4. Elementary Group Theory.

1. Let G be a group.
  (i) Given normal subgroups H and K of G, define a group homomorphism $G \to G/H \times G/K$ with kernel $H \cap K$.
  (ii) Can G have exactly *two* subgroups of index 2? Prove your answer.

2. Let H be a cyclic normal subgroup of order p in a group G.
  (i) Define a homomorphism $G \to \mathbf{F}_p^\times$ whose kernel is the centralizer of H.
  (ii) If $(p-1)$ is relatively prime to the index $[G : H]$ show that H is in the center of G.

3. Let $H \subset G$ be finite groups, and $K = N_G(H) \neq G$ be the normalizer of H.
  (i) If J denotes the intersection of all conjugates of K, show that $J \cap H$ is a normal subgroup of K.
  (ii) If H is the only (non-trivial) normal subgroup of K, show that G acts faithfully on the set of all conjugates of H.

4. Let A_n denote the alternating group on $n > 2$ letters.
  (i) Show that A_n is generated by 3-cycles [*Hint:* $(ij)(kl) = (ij)(jk)^2(kl)$].
  (ii) Conclude that A_n is in the kernel of any homomorphism $S_n \to S_2$, and hence is the only subgroup of index 2 in S_n.

5. Let G be a group of order 24 having no normal subgroup of order 3.
  (i) Show that G has 4 subgroups of order 6.
  (ii) Show that the subgroups of order 6 in $G = SL_2(\mathbf{F}_3)$ are not isomorphic to those in $G = S_4$.

6. Let p be any prime, and put $G = SL_2(\mathbf{F}_p)$.
  (i) Show that the matrix $A = I + E_{12}$ generates a Sylow subgroup of G.
  (ii) Concluding that all elements of order p in G have trace $= 2$, show that their number is $\leq p^2 - 1$, and hence determine the number of Sylow p-subgroups of G.

Light Entertainment.

- For any prime $p > 2$, show that every non-cyclic group of order $2p$ is dihedral.
- Show that a group of order 56 (or 312) cannot be simple.
- Show that all elements of order 4 in $GL_2(\mathbf{F}_3)$ are conjugate to one another.
- Prove or disprove: any group of order > 2 has a non-trivial automorphism.
- Show: no finite group can be the *union* of conjugates of a proper subgroup.
- Show that SO_3 *is* the union of conjugates of a proper subgroup.