8. Group Representations.

Throughout these exercises, F is a field, and G is a group whose (finite) order is prime to the characteristic of F. In Exercises 1 – 4, the group G is generated by two elements ρ and τ such that

$$\rho^4 = 1\,, \qquad \tau^2 = \rho^k\,, \quad \text{and} \quad \tau\rho = \rho^{-1}\tau\,, \tag{$*$}$$

where either $k = 0$ (hence $G = D_8$) or $k = 2$ (hence $G = Q_8$). In either case, let us label the elements of G as

$$1\,, \rho\,, \rho^2\,, \rho^3\,, \tau\,, \rho\tau\,, \rho^2\tau\,, \rho^3\tau\,. \tag{$**$}$$

1. Let $F = \mathbf{Q}$ and $G = D_8$.
 (i) Show that there are 4 inequivalent FG-modules of dimension 1. For each of them, determine the (scalar) action of ρ and τ.
 (ii) Find non-commuting matrices $\Delta(\rho)$, $\Delta(\tau)$ defining an irreducible FG-representation Δ. Prove the irreducibility of Δ, as well as its uniqueness (up to isomorphism).

2. Repeat Exercise 1 with $G = Q_8$.

3. Let $F = \mathbf{Q}(\sqrt{-1})$.
 (i) For $G = D_8$, show that the matrices $\Delta(\rho)$ and $\Delta(\tau)$ from Exercise 1 still define an irreducible representation.
 (ii) For $G = Q_8$, show that the matrices $\Delta(\rho)$ and $\Delta(\tau)$ from Exercise 2 no longer define an irreducible representation, and find (irreducible) replacements.

4. Let $G = Q_8$, and Δ be the irreducible representation of FG.
 (i) For $F = \mathbf{Q}(\sqrt{-1})$, describe the ring $\Delta(FG)$.
 (ii) Repeat (i) for $F = \mathbf{Q}$.

5. Let M and M' be FG-modules corresponding to the representations Δ and Δ'.
 (i) Show that $\mathrm{Hom}_{FG}(M, M') \neq 0$ if and only if M and M' have FG-submodules L and L' which are isomorphic.
 (ii) Let $E \supset F$ be a larger field, and M_E, M'_E the EG-mdules obtained by extension of scalars. Show that $\mathrm{Hom}_{FG}(M, M') \neq 0 \iff \mathrm{Hom}_{EG}(M_E, M'_E) \neq 0$. (Hint: think in terms of Δ and Δ'.)

6. Suppose V is a finitely generated $\mathbf{Z}$-module contained in a $\mathbf{Q}$-space W.
 (i) If $q \in \mathbf{Q}$ is such that $qV \subseteq V$, show that $q \in \mathbf{Z}$. (Hint: if $v_1, \ldots, v_m$ generate V, the equations $qv_i = \sum_i a_{ij} v_j$ force q to be a root of a certain characteristic polynomial.)
 (ii) In the notation of Section 2, show that $m_k \mid n$, if F contains $\mathbf{Q}$.