5. Basic Galois Theory.

1. Let K/k be a Galois extension with group G.
   (i) Given subgroups $H_1 \neq H_2$ of G, show that their fixed fields are distinct.
   (ii) If H_1 and H_2 are as above, but conjugate in G, show that their fixed fields are isomorphic.

2. Let $K = k(t)$ be the field of "rational functions" over the field k.
   (i) If k has characteristic 0, show that $\tau : f(t) \mapsto f(t+1)$ generates a proper subgroup $G \subset \mathrm{Aut}_k(K)$ whose fixed field is k.
   (ii) Show that $\mathrm{Aut}_k(K)$ consists of all transformations of the form $f(t) \mapsto f(at{+}b/ct{+}d)$ with $ad{-}bc \neq 0$ and $a, b, c, d \in k$.

3. Let $K = k(t)$ be as above, with $k = \mathbf{F}_3$.
   (i) Describe the group generated by $\tau : f(t) \mapsto f(t+1)$, and determine its fixed field.
   (ii) Ditto for the group of all tranfomations of the form $f(t) \mapsto f(at+b)$ with $a \neq 0$ and $a, b \in k$.

4. Let $k = \mathbf{F}_3$, $K = k(\sqrt{2})$, and consider the polynomial $f(x) = x^4 + x^3 + x + 2$ in $k[x]$.
   (i) Show that K is a splitting field for $f(x)$.
   (ii) Find a generator π of $K^\times$ and determine the roots of $f(x)$ in terms of π.

5. Let $k = \mathbf{Q}$ and consider the polynomial $f(x) = x^4 + 2x^2 - 5$ in $k[x]$.
   (i) Show that $f(x)$ is irreducible over k but has exactly 2 real roots $\pm\alpha$.
   (ii) Show that $K = k(\alpha, \sqrt{-5})$ is a splitting field for $f(x)$.

6. Let K/k and L/k be a field extensions of degree $[K : k] = m$ and $[L : k] = n$, respectively, with $L = k(\alpha)$.
   (i) Show that, if m and n are relatively prime, the degree $[K(\alpha) : k]$ equals $m \cdot n$.
   (ii) Show: if $f(x) \in k[x]$ is irreducible with degree relatively prime to m, it remains irreducible in $K[x]$.