean.

1. Let A be an $m \times n$ matrix over $\mathcal{Z}$. For every $\nu \leq \min(m,n)$, let $E_\nu(A) \subseteq \mathcal{Z}$ be the ideal generated by all $\nu \times \nu$ subdeterminants of A.
 (i) Show that $E_\nu(A) = E_\nu(MAN)$ if M and N are strongly regular matrices.
 (ii) Hence deduce the uniqueness of the invariant factors of A.

2. Let H be a $\mathcal{Z}$-module. For any $a \in \mathcal{Z}$, define the submodule $H(a) = \{x \in H \mid ax = 0\}$. Supposing that $a = dk$ with $(d,k) = 1$, show that
 (i) $H(d) \cap H(k)$ is trivial ($= 0$), and
 (ii) $H(a) = H(d) \oplus H(k)$ is a direct sum.

3. With H as above, we say that H is *p-primary* if $H = H(p^s)$ for some prime $p \in \mathcal{Z}$. If H is finitely generated, we know that $H \simeq \mathcal{Z}/p^{\nu_1}\mathcal{Z} \oplus \cdots \oplus \mathcal{Z}/p^{\nu_r}\mathcal{Z}$, with suitable $\nu_1 \geq \cdots \geq \nu_r > 0$ (see?), and we shall then say that H is of type $(p^{\nu_1}, \cdots, p^{\nu_r})$. Prove:
 (i) H is of type $(p^{\nu_1}, \cdots, p^{\nu_r}) \iff H(p)$ is of type $(p, \cdots, p)$ with r terms and pH is of type $(p^{\nu_1 - 1}, \cdots, p^{\nu_z - 1})$, where $z \leq r$ is the largest index with $\nu_z > 1$.
 (ii) Another $\mathcal{Z}$-module L is isomorphic to H if and only if it is p-primary of the same type.

4. Let F be a field and B an $n \times n$ matrix over F. Put $\mathcal{Z} = F[t]$ and consider the $\mathcal{Z}$-matrix $A = tI_n - B$. Let $d_1(t), \ldots, d_r(t)$ be the invariant factors of A, with $d_i(t) \mid d_{i+1}(t)$.
 (i) Show that $d_r(t)$ is the minimal polynomial of B.
 (ii) Deduce the Cayley-Hamilton Theorem.

5. In the notation above, let $F = \mathbf{Q}$, $n = 4$, and $B = 2E_{11} + E_{22} + 2E_{31} + E_{33} + E_{34} + E_{44}$.
 (i) Using elementary row and column operations, convert the matrix $A = tI_4 - B$ to diagonal form.
 (ii) Determine the invariant factors of A and the Jordan form of B.

6. Consider a 10×10 matrix B over $\mathbf{Q}$, with characteristic polynomial $(x-2)^4(x^2-3)^3$, minimal polynomial $(x-2)^2(x^2-3)^2$, and $(B - 2I)$ of rank 8.
 (i) Find the Jordan canonical form of B.
 (ii) Find the rational canonical form of B.

7. Let H be an (additive) abelian group with generators u, v, w, and the relations $8u + 9v = 2u - 20v + 22w = 27v - 24w = 0$.
 (i) Represent H as a direct sum of cyclic groups.
 (ii) Find the number of cyclic subgroups of H. Explain your answer.

8. Let $H = C_8 \times C_{12} \times C_{30}$, where C_n denotes a cyclic group of order n.
 (i) Does H admit homomorphism onto C_{45}, C_{60} ? Explain.
 (ii) Find the number of isomorphism classes of abelian groups having the same order as H. Explain your answer.