9. Group actions. If G is a group and A is a set, a *G-action* on A is a map $G \times A \longrightarrow A$ (denoted by $g, a \mapsto ga$) satisfying $(gh)a = g(ha)$ and $1a = a$. It induces the following equivalence relation on A: $x \equiv y \mod G \iff \exists g \in G$ such that $y = gx$. (Note how you need the group axioms for this!). Hence it partitions the set A into equivalence classes, which are sets of the form Gx ($x \in A$) known as *orbits*.

For any $x \in A$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *stabilizer* of x. Stabilizers for members of the same orbit are conjugate: if $y = gx$ one easily checks that $G_y = gG_xg^{-1}$.

A G-action is called *transitive* if it consists of a single orbit. Example: given a subgroup H, let $A = G/H = \{gH \mid g \in G\}$ with G acting by left multiplication; this is obviously transitive. Conversely, if $A = Gx$ is transitive, the map $G \longrightarrow A$, by $g \mapsto gx$, induces a G-isomorphism between A and G/H, where $H = G_x$ is the stabilizer. Indeed, $g_1x = g_2x \iff g_2^{-1}g_1x = x$.

This gives a bijection between conjugacy classes of subgroups on the one hand and isomorphism classes of transitive G-actions on the other.

An application: Sylow's Theorem.

THEOREM: Let $|G| = mp^r$, where p is a prime not dividing m. Then,

(i) G has a subgroup of order p^r (called a Sylow p-subgroup).
(ii) Any two such subgroups are conjugate in G, and every p-subgroup of G is contained in one of them.
(iii) The number of such subgroups is $\equiv 1 \mod p$ and divides m.

Proof: (Wielandt) Let G act by left multiplication on the family $\mathcal{A}$ of subsets $U \subset G$ such that $|U| = p^r$. Then one easily checks that

$$\text{(a)} \quad |\mathcal{A}| \equiv m \pmod p \qquad \text{(b)} \quad U \in \mathcal{A} \implies |G_U| \leq p^r.$$

Existence. Partition $\mathcal{A}$ into orbits. By (a), at least one of these, say $\mathcal{T} = \{gU \mid g \in G\}$, has cardinality prime to p. Thus the order of G_U must be divisible by p^r, hence must equal p^r, by (b).

Conjugacy. Let $H \subseteq G$ be any subgroup of order p^s, and observe its action on $\mathcal{T}$. Since all non-trivial orbits have p-power cardinality, there must also be trivial ones, i.e. $HV = V$ for some $V \in \mathcal{T}$. Hence $H \subseteq G_V$ (with equality if $s = r$), and G_V is conjugate to G_U.

Number. Finally let $H = G_U$ act by *conjugation* on the set $\mathrm{Syl}_p(G)$ of all Sylow p-subgroups of G. If this action had a fix-point $K \neq H$, the equation $HK = KH$ would imply that HK is a group with two distinct normal Sylow p-subgroups — an impossibility by (ii). Hence the only fix-point is H, all other orbits have p-divisible cardinalities, and therefore $|\mathrm{Syl}_p(G)| \equiv 1 \mod p$. This number divides m, because it is the cardinality of the orbit of H under conjugation by G (the stabilizer of H under this action is at least H).

Two Virtues of p-Groups.

Let $F \subseteq A$ be the set of fixed points (one-point orbits) of a group action $G \times A \longrightarrow A$, where G has order p^r. Since all the other orbits have p-power cardinalities, we have $|F| \equiv |A| \pmod p$. In particular, when G acts on itself by conjugation, the neutral element cannot be the only fixed point. In other words:

every non-trivial p-group G has a non-trivial centre $Z(G)$.

Since $G/Z(G)$ has order $\leq p^{r-1}$, it now follows by induction (starting with $r = 1$) that

every non-trivial p-group has a normal subgroup of index p.

More generally, any finite group is called *nilpotent*, if every non-trivial quotient group has a non-trivial centre; it is called *solvable* if every non-trivial subgroup has a normal subgroup of prime index. Nilpotency implies solvability (induction as above), but not vice versa — cf. S_3.

10. The Platonic groups. One of the key elements in this paragraph is the following simple diophantine equation

$$n+2 = n_1 + n_2 + n_3\,, \qquad \text{with} \quad n > n_i > 1 \qquad \text{and} \quad n_i \mid n\,. \tag{$*$}$$

Exercise 1: Prove that the only integer solutions (n, n_1, n_2, n_3) of $(*)$ are the infinite sequence $(2k, k, k, 2)$ and the three special solutions

$$\text{(i)} \quad (12, 6, 4, 4)\,, \qquad \text{(ii)} \quad (24, 12, 8, 6)\,, \qquad \text{(iii)} \quad (60, 30, 20, 12)\,. \tag{$\dagger$}$$

(*Hint:* Show that at least one of the n_i must be $= n/2$ and that, if none of them equals 2, another one must be $= n/3$. Then think about the remaining one.)

Our task is to study the patterns which can occur when a finite group $G < SO_3$ of rotations acts on the unit sphere $S \subset \mathbf{R}^3$. Every non-trivial $\rho \in G$ has an axis which meets S in two antipodal points. Since ρ leaves them fixed, each of these points has a non-trivial stabilizer. Such points are called "poles" of G; they form a finite subset $P \subset S$, invariant under G.

Concentrating on the group action $G \times P \longrightarrow P$, we suppose that it has orbits $T_1, \dots, T_r$ with cardinalities $n_1, \dots, n_r$, and that G has order n. These numbers are related by the formula:

$$2\,(n-1) = (n - n_1) + \cdots + (n - n_r). \tag{$**$}$$

To prove this, we count the number of non-trivial $\rho \in G$ in two ways. Every pole $w \in P$ is left fixed by $f_w = |G_w|$ elements of G (including the identity), and every non-trivial $\rho \in G$ belongs to exactly 2 antipodal stabilizers G_w (the poles of its axis). Hence, if we sum the numbers $(f_w - 1)$ over all $w \in P$, we are counting every $\rho \neq 1$ twice, and get $2(n-1)$. On the other hand, summing $(f_w - 1)$ over the orbit T_i, we obtain $n_i(f_i - 1) = (n - n_i)$, because all poles $w \in T_i$ have $f_w = f_i = n/n_i$. Equating the two counts yields the desired equation $(**)$. Let us refer to the $(r+1)$-tuple $(n, n_1, \dots, n_r)$, with $n > n_1 \geq n_2 \geq \cdots \geq n_r$, as the "signature" of G.

LEMMA 1: Unless the finite group $G < SO_3$ is cyclic or dihedral (i.e. essentially planar), its signature is one of the three displayed in $(\dagger)$.

Proof: If $n_r = 1$, there must be a pole w with $f_w = n$, i.e., a fix-point. Then all $\rho \in G$ have the same axis, and we are dealing with a cyclic group. Signature: $(n, 1, 1)$.

If $n_r = 2$, the equation $(**)$ reads $n = (n - n_1) + \cdots + (n - n_{r-1})$. Since $n_i \mid n$, we have $n_i \leq n/2$ for all i, which means that this equation can only be satisfied by putting $n_1 = n_2 = n/2$ and $r = 3$. Signature: $(n, n/2, n/2, 2)$. The two poles in the third orbit must be antipodal, and their common stabilizer be a cyclic group of order $n/2$. Hence G is dihedral.

If $n_r > 2$, we can sum the inequalities $n/2 \leq (n - n_i) < (n-2)$ and obtain $rn/2 \leq 2\,(n-1) < r\,(n-2)$, whence we conclude that $r = 3$ and $2 + n = n_1 + n_2 + n_3$. The result of Exercise 1 now yields the lemma.

THEOREM 1: If $G < SO_3$ has signature (i) or (ii), its action on $\mathrm{Syl}_3(G)$ defines an isomorphism of G with A_4 or S_4, respectively.

Proof: First we show that $|\mathrm{Syl}_3(G)| = 4$ in both cases. Indeed, every $\rho \in G$ of order 3 lies in the common stabilizer $G_v = G_w$ of a pair of antipodal poles whose orbits have cardinality divisible by $n/3$, hence equal to 4 in Case (i) and to 8 in Case (ii). In either case, a total of 8 poles and 4 stabilizers is involved. Since the latter are *exactly* of order 3, they are the Sylow 3-subgroups. Let us label them H_1, H_2, H_3, H_4.

The normalizers K_i of H_i — i.e., the stabilizers of G acting on $\mathrm{Syl}_3(G)$ — have order $n/4$. In Case (i), we have $K_i = H_i \simeq A_3$. In Case (ii), the order of K_i is 6, whence $K_i \simeq S_3$ because there are no poles of order $6k$ (as there are no orbits of cardinality $4/k$). In either case, the intersection of the K_i is the kernel N of the homomorphism $G \longrightarrow S_4$ given by this G-action. To finish the proof, we must show that $N = \{1\}$ and that A_4 is the only subgroup of index 2 in S_4. This will be done in the following two exercises.

Exercise 2: Let $H < G$ be finite groups, and suppose that H is the unique minimal non-trivial normal subgroup of its normalizer $K = N_H(G) \neq G$. Show that G acts faithfully (by conjugation) on the set of all conjugates of H.

Exercise 3: For $n > 2$ show that A_n is generated by 3-cycles [hint: $(ij)(kl) = (ij)(jk)^2(kl)$]. Conclude that A_n is in the kernel of any homomorphism $S_n \longrightarrow S_2$, and hence is the only subgroup of index 2 in S_n.

LEMMA 2: Let ρ, σ, and τ be distinct elements of order 2 in SO_3 such that $\sigma\rho = \rho\sigma$ and $\tau\rho = \rho\tau$. Then the axis of ρ is perpendicular to those of σ and τ but equal to that of the rotation $\sigma\tau$.

Proof: As commuting symmetric operators, ρ and σ are simultaneously diagonalizable. Since rotations through 180° are completely determined by their axes, the axis of ρ must be a non-axis eigenvector of σ, hence orthogonal to the axis of σ. Ditto for ρ and τ.

The axis of ρ is thus reversed by both σ and τ, hence left fixed by their product. (Incidentally, the angle of the rotation $\sigma\tau$ equals twice the angle between the axes of σ and τ.)

THEOREM 2: If $G < SO_3$ has signature (iii), its action on $\mathrm{Syl}_2(G)$ defines an isomorphism of G with A_5.

Proof: Since the signature shows no orbits of cardinality $15/k$, there are no elements of order 4. Hence every $H \in \mathrm{Syl}_2(G)$ consists of the identity and three "turns", i.e., elements of order 2. Another look at the signature shows that G contains exactly 15 such turns, each sharing its axis with no other element of G.

Let $\langle\rho,\sigma\rangle$ and $\langle\rho,\tau\rangle$ be non-trivially intersecting members of $\mathrm{Syl}_2(G)$. Since ρ shares its axis with no other element of G, Lemma 2 forces $\sigma\tau \in G$ to equal ρ, whence $\langle\rho,\sigma\rangle = \langle\rho,\tau\rangle$. The 15 turns of G therefore make up 5 Sylow 2-groups H_1, H_2, H_3, H_4, H_5.

As in the proof of Theorem 1, we now consider the normalizers K_i of these groups. They cannot be cyclic or dihedral, since the signature shows no orbits of cardinality $10/k$ — as it would if G had elements of order 6. Hence, by Lemma 1 and Theorem 1, each K_i is isomorphic to A_4 and contains the Klein 4-group H_i as its unique minimal non-trivial normal subgroup. By Exercise 2, the G-action on $\mathrm{Syl}_2(G)$ defines an injection $G \longrightarrow S_5$; by Exercise 3, its image is A_5.

Finite subgroups of $GL_3(\mathbf{R})$.

PROPOSITION: Every finite subgroup $G < SL_3(\mathbf{R})$ which is neither cyclic nor dihedral must be isomorphic to A_4, S_4, or A_5.

Proof: Let α be the sum of all $\mu^T\mu$ as μ ranges over G (here μ^T denotes the transpose of the matrix μ). Then it is easy to see that α is symmetric and positive definite, hence has a square root β with the same properties. Moreover, $\mu^T\alpha\mu = \alpha$ for all $\mu \in G$. Since $\beta^T = \beta$ and $\beta^2 = \alpha$, it follows for all $\mu \in G$ that

$$(\beta\mu\beta^{-1})^T(\beta\mu\beta^{-1}) = \beta^{-1}\mu^T\alpha\mu\beta^{-1} = \beta^{-1}\alpha\beta^{-1} = I,$$

i.e., that $\beta\mu\beta^{-1}$ is orthogonal. In other words, G is conjugate in $GL_3(\mathbf{R})$ to a finite group of rotations.

Exercise 4: Let $T < GL_3(\mathbf{R})$ be the subgroup consisting of $\pm I$. Show that every finite subgroup $H < GL_3(\mathbf{R})$ is contained in $SL_3(\mathbf{R}) \times T$. Conclude that H is either equal to $G \times T$ or isomorphic to G, where G is a suitable subgroup of $SL_3(\mathbf{R})$. (*Hint:* Restricted to H, the projection $SL_3(\mathbf{R}) \times T \longrightarrow SL_3(\mathbf{R})$ is either injective or has kernel T.)

Regular polyhedra.

Exercise 5: Remember the set $P \subset S$ of poles. We know $P = T_1 \cup T_2 \cup T_3$, with $|T_i| = n_i$. For each of the cases in (†), show that the points of T_3 are the vertices of a regular n_2-hedron with f_3 triangular faces around each vertex.

We shall do this exercise in Case (iii), leaving the two easier cases for the reader. Pick a pair v, $v_1 \in T_3$ with minimal angular distance $\delta(v, v_1) > 0$. Since $|G_v| = f_3 = 5$, the G_v-orbit of v_1 consists of 5 "neighbours" $\{v_1, \ldots, v_5\}$ of v. All these points lie in the "northern" hemisphere, whose pole is v, because $\delta(v, v_1) \leq \delta(v_1, v_2) \leq 72°$. A similar system of 6 points populates the southern hemisphere — and that accounts for all 12 elements of T_3.

Around v we have 5 "triangles" $\Delta_i = (v, v_i, v_{i+1})$, with $i \in \mathbf{F}_5$, and around every $\rho v \in T_3$, with $\rho \in G$, a congruent system $\rho\Delta_i$. To see that Δ_1 is equilateral, note that the great circles joining v to v_i and v_{i+1} make an angle of 72° at v. This must also happen at v_2, with respect to *its* neighbours, one of which is v. In particular, the two neighbours of v_2 adjacent to v must lie in the northern hemisphere, and hence can be none other than v_1 and v_3. Therefore $\delta(v, v_1) = \delta(v, v_2) = \delta(v_1, v_2)$.